QUANTUM APPROXIMATION ALGORITHMS

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Many-body Quantum Systems via Classical and Quantum Computation
Classical approaches for quantum Hamiltonians (DMRG, mean-field methods, everything else)

Quantum approaches for quantum Hamiltonians (e.g. AQC, QAOA for quantum Hamiltonians)

Quantum approaches for discrete optimization (AQC, QAOA for quantum Hamiltonians)

Quantum approaches for continuous optimization

WHAT IS QUANTUM OPTIMIZATION?
THANKS FOR THE SOAPBOX!

Quantum optimization problems aren’t worlds apart from classical ones

We should exploit connections between them for fun and profit
TALE AS OLD AS TIME

Alice
Analyst

Bob
Best-thing finder

Problem
Solution
TALE AS OLD AS TIME

Alice Analyst

Problem

Solution: x1

Bob Best-thing finder

+Secret Sauce

Solution: x2

Bob Best-thing finder

Alice Analyst
TALE AS OLD AS TIME

Alice Analyst

Bob Best-thing finder

+Secret Sauce

Solution: x3

Alice Analyst

Bob Best-thing finder

+Secret Sauce

Solution: x4
CLASSICAL SPIN ON A QUANTUM HAMILTONIAN

Transverse-field Ising Model:  
\[ H = \sum_{(u,v) \in E} Z_u Z_v - g \sum_{u \in V} X_u \]

Ground state is a classical distribution:  
\[ |\psi\rangle = \sum_{x \in \{0,1\}^n} \sqrt{p_x} |x\rangle \]
CLASSICAL SPIN ON A QUANTUM HAMILTONIAN

Transverse-field Ising Model: \[ H = \sum_{(u,v) \in E} Z_u Z_v - g \sum_{u \in V} X_u \]

Ground state is a classical distribution: \[ |\psi\rangle = \sum_{x \in \{0,1\}^n} \sqrt{p_x} |x\rangle \]

\[ \langle \psi | \sum_{(u,v)} Z_u Z_v |\psi\rangle = \sum_{x \in \{0,1\}^n} \sqrt{p_x} \langle x | Z_u Z_v |x\rangle = \sum_{z \in \{-1,1\}^n} \sqrt{p_z} Z_u Z_v = \mathbb{E}_z [Z_u Z_v] \]
CLASSICAL SPIN ON A QUANTUM HAMILTONIAN

Transverse-field Ising Model:  \( H = \sum_{(u,v) \in E} Z_u Z_v - g \sum_{u \in V} X_u \)

Ground state is a classical distribution:  \( |\psi\rangle = \sum_{x \in \{0,1\}^n} \sqrt{p_x} |x\rangle \)

\[
\langle \psi | \sum_{(u,v)} Z_u Z_v |\psi\rangle = \sum_{x \in \{0,1\}^n} \sqrt{p_x} \langle x | Z_u Z_v | x \rangle = \sum_{z \in \{-1,1\}^n} \sqrt{p_{z u} z v} = \mathbb{E}_z [z_u z_v]
\]

\[
\langle \psi | \sum_u X_u |\psi\rangle = \sum_{x,y \in \{0,1\}^n} \sqrt{p_x p_y} \langle x | \sum_u X_u | y \rangle = \sum_{x,y \text{ differ in 1 bit}} \sqrt{p_x p_y}
\]
A CLASSICAL SPIN ON A WELL-KNOWN HAMILTONIAN

$$H = \sum_{(u,v) \in E} Z_u Z_v - g \sum_{u \in V} X_u$$

$$\min_{\{z \in \{-1,1\}^n \text{ w.p. } p_z\}} \mathbb{E}_z[Z_u Z_v] - g \sum_{z, w \text{ differ in 1 bit}} \sqrt{p_z p_w}$$

**Problem**

|ψ⟩

**Quality**

**Diversity**

**Alice**

Analyst

**Bob**

Best-thing finder
A WELL UNDERSTOOD PROBLEM

Input: Graph G

Laplacian of G

\[ L(G) = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix} \]

Output: Max eigenvalue to \( \frac{1}{\text{poly}(|G|)} \) precision

\[ \lambda_{\text{max}}(L(G)) \]

Complexity: in P

What if G = cycle or complete graph?
SUCCINCTLY REPRESENTED GRAPHS

Input: Graph G

G implicitly represents exponentially larger $G'$

Output: Max eigenvalue to $\lambda_{\text{max}}(L(G'))$ precision

$\frac{1}{\text{poly}(|G|)}$

Complexity: Does succinct description of $G'$ help or hinder? How about only verifying the answer?
A HOME FOR SUCCINCT EIGENVALUE PROBLEMS

Input: Graph $G$ & $a \leq b$ with $b - a \geq \frac{1}{\text{poly}(|G|)}$

Implicitly represents exponentially larger $G'$

Output: Decide above, promised one holds $\lambda_{\max}(L(G')) \geq b$ OR $\lambda_{\max}(L(G')) \leq a$?
EXAMPLE: QUANTUM SPIN ON CLASSICAL PROBLEM

Generalized Johnson Graph, $G^k$: Vertices of $G^k$ are $S \subset V$ of size $k$

$\{S, T\}$ is an edge iff $S \Delta T = \{i, j\}$ is an edge of $G$

Quantum Max Cut: Given $G$, compute

$\text{Max}_{1 \leq k \leq n-1} \lambda_{\text{max}}(L(G^k))$

QMA Complete!
(Classical) Max-Cut
(Classical) Max-Cut

\[ G = (V, E) \]
(Classical) Max-Cut

$G = (V, E)$
(Classical) Max-Cut

\[ G = (V, E) \]
(Classical) Max-Cut

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\( G = (V, E) \)
(Classical) Max-Cut

\[ G = (V, E) \]

7 edges cut
(Classical) Max-Cut

\[ G = (V, E) \]

7 edges cut
(Classical) Max-Cut

\[ G = (V, E) \]

7 edges cut
(Classical) **Max-Cut**

\[
G = (V, E)
\]

7 edges cut
(Classical) Max-Cut

\[ G = (V, E) \]

7 edges cut
(Classical) Max-Cut

\[ G = (V, E') \]

12 edges cut
(Classical) Max-Cut

\[ G = (V, E) \]

12 edges cut (the max cut)
(Classical) Max-Cut

\[ G = (V, E) \]
(Classical) Max-Cut

Goal: find partition $f: V \rightarrow \{\square, \blacksquare\}$ maximizing $G = (V, E)$
(Classical) Max-Cut

**Goal:** find partition $f: V \rightarrow \{\square, \blacksquare\}$ maximizing

$$\sum_{(u,v) \in E} 1[ f(u) \neq f(v) ]$$

$G = (V, E)$
(Classical) Max-Cut

Goal: find partition $f : V \rightarrow \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} 1[f(u) \neq f(v)]$$

$G = (V, E)$
Goal: find partition $f: V \rightarrow \{+1, -1\}$ maximizing

$\sum_{(u,v) \in E} 1[f(u) \neq f(v)]$

$G = (V, E)$
(Classical) Max-Cut

Goal: find partition $f: V \to \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} 1[f(u) \neq f(v)]$$
Goal: find partition $f: V \rightarrow \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} \left( \frac{1 - f(u) \cdot f(v)}{2} \right)$$

$G = (V, E)$
(Classical) Max-Cut

Goal: find partition $f: V \rightarrow \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} \left( \frac{1 - f(u) \cdot f(v)}{2} \right) \quad \text{1 if } f(u) \neq f(v)$$

$G = (V, E)$
Goal: find partition $f: V \rightarrow \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} \left( \frac{1 - f(u) \cdot f(v)}{2} \right)$$

$G = (V, E)$

(Classical) Max-Cut
(Classical) Max-Cut

Goal: find partition $f : V \rightarrow \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} \left( \frac{1 - f(u) \cdot f(v)}{2} \right)$$

NP-hard to solve exactly!
(Classical) Max-Cut

Goal: find partition $f: V \to \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} \left( \frac{1 - f(u) \cdot f(v)}{2} \right)$$

NP-hard to solve exactly!

So instead look for approximation algorithms.

$G = (V, E)$
A $\alpha$-approximation algorithm runs in polynomial time, and for any instance $I$, delivers an approximate solution such that:

$$\frac{\text{Value(Approximate}_I)}{\text{Value(Optimal}_I)} \geq \alpha$$

where $\alpha$ is the largest "gap" between optimal and approximate solutions over all instances.
A $\alpha$-approximation algorithm runs in polynomial time, and for any instance $I$, delivers an approximate solution such that:

$$\frac{\text{Value(Approximate}_I)}{\text{Value(Optimal}_I)} \geq \alpha$$

**Heuristics**
- Guided by intuitive ideas
- Perform well on practical instances
- May perform very poorly in worst case
- Difficult to prove anything about performance

**Approximation Algorithms**
- Guided by worst-case performance
- May perform poorly compared to heuristics
- Rigorous bound on worst-case performance
- Designed with performance proof in mind
How far can we go?

0.87856 + \epsilon approximations are **NP-Hard**! (under Unique Games Conjecture)

\[
\frac{1}{2} + \frac{1}{2m} \quad \text{[Vitányi 1981]}
\]

\[
\frac{1}{2} + \frac{1}{2\Delta} \quad \text{[Hofmeister, Lefmann 1995]}
\]

0.87856

\[
\frac{1}{2} + \frac{1}{2n} \quad \text{[Haglin, Venkatesan 1991]}
\]

\[
\frac{1}{2} + \frac{1}{2m} \quad \text{[Goemans, Williamson 1995]}
\]

\[
\frac{1}{2} + \frac{1}{2\Delta} \quad \text{[Khot, Kindler Mossel, O’Donnell 2007]}
\]

 Slide courtesy of Yeongwoo Hwang
Quantum Max-Cut

Following slides courtesy of John Wright
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H = \sum_{(u,v) \in E} h_{uv} \]
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H = \sum_{(u,v) \in E} h_{uv}, \text{ where } h = \frac{1}{4} \cdot (I - XX - YY - ZZ) \]
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H = \sum_{(u,v) \in E} h_{uv}, \text{ where } h = \frac{1}{4} \cdot (I - XX - YY - ZZ) \]

only depends on \( G \)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} h_{uv}, \text{ where } h = \frac{1}{4} \cdot (I - XX - YY - ZZ) \]

only depends on \( G \)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

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Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Goal: Output the maximum energy state of \( H_G \)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

**Goal:** Output the **maximum energy state** of \( H_G \)

**Note:** max energy state of \( H_G \)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

**Goal:** Output the **maximum energy state** of \( H_G \)

**Note:** max energy state of \( H_G \)

= **min** energy state of \( \sum_{(u,v)\in E} (X_u X_v + Y_u Y_v + Z_u Z_v) \)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

**Goal:** Output the **maximum energy state** of \( H_G \)

**Note:** max energy state of \( H_G \)

\[ = \text{min energy state of } \sum_{(u,v) \in E} (X_u X_v + Y_u Y_v + Z_u Z_v) \]

(antiferromagnetic) **Heisenberg model**
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_uX_v - Y_uY_v - Z_uZ_v) \]

**Goal:** Output the **maximum energy state** of \( H_G \)

**Note:** max energy state of \( H_G \)

= **min** energy state of \( \sum_{(u,v) \in E} (X_uX_v + Y_uY_v + Z_uZ_v) \)

(antiferromagnetic) **Heisenberg model**

Dates back to [Heisenberg 1928]

Well-studied class of Hamiltonians
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Intuition
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Intuition

|\psi\rangle (n qubits)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Intuition

|\psi\rangle (n qubits)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

Intuition

$$\left| \psi \right\rangle$$ ($n$ qubits)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

**Intuition**

\[ |\psi\rangle (n \text{ qubits}) \]

**Term 1:** Does nothing
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

**Intuition**

$$|\psi\rangle (n \text{ qubits})$$

**Term 1:** Does nothing
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Intuition

Term 1: Does nothing

Term 2: Measure in X basis

\[ |\psi\rangle (n \text{ qubits}) \]
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

Intuition

Term 1: Does nothing

Term 2: Measure in $X$ basis

$|\psi\rangle$ ($n$ qubits)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Intuition

|\psi\rangle (n qubits)

Term 1: Does nothing

Term 2: Measure in X basis
- \(-1\) if same (++ or --)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

**Intuition**

|ψ⟩ (n qubits)

**Term 1:** Does nothing

**Term 2:** Measure in X basis
- −1 if same (+ + or − −)
- +1 if different (+ − or − +)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

Intuition

Term 1: Does nothing

Term 2: Measure in $X$ basis
- $-1$ if same ($++$ or $--$)
- $+1$ if different ($+-$ or $-+$)

want both different!

$|\psi\rangle$ ($n$ qubits)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[ H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v) \]

Intuition

Term 1: Does nothing

Term 2: Should be different in X basis
Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_uX_v - Y_uY_v + Z_uZ_v)$$

**Intuition**

**Term 1:** Does nothing

**Term 2:** Should be different in $X$ basis

**Term 3:** Should be different in $Y$ basis

$|\psi\rangle$ ($n$ qubits)
Quantum Max-Cut

Special case of 2-local Hamiltonian:

\[
H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)
\]

Intuition

**Term 1:** Does nothing

**Term 2:** Should be different in \(X\) basis

**Term 3:** Should be different in \(Y\) basis

**Term 4:** Should be different in \(Z\) basis

\(|\psi\rangle (n \text{ qubits})\)
Quantum Max-Cut
Special case of 2-local Hamiltonian:

\[
H_G = \sum_{(u,v) \in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)
\]

**Intuition**

**Term 1:** Does nothing

**Term 2:** Should be different in \( X \) basis

**Term 3:** Should be different in \( Y \) basis

**Term 4:** Should be different in \( Z \) basis

Like (classical) Max-Cut in \( X, Y, \) and \( Z \) bases!
Product states for QMax-Cut
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: ⓞ ⓞ ⓞ ⓞ ⓞ ⓞ ⓞ ⓞ ⓞ
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: 

$|\psi_u\rangle$ $|\psi_v\rangle$
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: ⚫⚫⚫⚫⚫⚫⚫⚫⚫

$|\psi_u\rangle$  $|\psi_v\rangle$

Product states possess no entanglement
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: $\bigotimes_{i=1}^{n} |\psi_i\rangle$

Product states possess no entanglement

But they can often be close to the ground state!
Product states for QMax-Cut

States of the form \( |\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle \)

\( n \) qubits: \( \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \)

\( |\psi_u\rangle \quad |\psi_v\rangle \)

Product states possess no entanglement

But they can often be close to the ground state!

[Brandao Harrow 2016]: The ground state is close to product if \( G \) is high degree.
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: $\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$

$|\psi_u\rangle$ $|\psi_v\rangle$
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: • • • • • • • • • •

$|\psi_u\rangle$ $|\psi_v\rangle$

Useful to look at **Bloch sphere** representation.
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: □ □ □ □ □ □ □ □

$|\psi_u\rangle \quad |\psi_v\rangle$

Useful to look at **Bloch sphere** representation.

**Bloch sphere:** Each single-qubit state $|\psi_u\rangle$ can be associated with a real vector $(c_X, c_Y, c_Z)$ such that $c_X^2 + c_Y^2 + c_Z^2 = 1$. 
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits:

Useful to look at **Bloch sphere** representation.

**Bloch sphere:** Each single-qubit state $|\psi_u\rangle$ can be associated with a real vector $(c_X, c_Y, c_Z)$ such that $c_X^2 + c_Y^2 + c_Z^2 = 1$.

Set $f(u) = (c_X, c_Y, c_Z)$. Then $f: V \rightarrow S^2$. 

|ψ_u⟩ |ψ_v⟩
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: $\bigotimes_{u \in V} |\psi_u\rangle$ $|\psi_v\rangle$

Useful to look at **Bloch sphere** representation.

**Bloch sphere:** Each single-qubit state $|\psi_u\rangle$ can be associated with a real vector $(c_X, c_Y, c_Z)$ such that $c_X^2 + c_Y^2 + c_Z^2 = 1$.

Set $f(u) = (c_X, c_Y, c_Z)$. Then $f : V \rightarrow S^2$. Unit sphere in $\mathbb{R}^3$. 
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

$n$ qubits: $\bigotimes_{u \in V} |\psi_u\rangle$ | $|\psi_v\rangle$

Useful to look at **Bloch sphere** representation.

**Bloch sphere:** Each single-qubit state $|\psi_u\rangle$ can be associated with a real vector $(c_X, c_Y, c_Z)$ such that $c_X^2 + c_Y^2 + c_Z^2 = 1$.

Set $f(u) = (c_X, c_Y, c_Z)$. Then $f: V \rightarrow S^2$.

unit sphere in $\mathbb{R}^3$
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

**Bloch sphere** representation: $f: V \rightarrow S^2$. 
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

**Bloch sphere** representation: $f : V \rightarrow S^2$.

Product state energy formula:

$$\langle \psi | H_G |\psi \rangle = \sum_{(u,v) \in E} \left( \frac{1 - \langle f(u), f(v) \rangle}{4} \right)$$
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

**Bloch sphere** representation: $f : V \rightarrow S^2$.

Product state energy formula:

$$\langle \psi | H_G | \psi \rangle = \sum_{(u,v) \in E} \left(1 - \frac{\langle f(u), f(v) \rangle}{4}\right)$$

“Want” neighboring $f(u)$ and $f(v)$ to point in opposite directions.
Product states for QMax-Cut

States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

**Bloch sphere** representation: $f: V \rightarrow S^2$.

Product state energy formula:

$$\langle \psi | H_G | \psi \rangle = \sum_{(u,v) \in E} \left( \frac{1 - \langle f(u), f(v) \rangle}{4} \right)$$

"Want" neighboring $f(u)$ and $f(v)$ to point in opposite directions.

Like **(classical)** Max-Cut! There, $f: V \rightarrow \{\pm 1\} = S^0$. 
How far can we go?

Approximation algorithms for quantum Max

0.498 (optimal for basic SDP) [Gharibian, P 2019]

0.533 (2nd level SDP) [P, Thompson 2021]

0.5 (optimal for product states) [P, Thompson 2022]

0.53 (SDP + Circuit) [Lee 2022]

Beyond product states: 0.53 [Anshu, Gossett, Morenz 2020]

Beyond product states: 0.53 [Anshu, Gossett, Morenz 2020]

0.562 (As Lee, but Δ-free) [King 2022]

0.582 (SDP + Circuit) [Lee 2022]

0.956 [Hwang, Neeman, P, Thompson, Wright 2022]

Classical Intuition: Best possible?

NP-hardness Barrier

Slide courtesy of Yeongwoo Hwang
First approximations for Max k-Local Hamiltonian

Classical approximation scheme for planar graphs:

First nontrivial general approximations:
Classical approximation scheme for dense instances

Near-optimal product-state approx for special cases:
Uses semidefinite programming (SDP) for bounds

Approximation w.r.t. number of terms and degree:

[Bansal, Bravyi, Terhal 2007: arXiv 0705.1115]

[Gharibian, Kempe 2011: arXiv 1101.3884]

[Brandao, Harrow 2013: arXiv 1310.0017]

[Harrow, Montanaro 2015: arXiv 1507.00739]

All of these results use product states
## Recent approximations for Max 2-Local Hamiltonian

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td><strong>Max traceless 2-LH:</strong> $\sum_{ij} H_{ij}$, $H_{ij}$ traceless</td>
<td><strong>Max Ising:</strong> Max $-\sum_{ij} z_i z_j$, $z_i \in {-1,1}$</td>
<td>$\Omega(1/\log n)$ [Charikar, Wirth ‘04]</td>
<td>$\Omega(1/\log n)$ [Bravyi, Gosset, Koenig, Temme ‘18] 0.184 (bipartite, no 1-local terms) [P, Thompson ‘20]</td>
</tr>
<tr>
<td><strong>Max positive 2-LH:</strong> $\sum_{ij} H_{ij}$, $H_{ij} \succeq 0$</td>
<td><strong>Max 2-CSP</strong></td>
<td>0.874 [Lewin, Livnat, Zwick ‘02]</td>
<td>0.25 [Random assignment] 0.282 [Hallgren, Lee ‘19] 0.328 [Hallgren, Lee, P ‘20] 0.387 / 0.498 (numerical) [P, Thompson ‘20] 0.5 (best possible via product states) [P, Thompson ‘21]</td>
</tr>
<tr>
<td><strong>Quantum Max Cut:</strong> $\sum_{ij} I - X_i X_j - Y_i Y_j - Z_i Z_j$ (special case of above)</td>
<td><strong>Max Cut:</strong> Max $\sum_{ij} I - z_i z_j$, $z_i \in {-1,1}$</td>
<td>0.878 [Goemans, Williamson ‘95]</td>
<td>0.498 [Gharibian, P ‘19] 0.5 [P, Thompson ‘22] 0.53* [Anshu, Gosset, Morenz ‘20] 0.533* [P, Thompson ‘21] 0.562* [Lee ‘22] (also [King ‘22])</td>
</tr>
<tr>
<td><strong>Max 2-Quantum SAT:</strong> $\sum_{ij} H_{ij}$, $H_{ij} \succeq 0$, rank 3</td>
<td><strong>Max 2-SAT</strong></td>
<td>0.940 [Lewin, Livnat, Zwick ‘02]</td>
<td>0.75 [Random Assignment] 0.764 / 0.821 (numerical) [P, Thompson ‘20] 0.833… best possible via product states</td>
</tr>
</tbody>
</table>


* These results are not product-state based
Quantum Max Cut

Model 2-Local Hamiltonian?

Has driven advances in quantum approximation algorithms, based on generalizations of classical approaches

QMA-hard and each term is maximally entangled
[Cubitt, Montanaro 2013]

Recent approximation algorithms
[Gharibian and P. 2019], [Anshu, Gosset, Morentz 2020],
[P. and Thompson 2021, 2021, 2022]

Evidence of unique games hardness
[Hwang, Neeman, P., Thompson, Wright 2021]

Likely that approximation/hardness results transfer to 2-LH with positive terms
[P., Thompson 2021, 2022]
Max Cut vs Quantum Max Cut

Relaxation

$$\text{Max } \sum_{ij \in E} \frac{(1 - v_i \cdot v_j)}{2}$$

$$\|v_i\| = 1, \text{ for all } i \in V$$

$$v_i \in \mathbb{R}^n$$

(upper bound)

$$\text{Max } \sum_{ij \in E} \frac{(1 - 3v_i \cdot v_j)}{4}$$

$$\|v_i\| = 1, \text{ for all } i \in V$$

$$v_i \in \mathbb{R}^n$$

Rounding

$$v_i \in \mathbb{R}^n \rightarrow \alpha_i = \frac{r^T v_i}{\|r^T v_i\|}$$

Approximability

0.878 Lasserre 1

(optimal under unique games conjecture)

0.498 Lasserre 1

0.5 Lasserre 2 (optimal using product states)

(0.533 using 1- & 2-qubit ansatz)
To learn more about Quantum Max Cut...

Optimal product-state approximations:  

Best-known Quantum Max Cut (QMC) approximations:

Lasserre hierarchy in 2-LH approximations:

Prospects for unique-games hardness:

Connections in approximating QMC and 2-LH:

Optimal space-bounded QMC approximations:  

(no quantum advantage possible!)

[P., Thompson 2022: arXiv 2206.08342] (Sections 2,3)


[P., Thompson 2021: arXiv 2105.05698]

[Lee 2022: arXiv 2209.00789]

[King 2022: arXiv 2209.02589]

[Hwang, Neeman, P., Thompson, Wright 2021: arXiv 2111.01254] (Start here: intro and Section 7)

[P., Thompson 2021, 2022 above]


[Anshu, Gosset, Morenz-Korol, Soleimanifar: arXiv 2105.01193]

[Kallaugher, P. 2022: arXiv 2206.00213]
State on \( n \) qubits

\[ \langle \psi | \in \mathbb{C}^{2^n} \]

\[
V = \begin{bmatrix}
\langle x_1 | = \langle \psi | X_1 \\
\langle y_1 | = \langle \psi | Y_1 \\
\langle z_1 | = \langle \psi | Z_1 \\
\vdots \\
\langle x_n | = \langle \psi | X_n \\
\langle y_n | = \langle \psi | Y_n \\
\langle z_n | = \langle \psi | Z_n \\
\end{bmatrix},
M_{ij} = \begin{bmatrix}
\langle \psi | X_i X_j | \psi \rangle & \langle x_i | y_j \rangle & \langle x_i | z_j \rangle \\
\langle y_i | x_j \rangle & \langle y_i | y_j \rangle & \langle y_i | z_j \rangle \\
\langle z_i | x_j \rangle & \langle z_i | y_j \rangle & \langle z_i | z_j \rangle
\end{bmatrix}
\]

Entries of this \( 3n \times 3n \) moment matrix are expectation values of all 2-local Pauli terms

\[
= VV^\dagger \succeq 0 \Rightarrow Re(VV^\dagger) \succeq 0
\]
Quantum Max Cut SDP Relaxation

\[
\begin{bmatrix}
X_1 & Y_1 & Z_1 & X_2 & Y_2 & Z_2 & X_3 & Y_3 & Z_3 \\
1 & 0 & 0 & M_{12} & M_{13} & & & & \\
0 & 1 & 0 & & & & & & \\
0 & 0 & 1 & & & & & & \\
1 & 0 & 0 & M_{12} & & M_{23} & & & \\
0 & 1 & 0 & & M_{23} & & & & \\
0 & 0 & 1 & & & & M_{13} & & \\
1 & 0 & 0 & & & & & M_{23} & \\
0 & 0 & 1 & & & & & & \ldots
\end{bmatrix}
\]

Real part of moment matrix

**Quantum Max Cut vector relaxation**

Max \( \sum_{ij \in E} (1 - x_i \cdot x_j - y_i \cdot y_j - z_i \cdot z_j)/4 \)

\( \|x_i\|, \|y_i\|, \|z_i\| = 1, \text{ for all } i \in V \)

\( x_i \cdot y_i = x_i \cdot z_i = y_i \cdot z_i = 0, \text{ for all } i \in V \)

\( (v_i \in \mathbb{R}^{3n}) \)

\( v_i = (x_i \oplus y_i \oplus z_i)/\sqrt{3} \)

**Max Cut vector relaxation**

Max \( \sum_{ij \in E} (1 - v_i \cdot v_j)/4 \)

\( \|v_i\| = 1, \text{ for all } i \in V \)

\( (v_i \in \mathbb{R}^n) \)
Quantum Lasserre Hierarchy

\[
\begin{array}{cccccc}
I & X_1 & Y_1 & Z_1 & X_2 & \ldots \\
X_1 & 1 & 1 & 1 & \ldots & \\
Y_1 & 1 & \ddots & \ddots & \ddots & \\
Z_1 & & \ddots & \ddots & \ddots & \\
X_2 & & & \ddots & \ddots & \ddots \\
\vdots & & & & \ddots & \ddots \\
Z_n & & & & & \ddots \\
X_1X_2 & & & & & 1 \\
\vdots & & & & & \ddots \\
Z_{n-1}Z_n & & & & & 1 \\
X_1X_2X_3 & & & & & 1 \\
\vdots & & & & & \ddots \\
Z_1Z_2 & \ldots & & & & 1 \\
\vdots & & & & & \ddots \\
Z_1 & \ldots & \ldots & \ldots & \ldots & Z_n \\
\end{array}
\]

\[
\text{Max } Tr[H\tilde{\rho}] \\
Tr[\tilde{\rho}] = 1 \\
Tr[\tilde{\rho} S^\dagger S] \geq 0, \forall \text{ deg-}k S
\]

\tilde{\rho} \text{ is called degree-}k \text{ pseudo density}

Classical  
[Lasserre 2001]  
[Parillo 2003]

Non-commutative/Quantum  
Rounding Infeasible Solutions

**α-Approximation Algorithm**

Round optimal non-positive pseudo-density \( \tilde{\rho} \) to sub-optimal positive density \( \rho \) so that:

\[
Tr[H\rho] \geq \alpha Tr[H\tilde{\rho}] \geq \alpha \lambda_{max}(H)
\]

\( \tilde{\rho} \) is called degree-k pseudo density
QUANTUM STREAMING ADVANTAGES
Space Efficiency

We would like algorithms that need very few bits/qubits

Ideally a number \textit{sublinear} in the size of the input, e.g. \(O(\sqrt{n})\) or \(O(\log(n))\) for a size-\(n\) input
Why Space-Efficient Algorithms?

Two reasons, pointing to different kinds of algorithm:

Qubits are expensive

- Even under the most optimistic assumptions, qubits will continue to be much more expensive than classical bits
- Motivates algorithms that use very few *qubits*, but maybe many classical bits

Qubits can be exponentially more powerful than classical bits

- We know there are problems that require exponentially fewer qubits than bits
- This is provable! (unlike with time complexity)
- Motivates looking at algorithms that use very little total space (bits + qubits) (and impossibility results)

Our focus has been on the second case
Streaming Algorithms

When dealing with very small space algorithms, it matters *how you receive the input dataset*

**Streaming**
- Dataset is built up by a “stream” of small updates
- Answer is expected at the end of the stream

Examples
- Calculating traffic statistics on a router
- Estimating properties of a large social networking graph given as a sequence of friendships
QUANTUM STREAMING ADVANTAGES FOR GRAPH PROBLEMS

Exponential advantage for Boolean Hidden Matching
[Gavinsky, Kempe, Kerenidis, Raz, and de Wolf 2008]

First natural problem: polynomial advantage for triangle counting
[Kallaugher 2021]

No quantum advantage possible: Max Cut or Quantum Max Cut
[Kallaugher, P 2022]

Exponential advantage for natural problem: Directed Max Cut
[Kallaugher, P, Voronova 2023]
QUANTUM GENERALIZATIONS OF VERTEX COVER
VERTEX COVER

$G = (V, E)$

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint
VERTEX COVER

\[ G = (V, E) \]

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint.
VERTEX COVER

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

\[ G = (V, E) \]
VERTEX COVER

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

\[ G = (V, E) \]
VERTEX COVER

\[ G = (V, E) \]

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint
**VERTEX COVER**

\[ G = (V, E) \]

**6 vertices colored**

Optimal since each 5-cycle needed 3!

**Goal:** color minimum number of vertices, so each edge has at least 1 colored endpoint
**VERTEX COVER**

\[ G = (V, E) \]

**Goal:** *color* minimum number of vertices, so each edge has at least 1 colored endpoint

**NP-hard:** one of Karp’s original 21 problems

Several *2-approximations* known
e.g. [Bar-Yehuda, Bendel, Freund, Rawitz 2004]

Best possible under *Unique Games Conjecture* [Khot, Regev 2008]
VERTEX COVER AS CONSTRAINED LOCAL HAMILTONIAN

\[
\min_{|\psi\rangle} \langle \psi | \sum_u |1\rangle\langle 1|_u |\psi\rangle
\]

\[
\langle \psi | 00\rangle\langle 00|_{uv} |\psi\rangle = 0 \quad \text{for all edges (u,v)}
\]

Unhappy edge

Happy edge
PUT A TRANSVERSE FIELD ON IT

\[
\min \langle \psi | \sum_u (I - Z_u)/2 | \psi \rangle \\
\langle \psi | (1 - Z_u)(I + Z_u)/4 | \psi \rangle = 0 \text{ for all edges } (u,v)
\]
PUT A TRANSVERSE FIELD ON IT

\[
\min_{|\psi\rangle} \langle \psi | \sum_u (I - Z_u)/2 |\psi\rangle + \sum_u X_u
\]

\[
\langle \psi | (I + Z_u)(I + Z_v)/4|\psi\rangle = 0 \quad \text{for all edges } (u,v)
\]

Equivalent to PXP model (Rydberg blockade interactions)

We show Transverse Vertex Cover/PXP are StoqMA-complete

Simple \((2 + \sqrt{2})\)-approximation via quantum version of local ratio

[P, Rayudu, Thompson 2023]
QUANTUM SCARS
Thanks for staying awake to read this!