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QuAAC

QUANTUM APPROXIMATION ALGORITHMS

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ipam Many-body Quantum Systems via Classical and Quantum Computation

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U.S. DEPARTMENT OF ENERGY

NNSA

WHAT IS QUANTUM OPTIMIZATION?

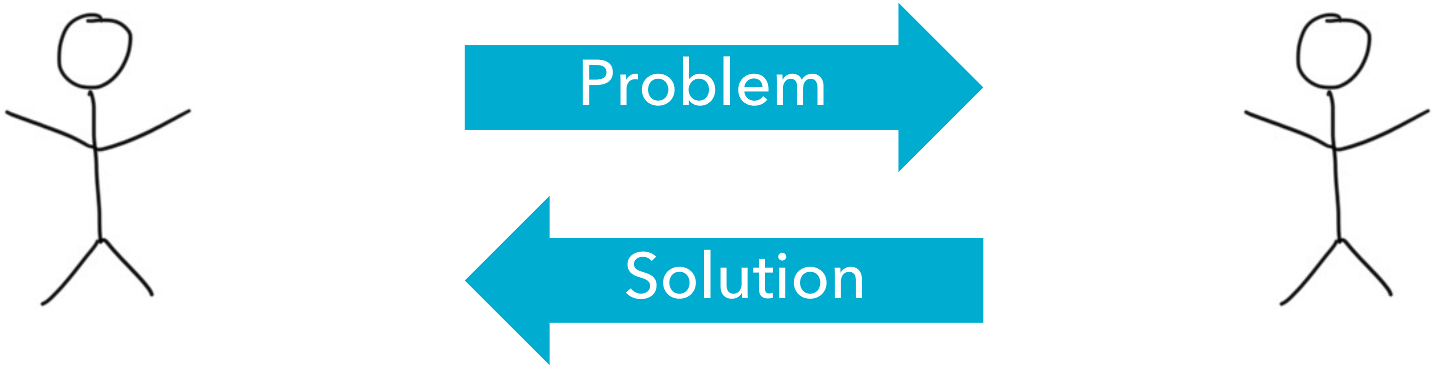
Algorithm \ Problem	Classical	Quantum
Classical	Classical optimization	Classical approaches for quantum Hamiltonians (DMRG, mean-field methods, everything else)
Quantum	Quantum approaches for discrete optimization (AQC, QAOA for quantum Hamiltonians) Quantum approaches for continuous optimization	Quantum approaches for quantum Hamiltonians (e.g. AQC, QAOA for quantum Hamiltonians)

THANKS FOR THE SOAPBOX!

Quantum optimization problems aren't worlds apart from classical ones

We should exploit connections between them for fun and profit

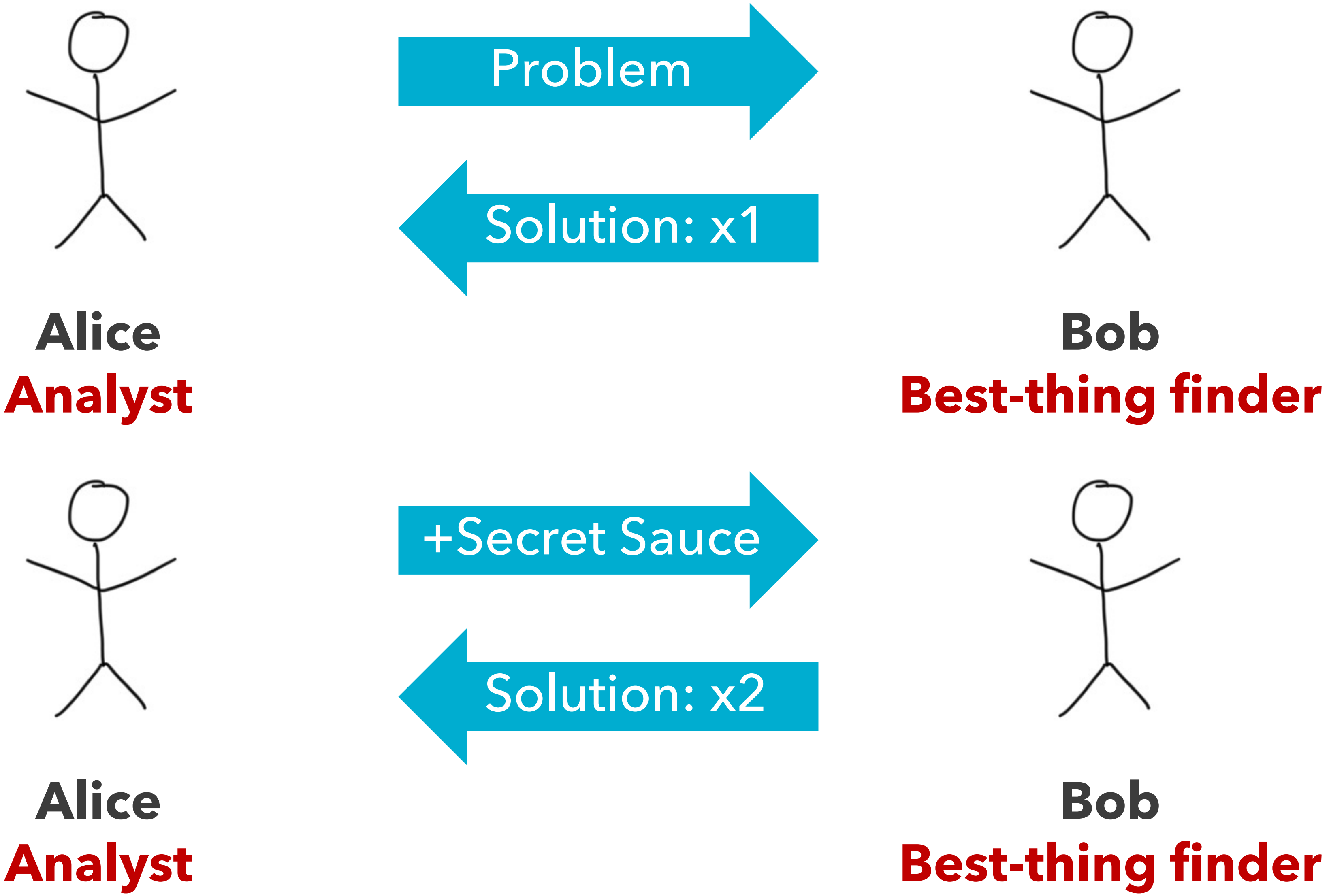
TALE AS OLD AS TIME



Alice
Analyst

Bob
Best-thing finder

TALE AS OLD AS TIME



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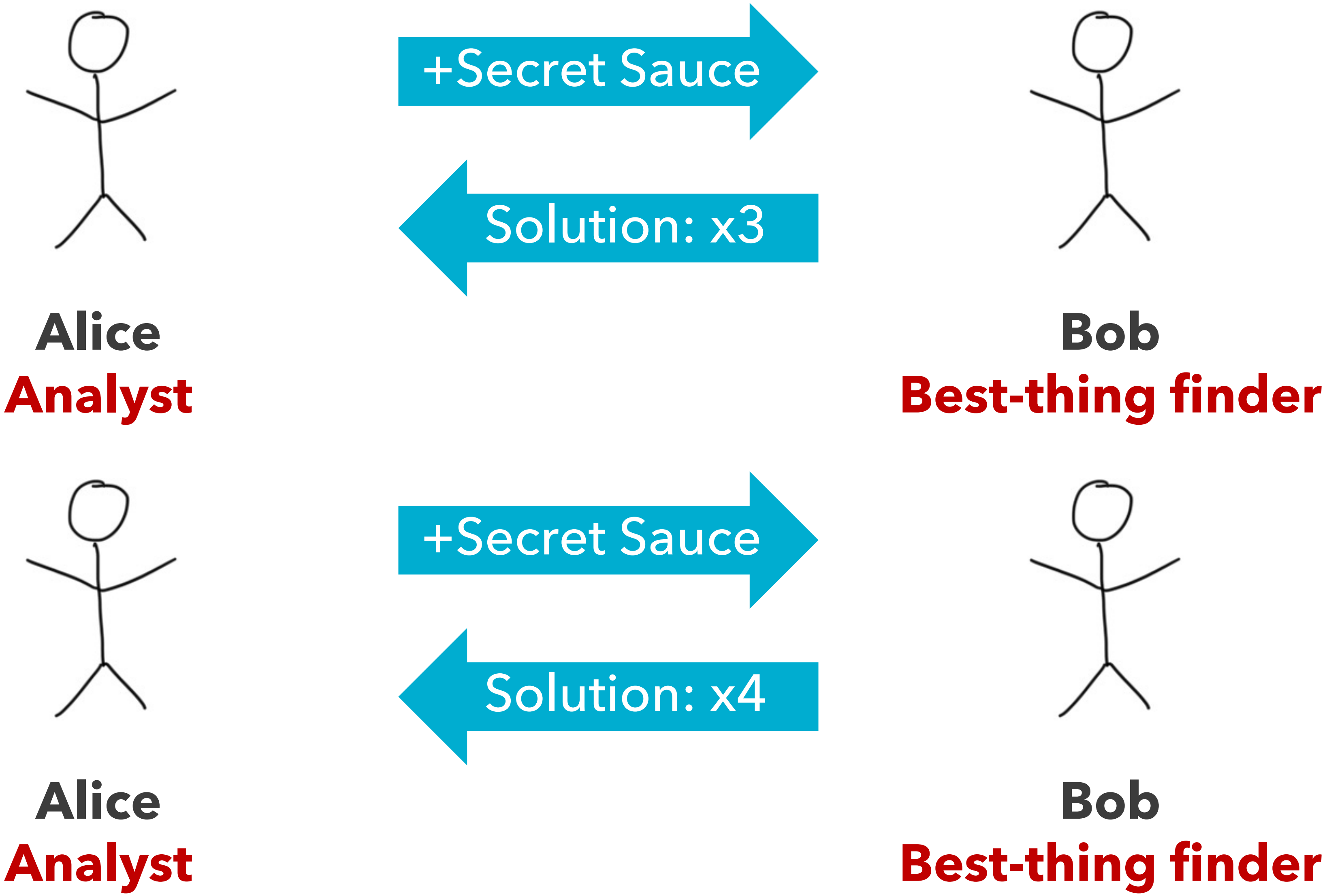

CLASSICAL SPIN ON A QUANTUM HAMILTONIAN

Transverse-field Ising Model: $H = \sum_{(u,v)\in E} Z_u Z_v - g \sum_{u\in V} X_u$

Ground state is a classical distribution: $|\psi\rangle = \sum_{x\in\{0,1\}^n} \sqrt{p_x}\,|x\rangle$

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$$\langle\psi| \textstyle\sum_{(u,v)} Z_u Z_v |\psi\rangle = \sum_{x\in\{0,1\}^n} \sqrt{p_x} \langle x | Z_u Z_v | x \rangle = \sum_{z\in\{-1,1\}^n} \sqrt{p_z} z_u z_v = \mathbb{E}_z[z_u z_v]$$

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$$\langle\psi|\sum_{(u,v)} Z_u Z_v|\psi\rangle = \sum_{x\in\{0,1\}^n} \sqrt{p_x}\langle x|Z_u Z_v|x\rangle = \sum_{z\in\{-1,1\}^n} \sqrt{p_z} z_u z_v = \mathbb{E}_z[z_u z_v]$$

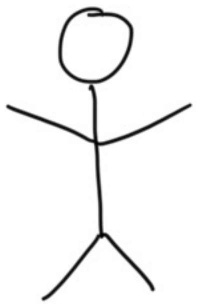
$$\langle\psi|\sum_u X_u|\psi\rangle = \sum_{x,y\in\{0,1\}^n} \sqrt{p_x p_y}\,\langle x|\sum_u X_u|y\rangle = \sum_{x,y \text{ differ in 1 bit}} \sqrt{p_x p_y}$$

A CLASSICAL SPIN ON A WELL-KNOWN HAMILTONIAN

$$H = \sum_{(u,v)\in E} Z_u Z_v - g \sum_{u \in V} X_u \;\Rightarrow\; \min_{\{z\in\{-1,1\}^n \text{ w.p. } p_z\}} \mathbb{E}_z[z_u z_v] - g \sum_{z,w \text{ differ in 1 bit}} \sqrt{p_z p_w}$$

Quality

Diversity



Problem

$|\psi\rangle$

Alice
Analyst

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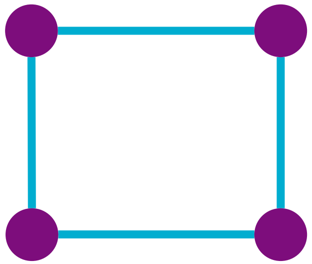
QUANTUM MAX CUT

A WELL UNDERSTOOD PROBLEM



$$L(G) = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\lambda_{max}(L(G))$$

Input: Graph G

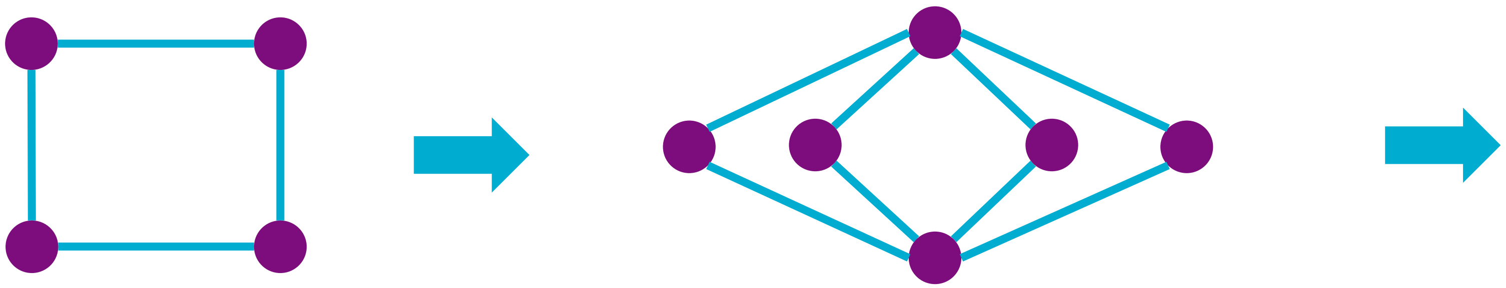
Laplacian of G

Output: Max eigenvalue to $\frac{1}{\mathrm{poly}(|G|)}$ precision

Complexity: in P

What if G = cycle or complete graph?

SUCCINCTLY REPRESENTED GRAPHS



$\lambda_{max}(L(G'))$

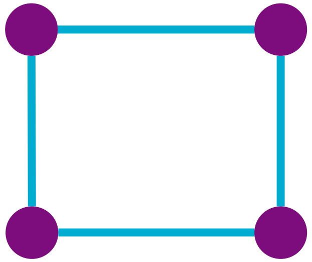
Input: Graph G

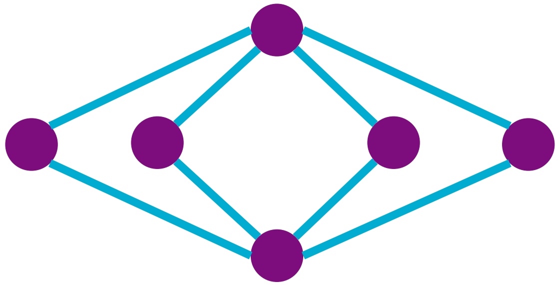
G implicitly represents exponentially larger G′

Output: Max eigenvalue to $\frac{1}{\text{poly}(|G|)}$ precision

Complexity: Does succinct description of G′ help or hinder?
How about only verifying the answer?

A HOME FOR SUCCINCT EIGENVALUE PROBLEMS





$$\lambda_{max}(L(G')) \geq b$$
OR
$$\lambda_{max}(L(G')) \leq a?$$

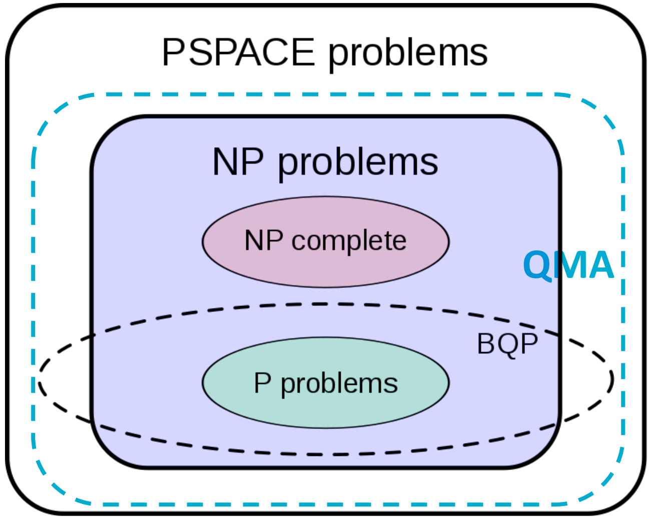
Input: Graph G &
$a \leq b$ with
$$b - a \geq \frac{1}{\text{poly}(|G|)}$$

Implicitly represents exponentially larger G′

Output: Decide above, promised one holds

$|v\rangle$ → **Polynomial-time quantum computation** →

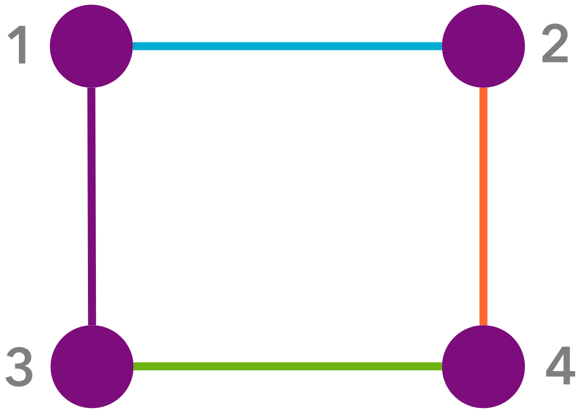
$$v^{\dagger}L(G')v \geq b$$
OR
$$v^{\dagger}L(G')v \leq a$$

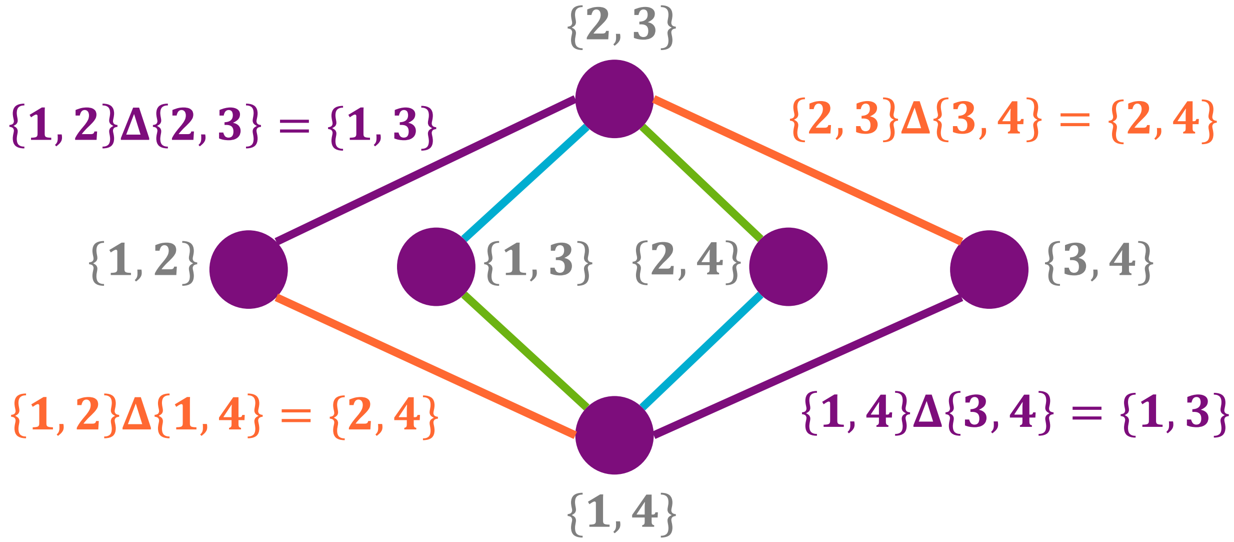


https://en.wikipedia.org/wiki/BQP

Poly-time quantum verifier puts problem in QMA

EXAMPLE: QUANTUM SPIN ON CLASSICAL PROBLEM





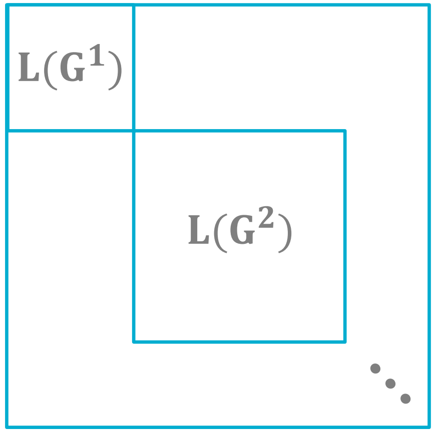
Generalized Johnson Graph, G^k: Vertices of G^k are $S \subset V$ of size k

$\{S, T\}$ is an edge iff $S \Delta T = \{i, j\}$ is an edge of G

Quantum Max Cut: Given G, compute

$$\mathrm{Max}_{1 \le k \le n-1}\, \lambda_{max}(L(G^k))$$

$H =$



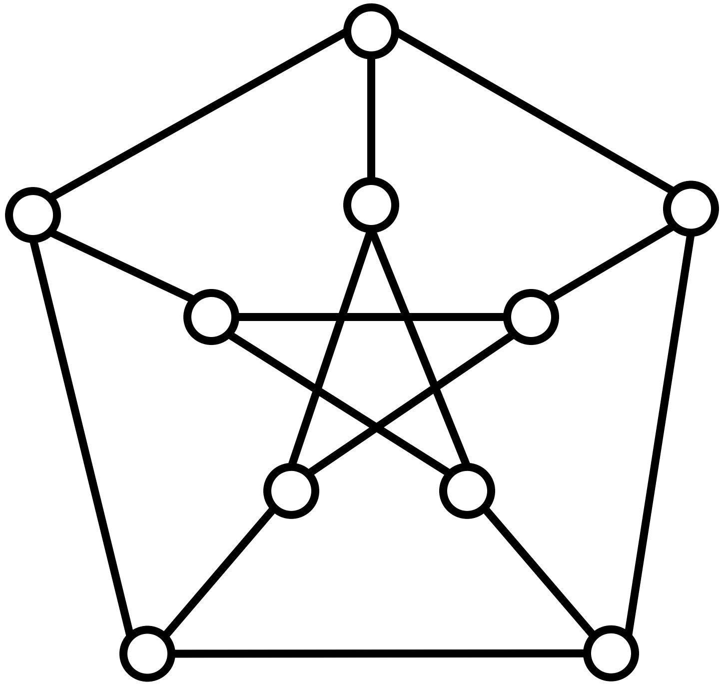
compute $\lambda_{max}(H)$

QMA Complete!

(Classical) Max-Cut

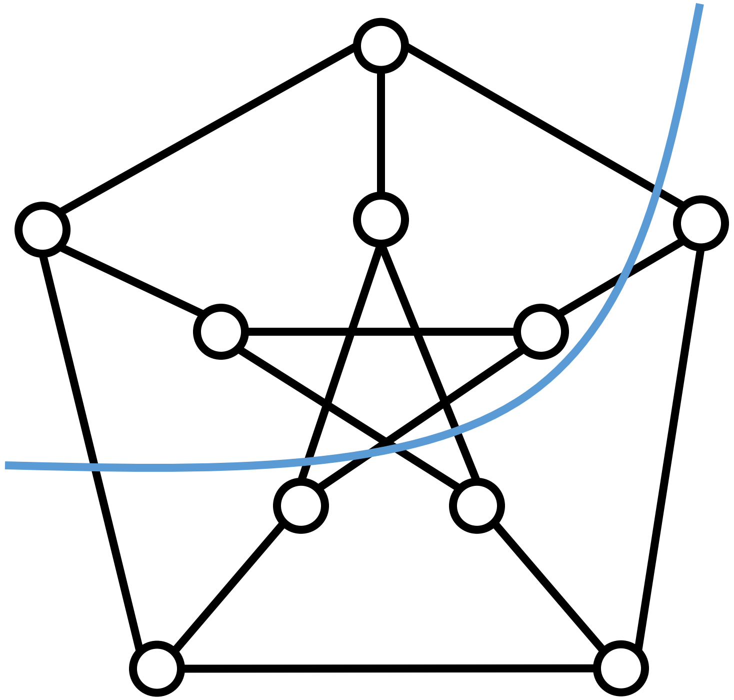
Following slides courtesy of John Wright

(Classical) Max-Cut



$G = (V, E)$

(Classical) Max-Cut



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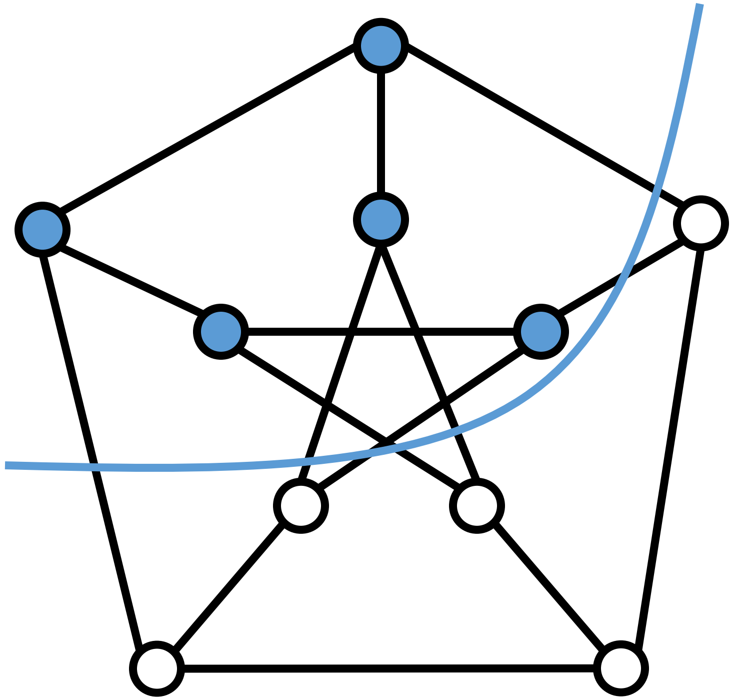

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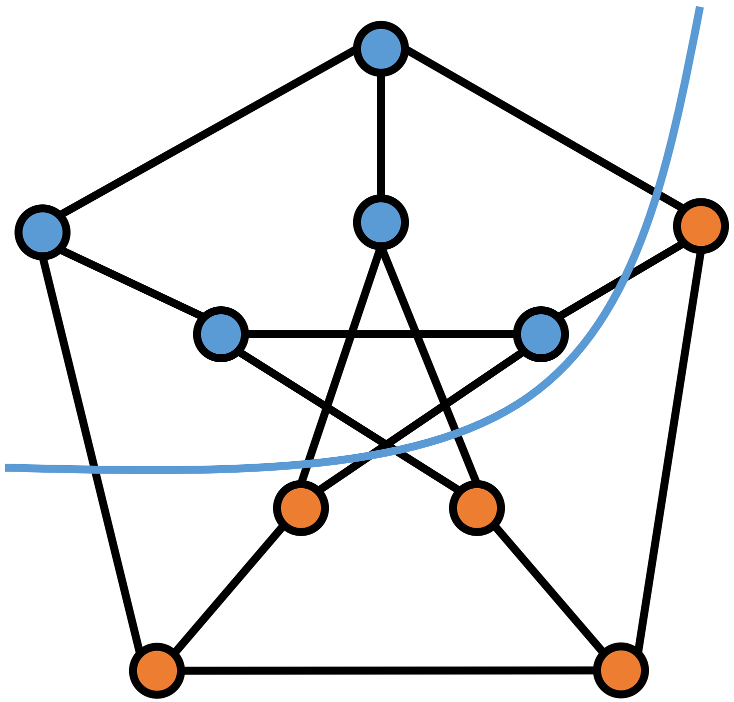

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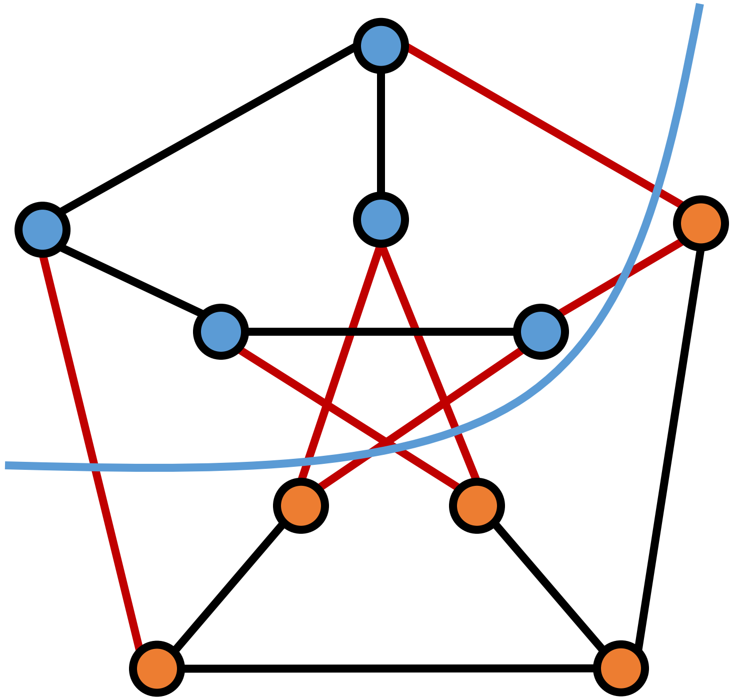

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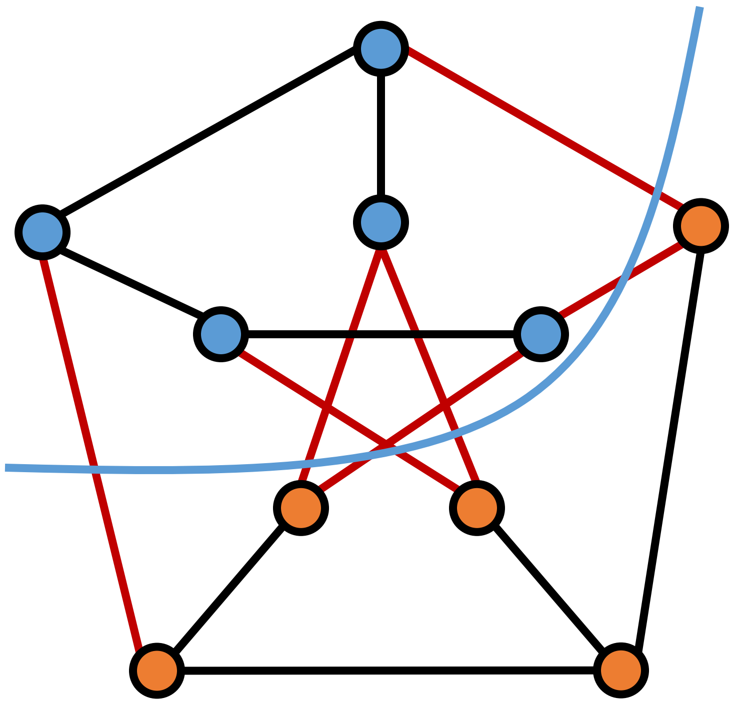
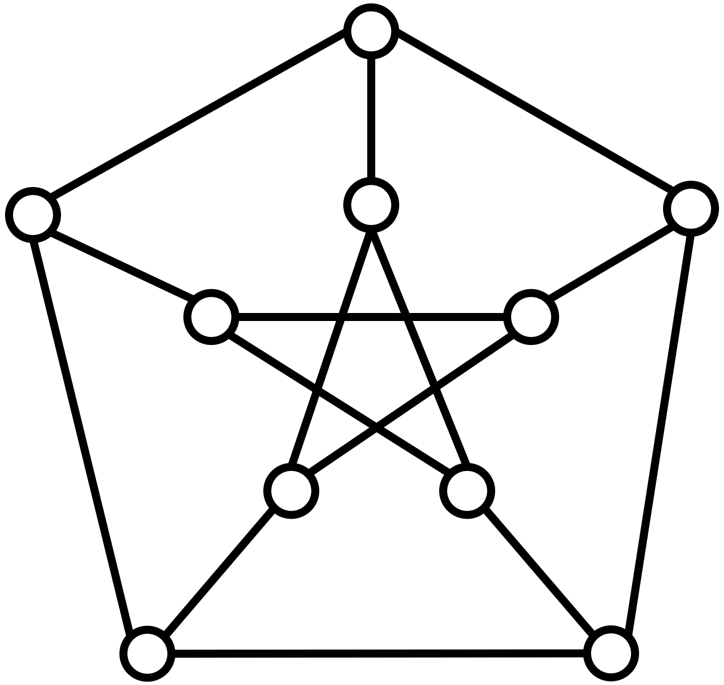


7 edges cut

$G = (V, E)$

(Classical) Max-Cut



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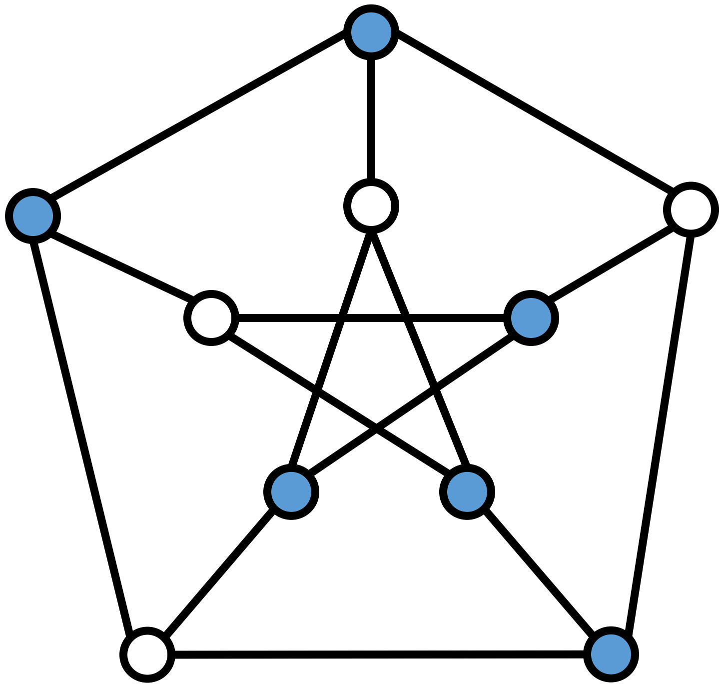

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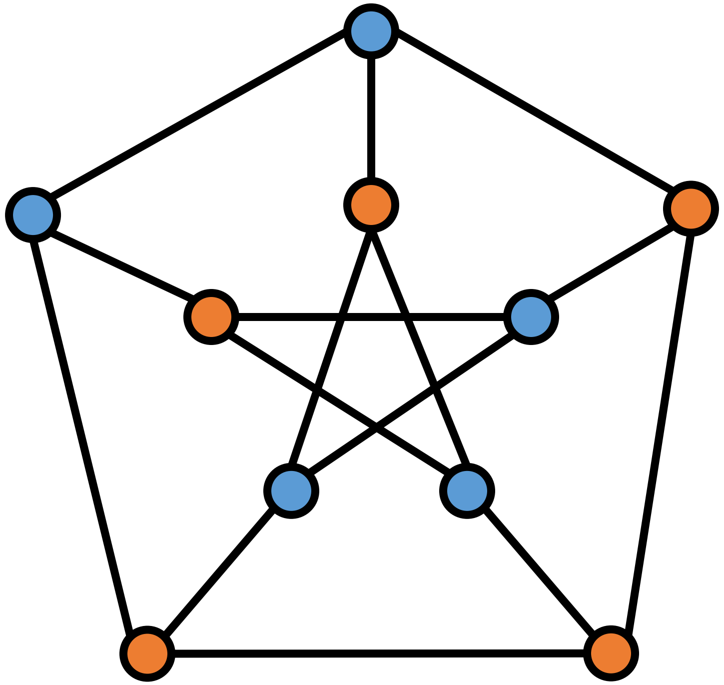

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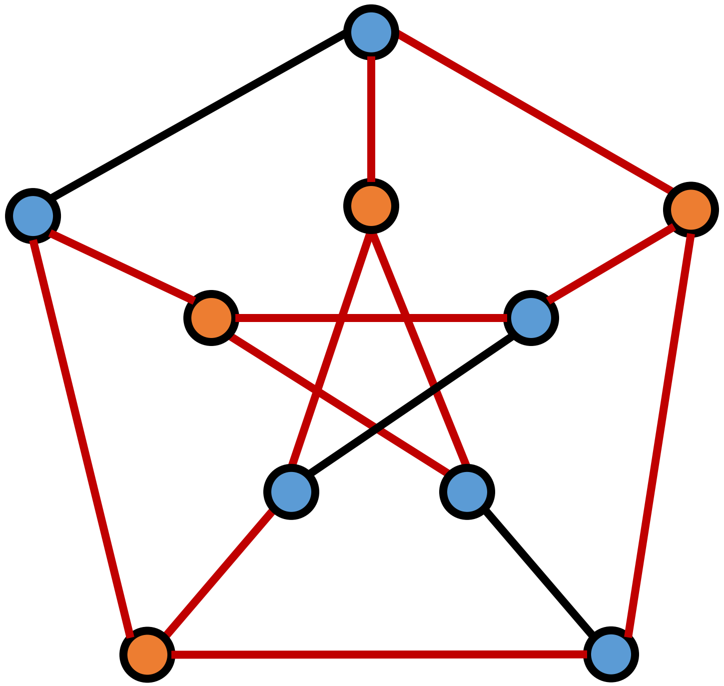

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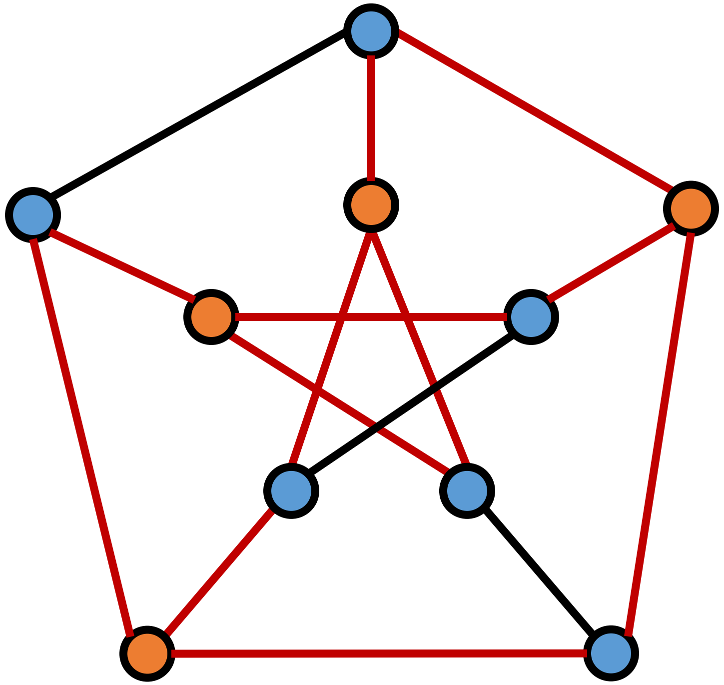
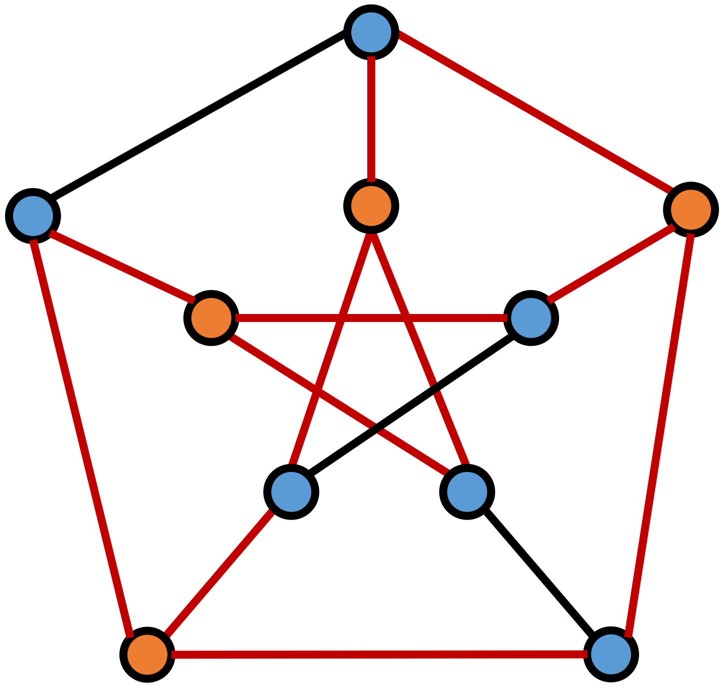


$G = (V, E)$

12 edges cut

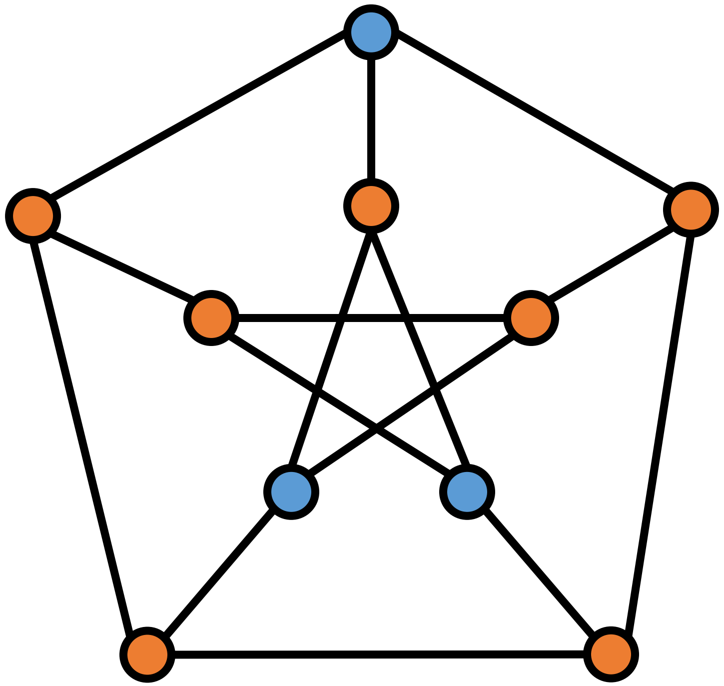
(Classical) Max-Cut



$G = (V, E)$

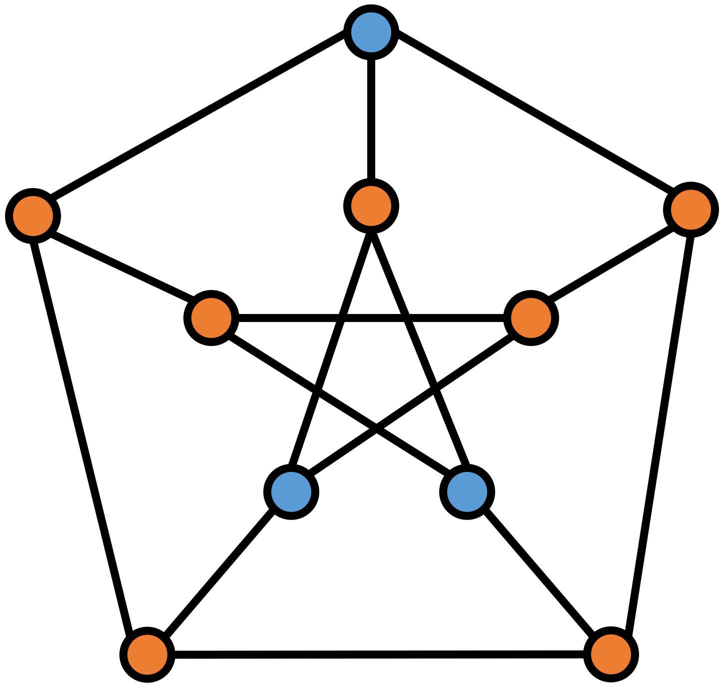
12 edges cut
(the **max cut**)

(Classical) Max-Cut



$G = (V, E)$

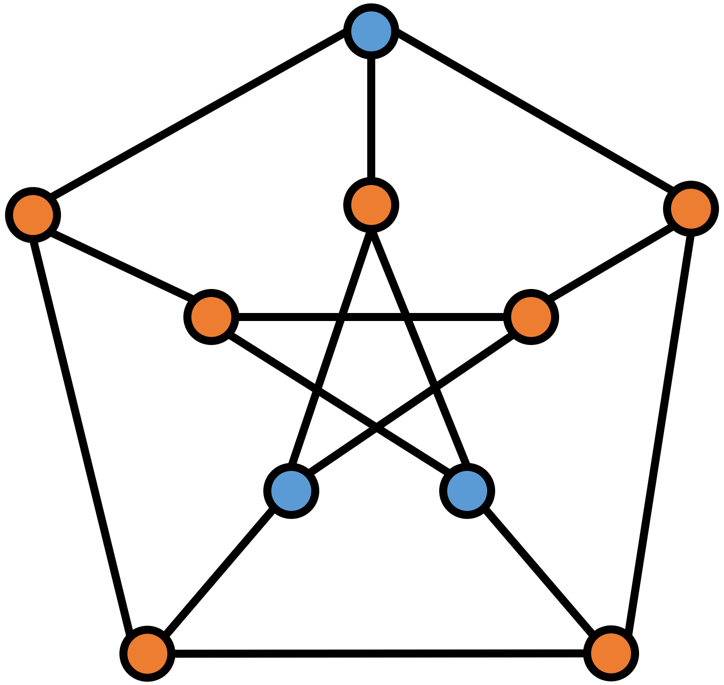
(Classical) Max-Cut



$G = (V, E)$

Goal: find partition $f : V \to \{■, ■\}$
maximizing

(Classical) Max-Cut



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Goal: find partition $f: V \to \{■, ■\}$ maximizing

$$\sum_{(u,v) \in E} \mathbf{1}[f(u) \neq f(v)]$$

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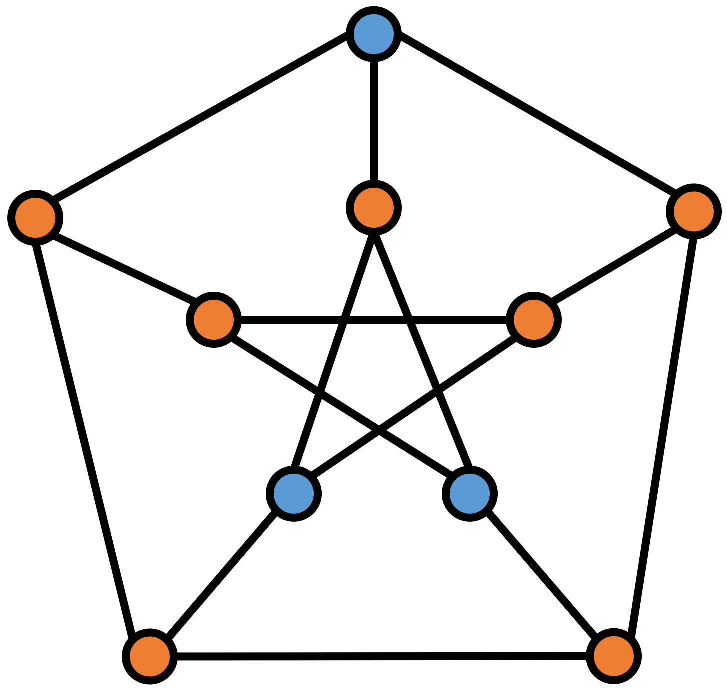

$G = (V, E)$

Goal: find partition $f: V \to \{+1, -1\}$ maximizing

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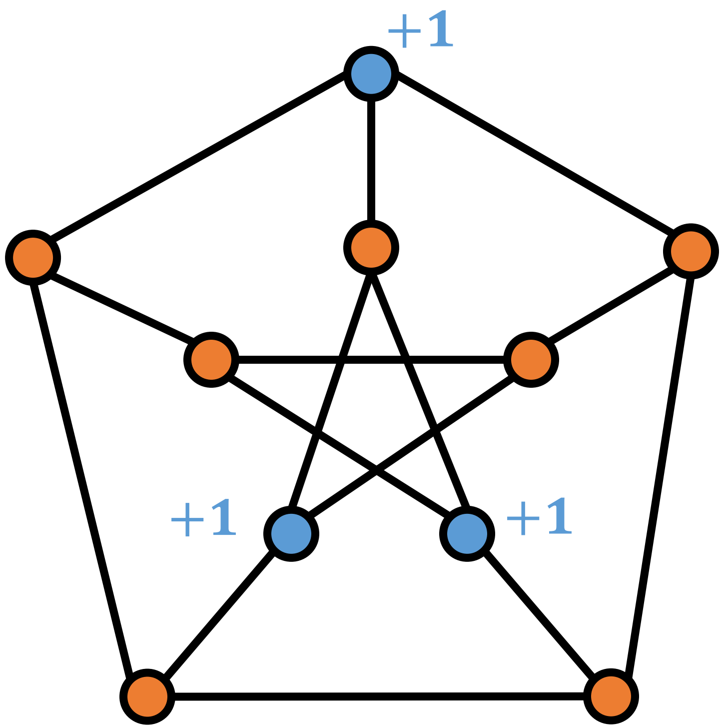
(Classical) Max-Cut



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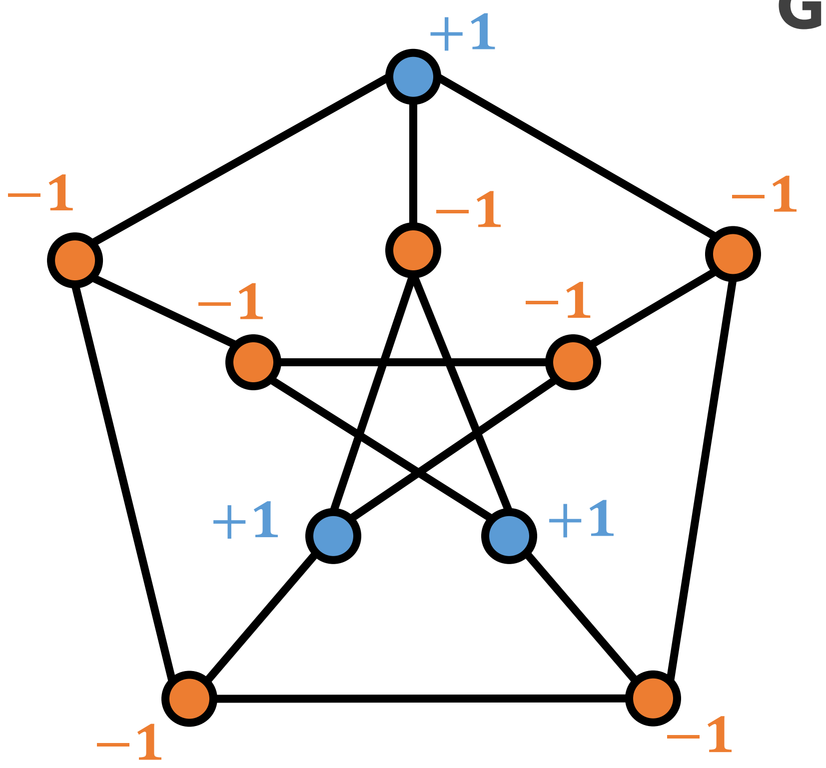

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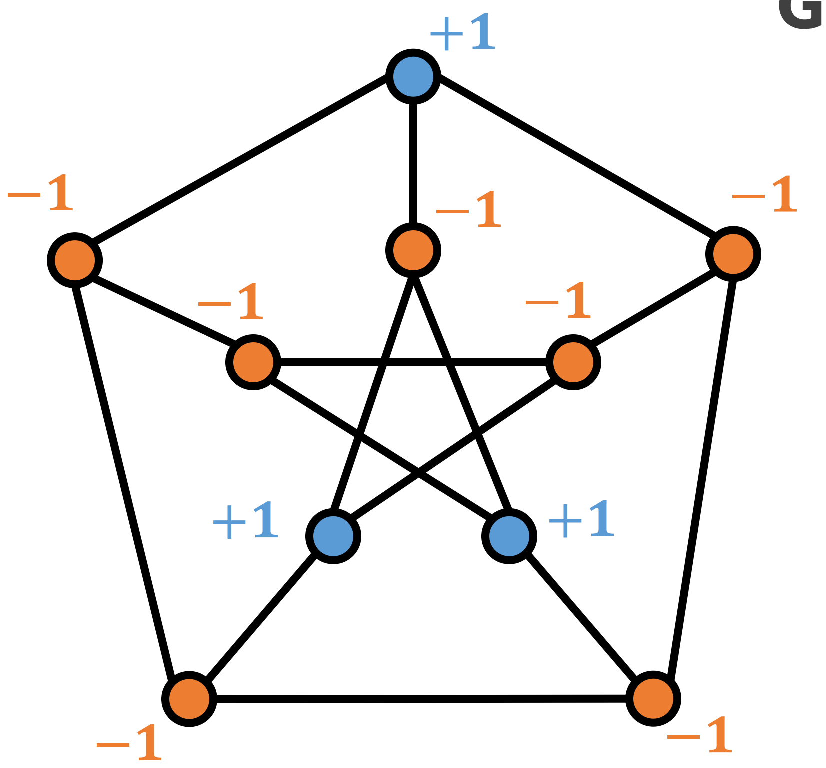
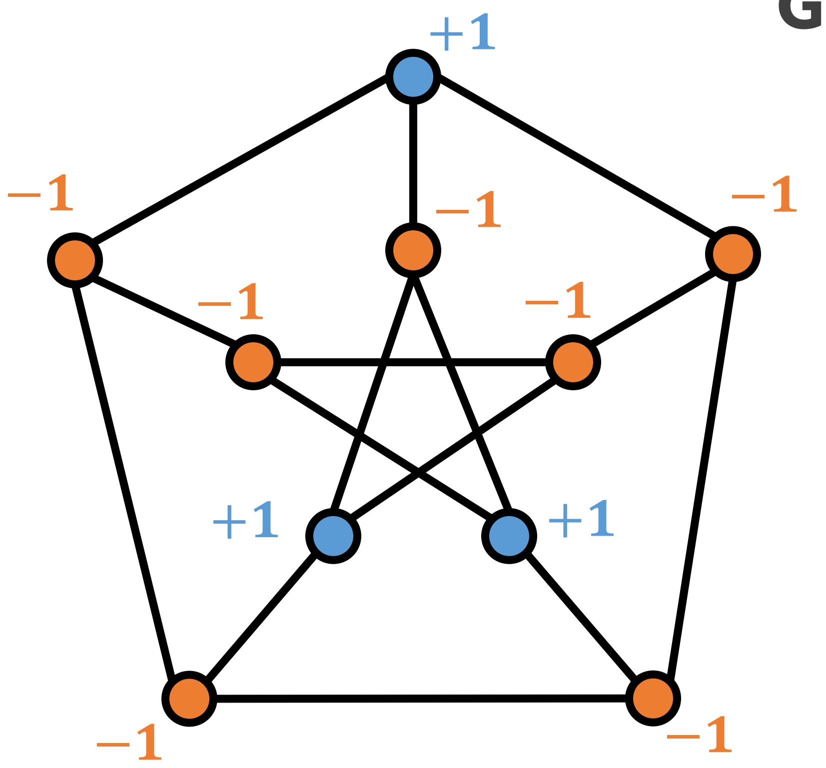


$G = (V, E)$

Goal: find partition $f: V \to \{+1, -1\}$ maximizing

$$\sum_{(u,v) \in E} \left(\frac{1 - f(u) \cdot f(v)}{2} \right)$$

(Classical) Max-Cut



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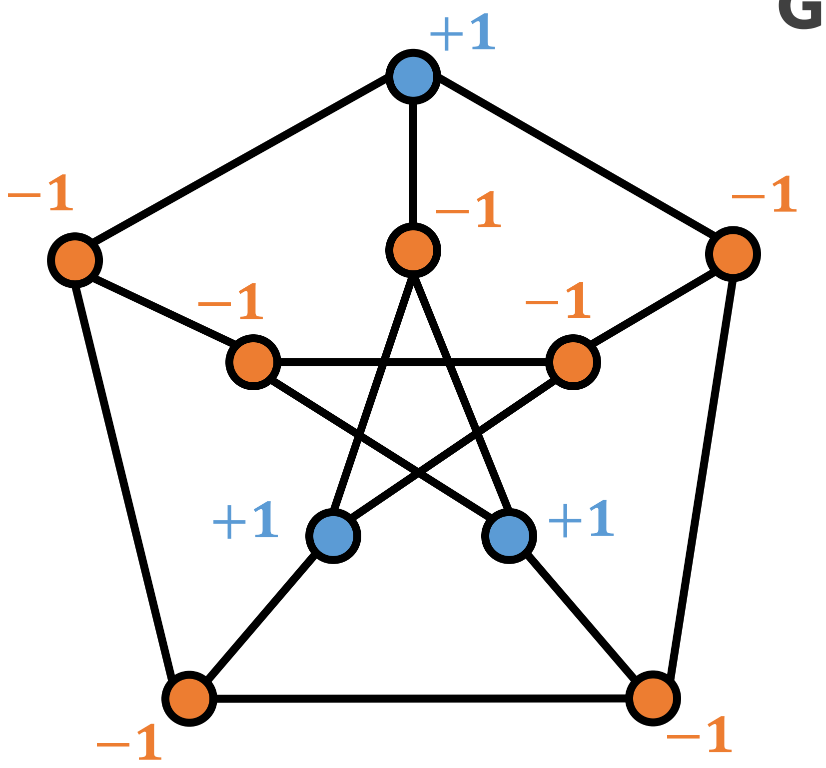
$$\sum_{(u,v) \in E} \underbrace{\left(\frac{1 - f(u) \cdot f(v)}{2} \right)}_{1 \text{ if } f(u) \neq f(v)}$$

(Classical) Max-Cut



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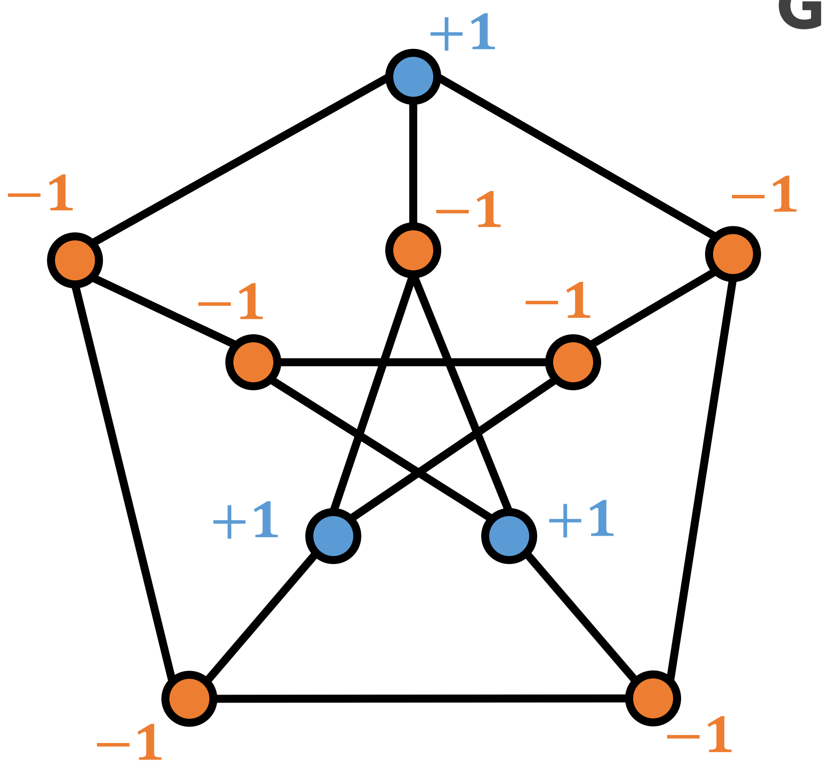

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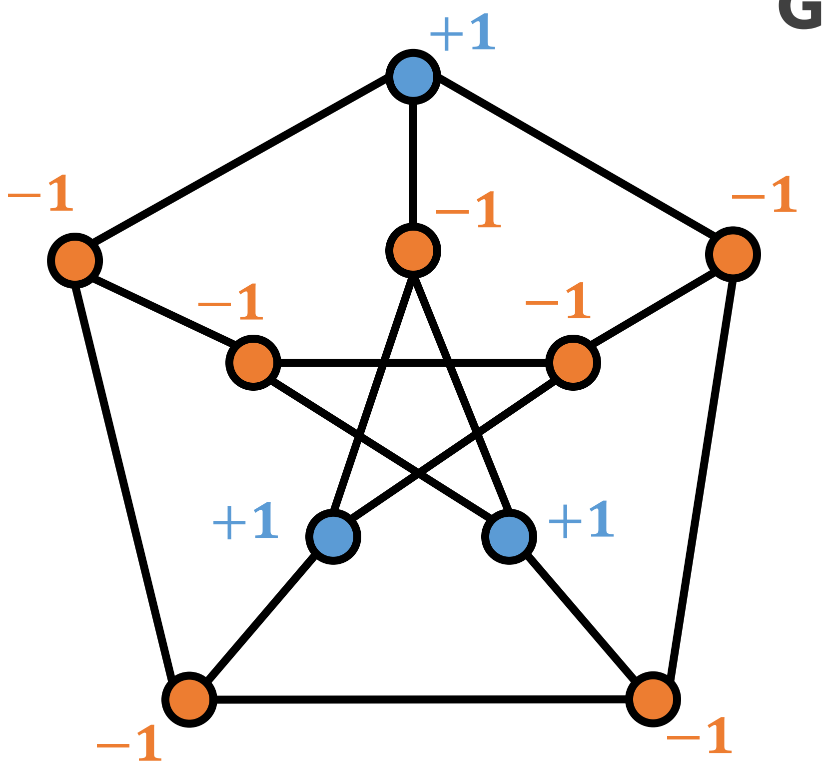
NP-hard to solve exactly!

(Classical) Max-Cut



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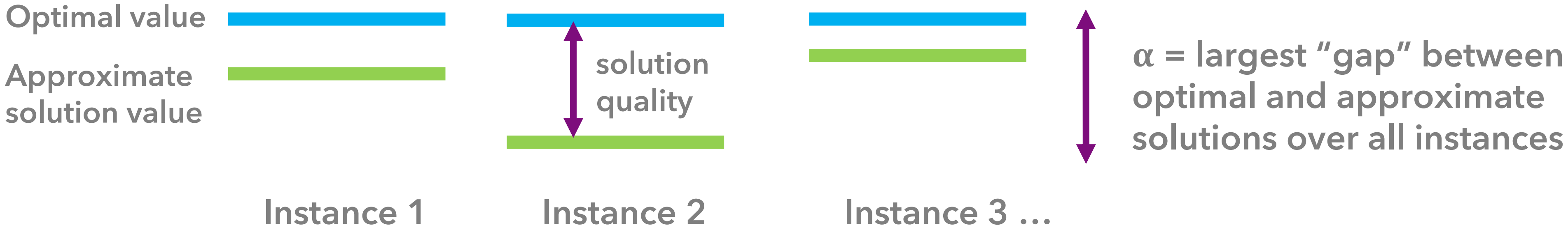
NP-hard to solve exactly!

So instead look for **approximation algorithms**.

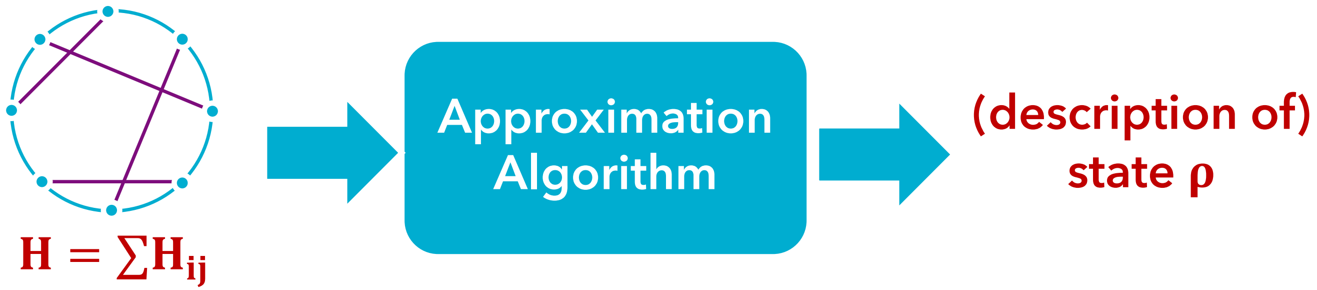
(QUANTUM) APPROXIMATION ALGORITHMS

A α-approximation algorithm runs in polynomial time, and for any instance I, delivers an approximate solution such that:

$$\frac{\text{Value}(\text{Approximate}_I)}{\text{Value}(\text{Optimal}_I)} \geq \alpha$$



(QUANTUM) APPROXIMATION ALGORITHMS



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Heuristics

- Guided by intuitive ideas
- Perform well on practical instances
- May perform very poorly in worst case
- Difficult to prove anything about performance

Approximation Algorithms

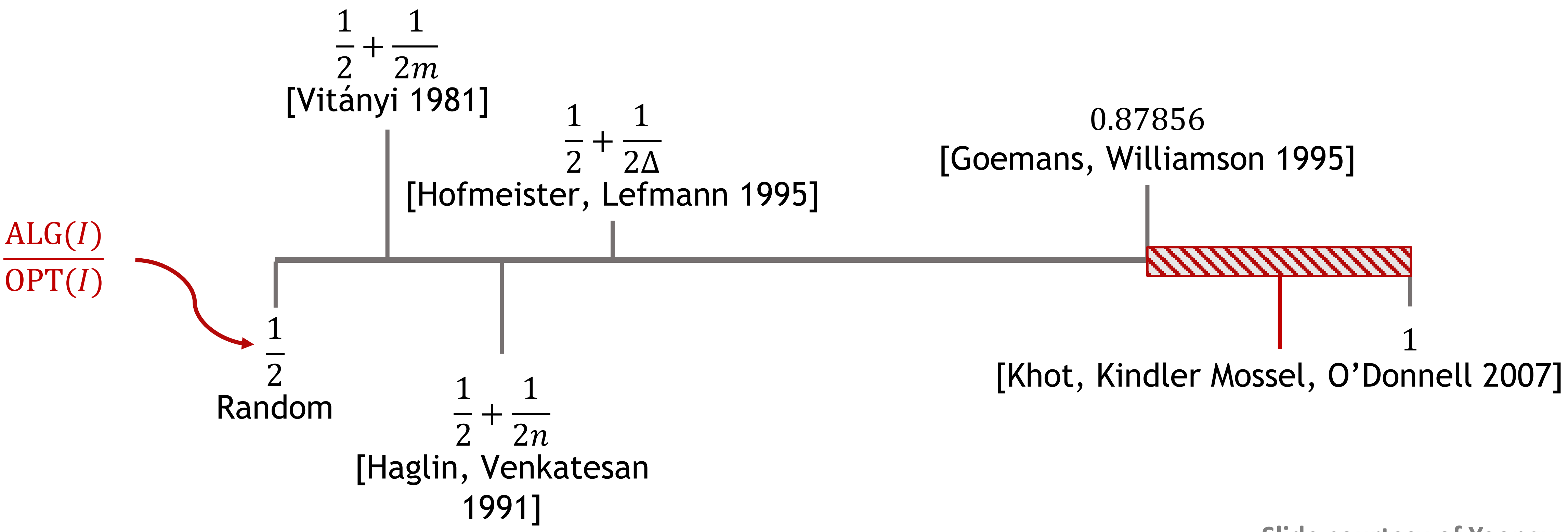
- Guided by worst-case performance
- May perform poorly compared to heuristics
- Rigorous bound on worst-case performance
- Designed with performance proof in mind

APPROXIMATION ALGORITHMS FOR MAX CUT

How far can we go?

Four approximation algorithms

$0.87856 + \epsilon$ approximations are **NP-Hard!** (under Unique Games Conjecture)



Slide courtesy of Yeongwoo Hwang

Quantum Max-Cut

Following slides courtesy of John Wright

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H = \sum_{(u,v)\in E} h_{uv}$$

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H = \sum_{(u,v)\in E} h_{uv}\ , \text{ where } h = \frac{1}{4}\cdot(I - XX - YY - ZZ)$$

Quantum Max-Cut

Special case of 2-local Hamiltonian:

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only depends on $\boldsymbol{G}$

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_{\boldsymbol{G}} = \sum_{(u,v)\in E} h_{uv}\text{ , where } h = \frac{1}{4}\cdot(I\ - XX\ - YY\ - ZZ)$$

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Quantum Max-Cut

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Quantum Max-Cut

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$$H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

Quantum Max-Cut

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Goal: Output the **maximum energy state** of $\boldsymbol{H_G}$

Quantum Max-Cut

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Note: max energy state of $\boldsymbol{H_G}$

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Quantum Max-Cut

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(antiferromagnetic) Heisenberg model

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(antiferromagnetic) Heisenberg model

Dates back to **[Heisenberg 1928]**

Well-studied class of Hamiltonians

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

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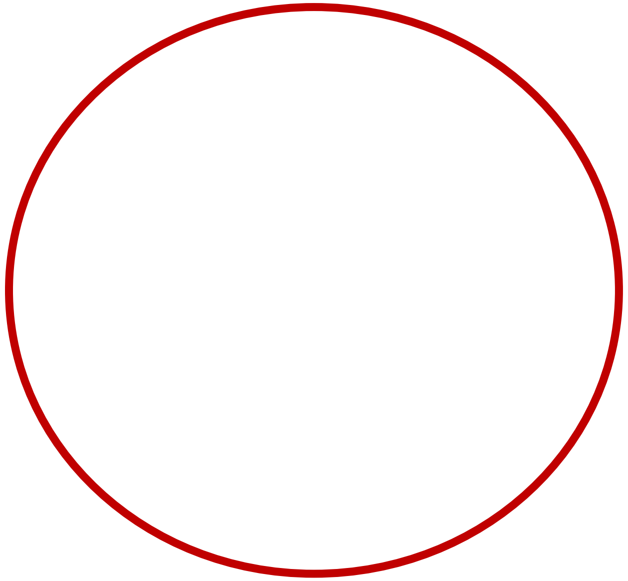

$|\psi\rangle$ ($\boldsymbol{n}$ qubits)

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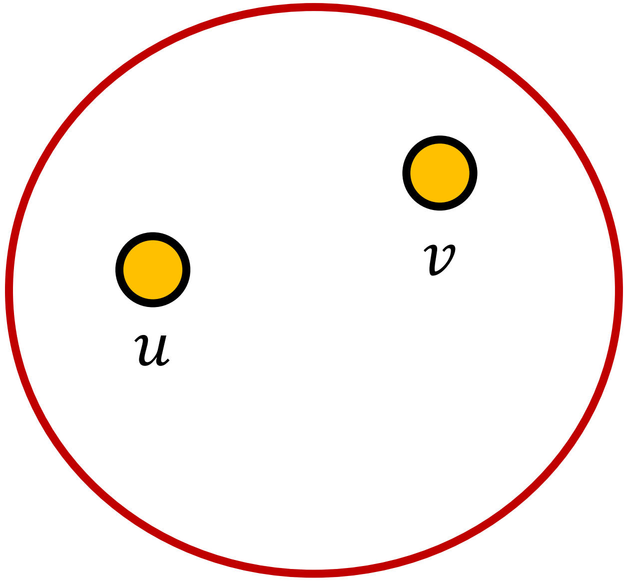

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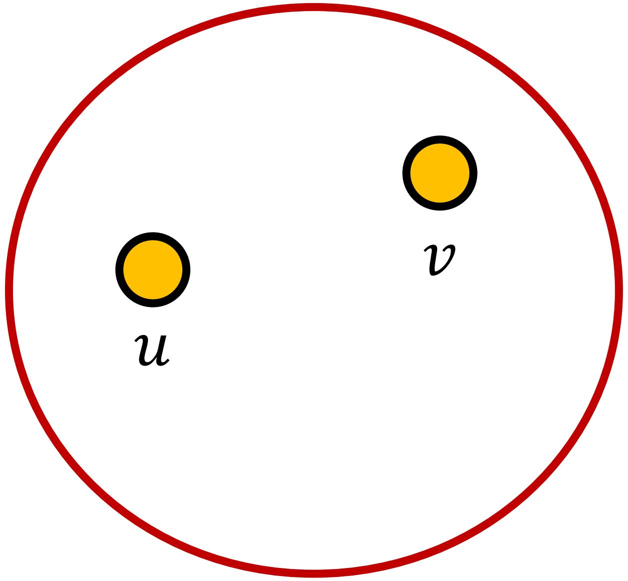

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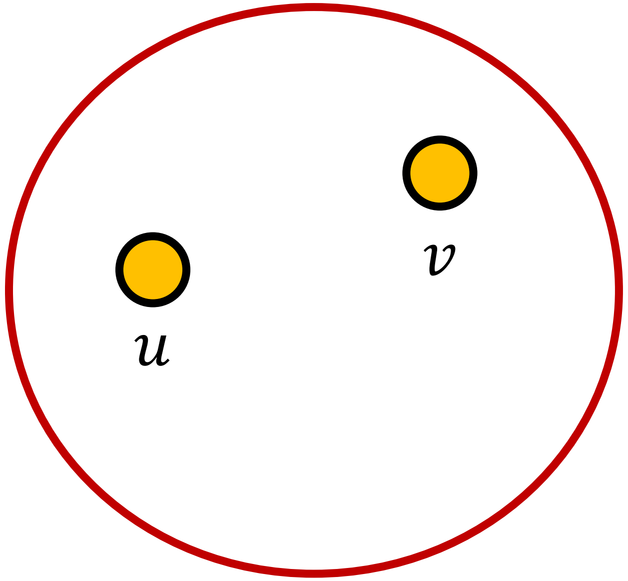

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Term 1: Does nothing

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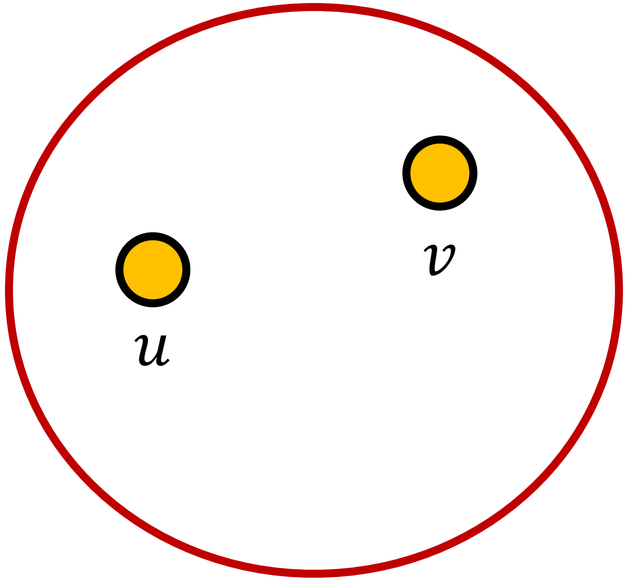

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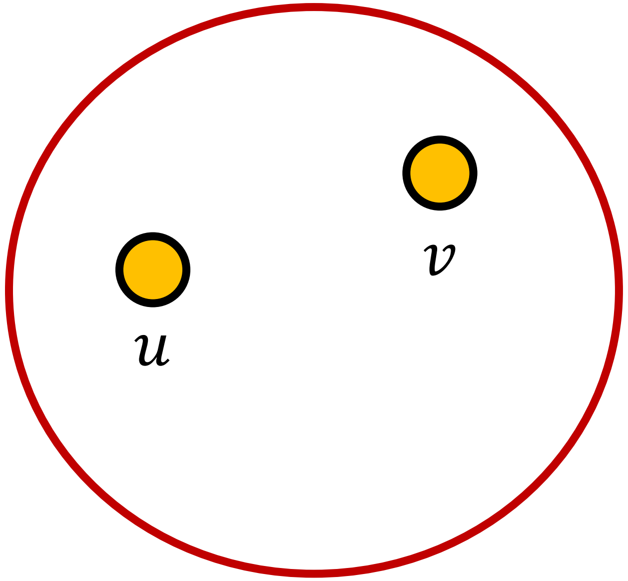


$|\psi\rangle$ (***n*** qubits)

Term 1: Does nothing

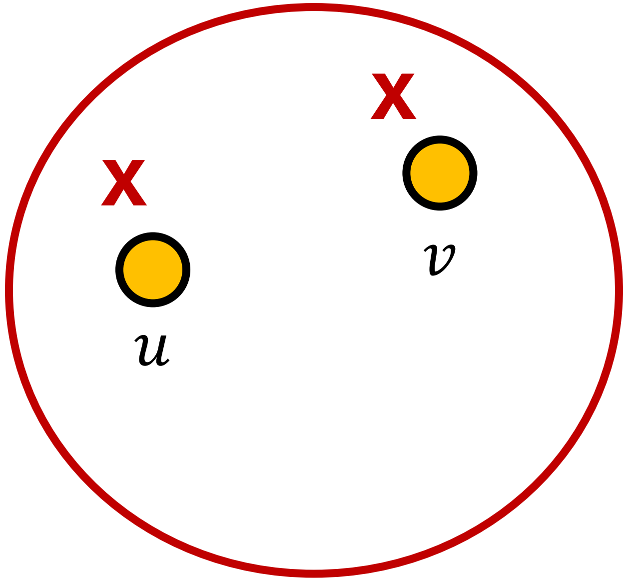
Term 2: Measure in **X** basis

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v)\in E} \tfrac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

Intuition



$|\psi\rangle$ (**n** qubits)

Term 1: Does nothing

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Quantum Max-Cut

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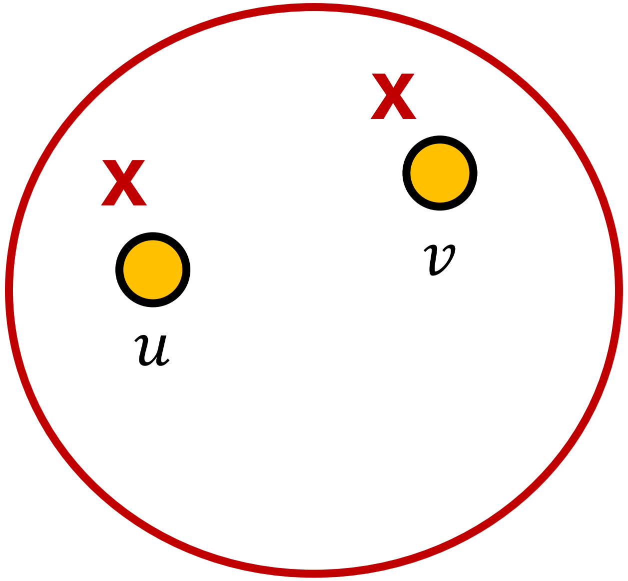


$|\psi\rangle$ ($\boldsymbol{n}$ qubits)

Term 1: Does nothing

Term 2: Measure in **X** basis

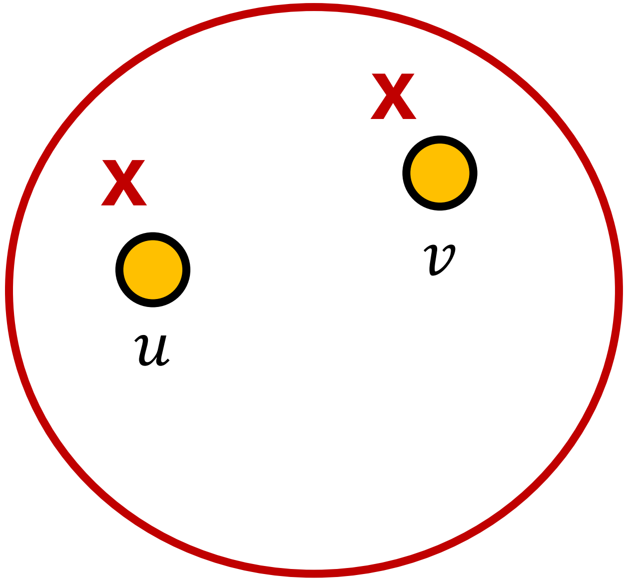
- **−1** if same (+ + or − −)

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot \left(I \boxed{- X_u X_v} - Y_u Y_v - Z_u Z_v\right)$$

Intuition



$|\psi\rangle$ (***n*** qubits)

Term 1: Does nothing

Term 2: Measure in **X** basis

- $-\mathbf{1}$ if same ($+\,+$ or $-\,-$)
- $+\mathbf{1}$ if different ($+\,-$ or $-\,+$)

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I \boxed{- X_u X_v} - Y_u Y_v - Z_u Z_v)$$

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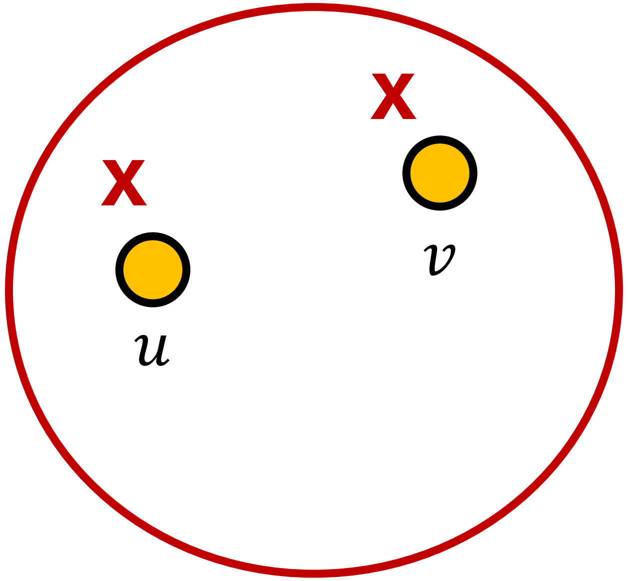


$|\psi\rangle$ (***n*** qubits)

Term 1: Does nothing

Term 2: Measure in **X** basis

- -1 if same ($+\,+$ or $-\,-$)
- $+1$ if different ($+\,-$ or $-\,+$)

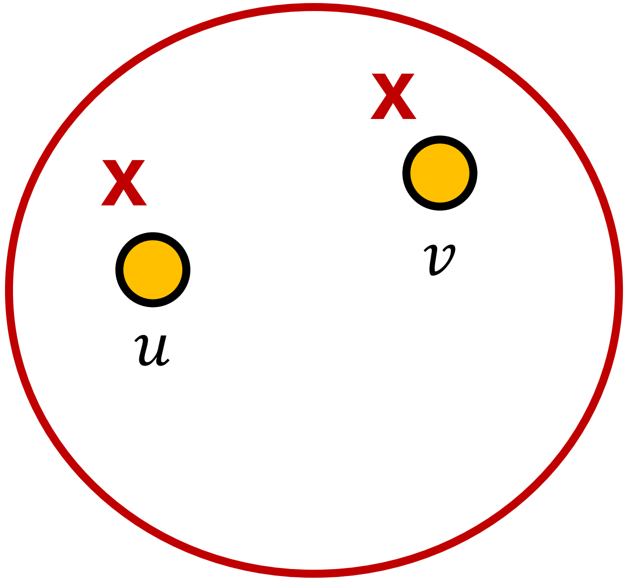
→ want both different!

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v - Z_u Z_v)$$

Intuition



$|\psi\rangle$ ($\boldsymbol{n}$ qubits)

Term 1: Does nothing

Term 2: Should be different in **X** basis

Quantum Max-Cut

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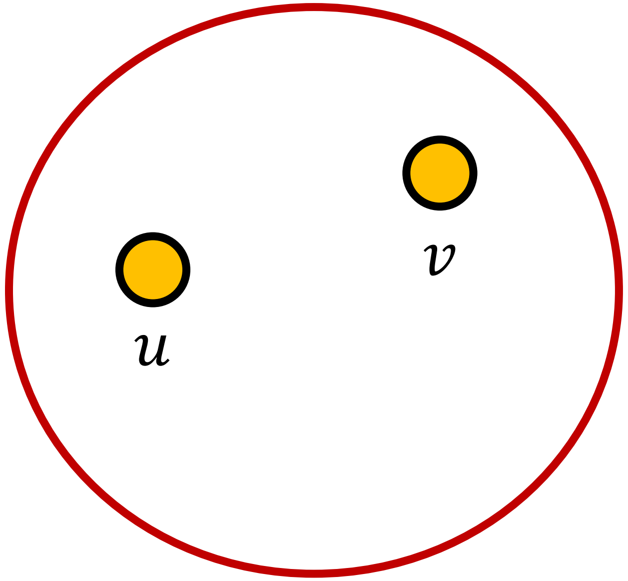


$|\psi\rangle$ (***n*** qubits)

Term 1: Does nothing

Term 2: Should be different in **X** basis

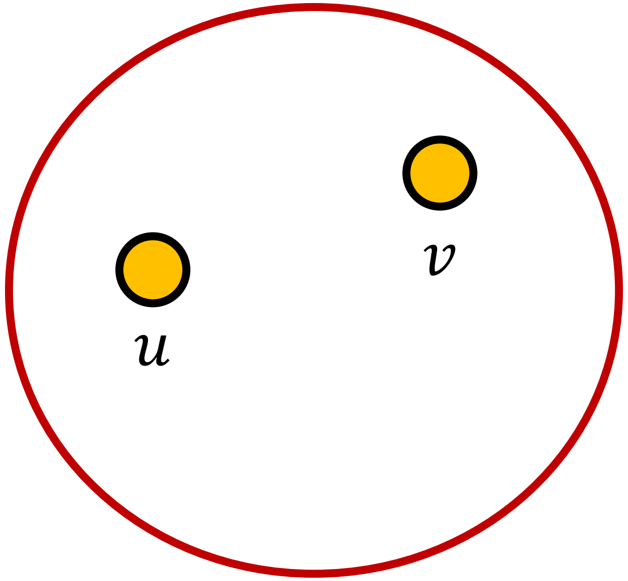
Term 3: Should be different in **Y** basis

Quantum Max-Cut

Special case of 2-local Hamiltonian:

$$H_G = \sum_{(u,v)\in E} \frac{1}{4} \cdot (I - X_u X_v - Y_u Y_v \boxed{- Z_u Z_v)}$$

Intuition



$|\psi\rangle$ ($\boldsymbol{n}$ qubits)

Term 1: Does nothing

Term 2: Should be different in **X** basis

Term 3: Should be different in **Y** basis

Term 4: Should be different in **Z** basis

Quantum Max-Cut

Special case of 2-local Hamiltonian:

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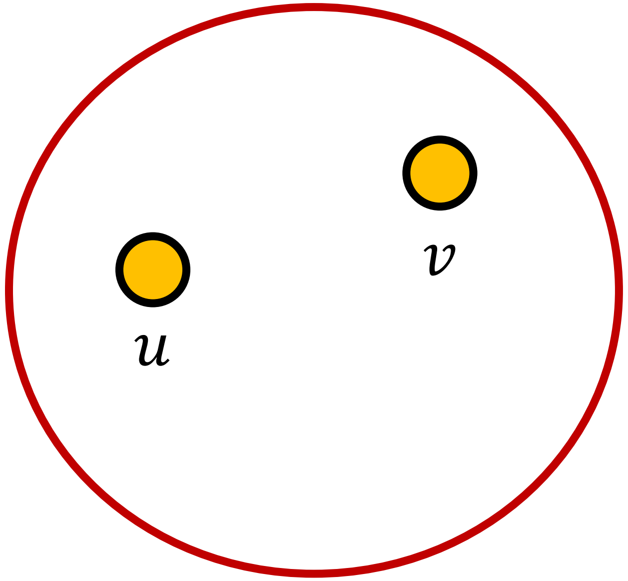

$|\psi\rangle$ ($\boldsymbol{n}$ qubits)

Term 1: Does nothing

Term 2: Should be different in **X** basis

Term 3: Should be different in **Y** basis

Term 4: Should be different in **Z** basis

Like **(classical)** Max-Cut
in **X**, **Y**, and **Z** bases!

Product states for QMax-Cut

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States of the form $|\psi\rangle = \bigotimes_{u \in V} |\psi_u\rangle$

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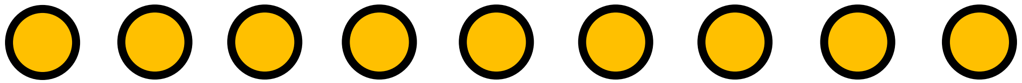
n qubits: 

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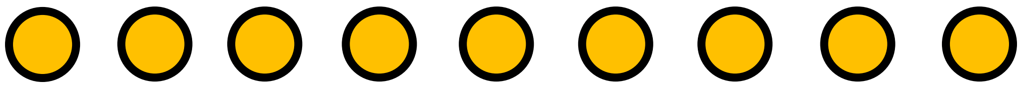

$|\psi_u\rangle$ $|\psi_v\rangle$

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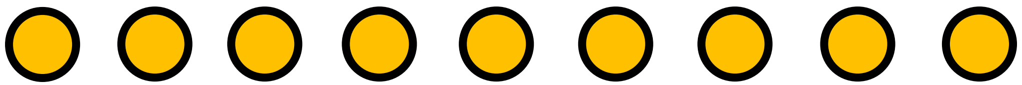

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Product states possess no entanglement

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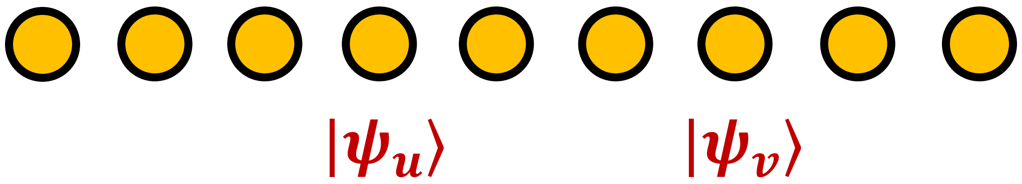

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But they can often be
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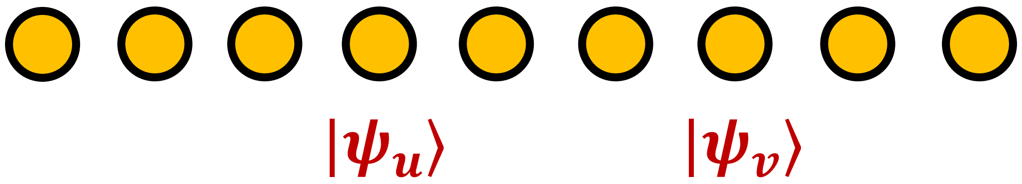

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[Brandao Harrow 2016]: The ground state is close to product if G is **high degree**.

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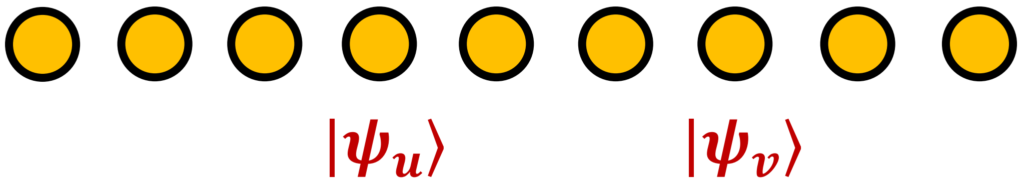

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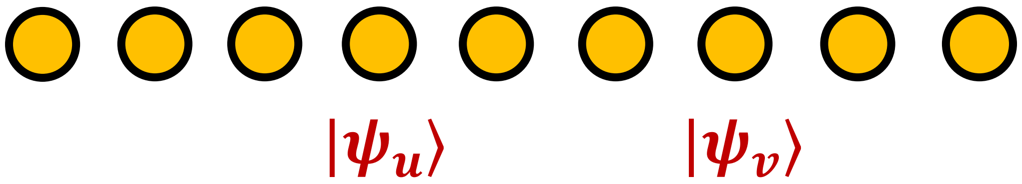
n qubits: 

Useful to look at **Bloch sphere** representation.

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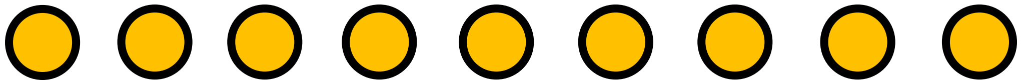
n qubits: 

$|\psi_u\rangle$ $|\psi_v\rangle$

Useful to look at **Bloch sphere** representation.

Bloch sphere: Each single-qubit state $|\psi_u\rangle$ can be associated with a real vector (c_X, c_Y, c_Z) such that $c_X^2 + c_Y^2 + c_Z^2 = 1$.

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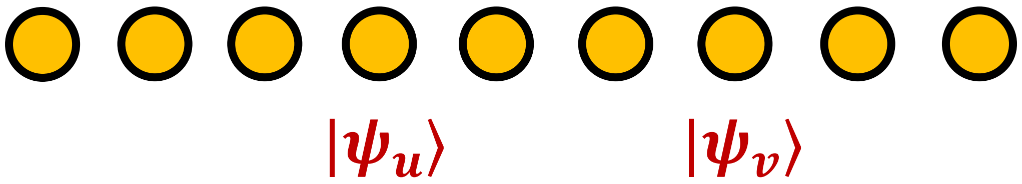

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Set $f(u) = (c_X, c_Y, c_Z)$. Then $f\colon V \to S^2$.

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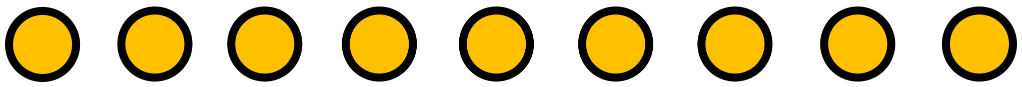
n qubits: 

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↑

unit sphere in $\mathbb{R}^3$

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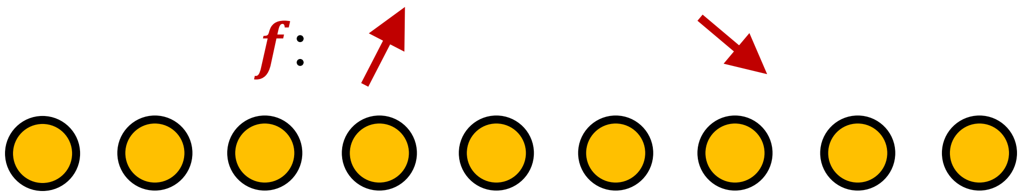
f:

n qubits:



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Bloch sphere representation: $f : V \rightarrow S^2$.

Product state energy formula:

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"Want" neighboring $f(u)$ and $f(v)$
to point in opposite directions.

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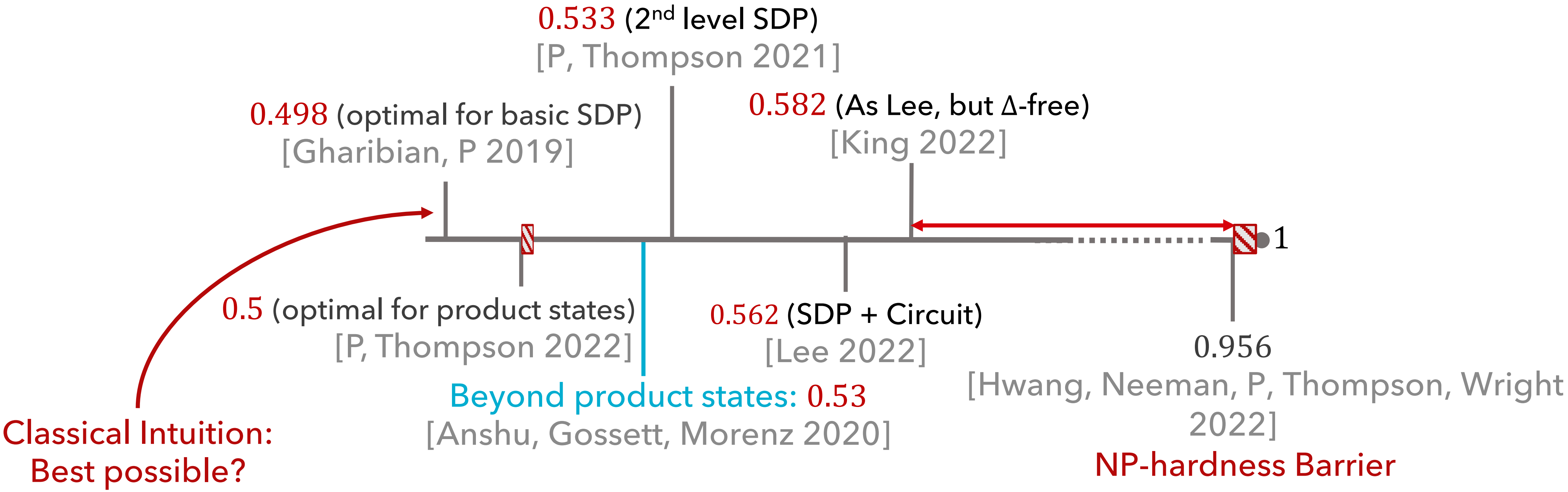
Like **(classical)** Max-Cut! There, $f : V \to \{\pm 1\} = S^0$.

APPROXIMATION ALGORITHMS FOR QUANTUM MAX

How far can we go?



Slide courtesy of Yeongwoo Hwang

First approximations for Max k-Local Hamiltonian

Classical approximation scheme for planar graphs: [Bansal, Bravyi, Terhal 2007: arXiv 0705.1115]

First nontrivial general approximations: [Gharibian, Kempe 2011: arXiv 1101.3884]
Classical approximation scheme for dense instances

Near-optimal product-state approx for special cases: [Brandao, Harrow 2013: arXiv 1310.0017]
Uses semidefinite programming (SDP) for bounds

Approximation w.r.t. number of terms and degree: [Harrow, Montanaro 2015: arXiv 1507.00739]

All of these results use product states

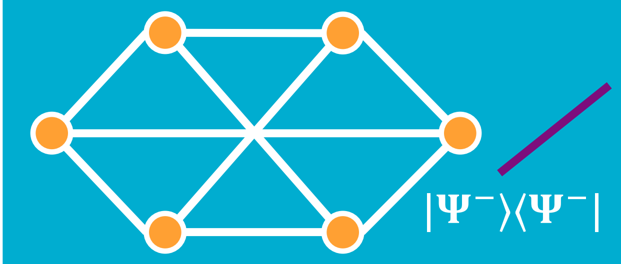
Recent approximations for Max 2-Local Hamiltonian

QMA-hard 2-LH problem class	NP-hard specialization	P approximation for NP-hard specialization	(Product-state) Approximation for QMA-hard 2-LH problem
Max traceless 2-LH: $\sum_{ij} H_{ij}$, H_{ij} traceless	**Max Ising:** Max $-\sum_{ij} z_i z_j$, $z_i \in \{-1,1\}$	$\Omega(1/\log n)$ [Charikar, Wirth '04]	$\Omega(1/\log n)$ [Bravyi, Gosset, Koenig, Temme '18] 0.184 (bipartite, no 1-local terms) [P, Thompson '20]
Max positive 2-LH: $\sum_{ij} H_{ij}$, $H_{ij} \succcurlyeq 0$	**Max 2-CSP**	0.874 [Lewin, Livnat, Zwick '02]	0.25 [Random assignment] 0.282 [Hallgren, Lee '19] 0.328 [Hallgren, Lee, P '20] 0.387 / 0.498 (numerical) [P, Thompson '20] **0.5 (best possible via product states) [P, Thompson '21]**
Quantum Max Cut: $\sum_{ij} I - X_iX_j - Y_iY_j - Z_iZ_j$ (special case of above)	**Max Cut:** Max $\sum_{ij} I - z_i z_j$, $z_i \in \{-1,1\}$	0.878 [Goemans, Williamson '95]	0.498 [Gharibian, P '19] **0.5 [P, Thompson '22]** 0.53* [Anshu, Gosset, Morenz '20] 0.533* [P, Thompson '21] 0.562* [Lee '22] (also [King '22])
Max 2-Quantum SAT: $\sum_{ij} H_{ij}$, $H_{ij} \succcurlyeq 0$, rank 3	**Max 2-SAT**	0.940 [Lewin, Livnat, Zwick '02]	0.75 [Random Assignment] 0.764 / 0.821 (numerical) [P, Thompson '20] **0.833… best possible via product states**

See [P, Thompson.; arXiv:2012.12347] for table

*** These results are not product-state based**

Quantum Max Cut



Instance of 2-Local Hamiltonian

Find max eigenvalue of $H = \sum H_{ij}$,

$$H_{ij} = \left(I - X_i X_j - Y_i Y_j - Z_i Z_j\right)/4$$

Each term is singlet projector:

$H_{ij} = |\Psi^-\rangle\langle\Psi^-|$

$|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$

Model 2-Local Hamiltonian?

Has driven advances in quantum approximation algorithms, based on generalizations of classical approaches

QMA-hard and each term is maximally entangled
[Cubitt, Montanaro 2013]

Recent approximation algorithms
[Gharibian and P. 2019], [Anshu, Gosset, Morentz 2020],
[P. and Thompson 2021, 2021, 2022]

Evidence of unique games hardness
[Hwang, Neeman, P., Thompson, Wright 2021]

Likely that approximation/hardness results transfer to 2-LH with positive terms
[P., Thompson 2021, 2022]

Max Cut vs Quantum Max Cut

Relaxation (upper bound)

Max Cut:

$$\text{Max} \sum_{ij\in E} (1 - v_i \cdot v_j)/2$$

$\|v_i\| = 1$, for all $i \in V$

$(v_i \in \mathbb{R}^n)$

Quantum Max Cut:

$$\text{Max} \sum_{ij\in E} (1 - 3v_i \cdot v_j)/4$$

$\|v_i\| = 1$, for all $i \in V$

$(v_i \in \mathbb{R}^n)$

Rounding

Max Cut:

$$v_i \in \mathbb{R}^n \longrightarrow \alpha_i = \frac{r^T v_i}{|r^T v_i|}$$

Quantum Max Cut:

$$v_i \in \mathbb{R}^{3n} \longrightarrow (\alpha_i, \beta_i, \gamma_i) = \left(\frac{r_x^T v_i}{\| r_x^T v_i \|}, \frac{r_y^T v_i}{\| r_y^T v_i \|}, \frac{r_z^T v_i}{\| r_z^T v_i \|} \right)$$

Approximability

Max Cut:

0.878 Lasserre 1

(optimal under unique games conjecture)

Quantum Max Cut:

0.498 Lasserre 1

0.5 Lasserre 2 (optimal using product states)

(0.533 using 1- & 2-qubit ansatz)

To learn more about Quantum Max Cut…

Optimal product-state approximations: [P., Thompson 2022: arXiv 2206.08342] (Sections 2,3)

Best-known Quantum Max Cut (QMC) approximations: [Anshu, Gosset, Morenz-Korol 2020: arXiv 2003.14394]
[P., Thompson 2021: arXiv 2105.05698]
[Lee 2022: arXiv 2209.00789]
[King 2022: arXiv 2209.02589]

Lasserre hierarchy in 2-LH approximations: [P., Thompson 2021, 2022 above]

Prospects for unique-games hardness: [Hwang, Neeman, P., Thompson, Wright 2021: arXiv 2111.01254] (Start here: intro and Section 7)

Connections in approximating QMC and 2-LH: [P., Thompson 2022 above, 2020: arXiv 2012.12347]
[Anshu, Gosset, Morenz-Korol, Soleimanifar: arXiv 2105.01193]

Optimal space-bounded QMC approximations: (no quantum advantage possible!) [Kallaugher, P. 2022: arXiv 2206.00213]

Quantum Moment Matrices are Positive

State on n qubits

$\langle\psi| \in \mathbb{C}^{2^n}$

$$V = \begin{bmatrix} \langle x_1| = \langle\psi|X_1 \\ \langle y_1| = \langle\psi|Y_1 \\ \langle z_1| = \langle\psi|Z_1 \\ \vdots \\ \langle x_n| = \langle\psi|X_n \\ \langle y_n| = \langle\psi|Y_n \\ \langle z_n| = \langle\psi|Z_n \end{bmatrix}, \quad M_{ij} = \begin{bmatrix} \langle\psi|X_iX_j|\psi\rangle & \langle x_i|y_j\rangle & \langle x_i|z_j\rangle \\ \langle y_i|x_j\rangle & \langle y_i|y_j\rangle & \langle y_i|z_j\rangle \\ \langle z_i|x_j\rangle & \langle z_i|y_j\rangle & \langle z_i|z_j\rangle \end{bmatrix}$$

Columns: $X_1\ Y_1\ Z_1\ X_2\ Y_2\ Z_2\ X_3\ Y_3\ Z_3$; rows: $X_1\ Y_1\ Z_1\ X_2\ Y_2\ Z_2\ X_3\ Y_3\ Z_3$

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots \\ M_{12}^{\dagger} & M_{22} & M_{23} & \\ M_{13}^{\dagger} & M_{23}^{\dagger} & M_{33} & \\ \vdots & & & \ddots \end{bmatrix} = VV^{\dagger} \succcurlyeq 0 \Longrightarrow Re\left(VV^{\dagger}\right) \succcurlyeq 0$$

Entries of this $3n\times3n$ *moment matrix* are expectation values of all 2-local Pauli terms

Quantum Max Cut SDP Relaxation

$$
\begin{array}{c} \\ X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \\ X_3 \\ Y_3 \\ Z_3 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} X_1 & Y_1 & Z_1 & X_2 & Y_2 & Z_2 & X_3 & Y_3 & Z_3 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & & & & & & & \cdots \\
0 & 1 & 0 & & M_{12} & & & M_{13} & & \\
0 & 0 & 1 & & & & & & & \\
& & & 1 & 0 & 0 & & & & \\
& M_{12}^T & & 0 & 1 & 0 & & M_{23} & & \\
& & & 0 & 0 & 1 & & & & \\
& & & & & & 1 & 0 & 0 & \\
& M_{13}^T & & & M_{23}^T & & 0 & 1 & 0 & \\
& & & & & & 0 & 0 & 1 & \\
\vdots & & & & & & & & & \ddots
\end{bmatrix}
\end{array} \succcurlyeq 0
\qquad
M_{ij} = \begin{bmatrix} x_i \cdot x_j & x_i \cdot y_j & x_i \cdot z_j \\ y_i \cdot x_j & y_i \cdot y_j & y_i \cdot z_j \\ z_i \cdot x_j & z_i \cdot y_j & z_i \cdot z_j \end{bmatrix}
$$

Real part of moment matrix

Quantum Max Cut vector relaxation

$$\text{Max } \sum_{ij \in E} (1 - x_i \cdot x_j - y_i \cdot y_j - z_i \cdot z_j)/4$$

$\|x_i\|, \|y_i\|, \|z_i\| = 1$, for all $i \in V$

$x_i \cdot y_i = x_i \cdot z_i = y_i \cdot z_i = 0$, for all $i \in V$

$(v_i \in \mathbb{R}^{3n})$

$v_i = (x_i \oplus y_i \oplus z_i)/\sqrt{3}$

↔

$x_i = v_i \oplus 0 \oplus 0$

$y_i = 0 \oplus v_i \oplus 0$

$z_i = 0 \oplus 0 \oplus v_i$

$$\text{Max } \sum_{ij \in E} (1 - 3 v_i \cdot v_j)/4$$

$\|v_i\| = 1$, for all $i \in V$

$(v_i \in \mathbb{R}^n)$

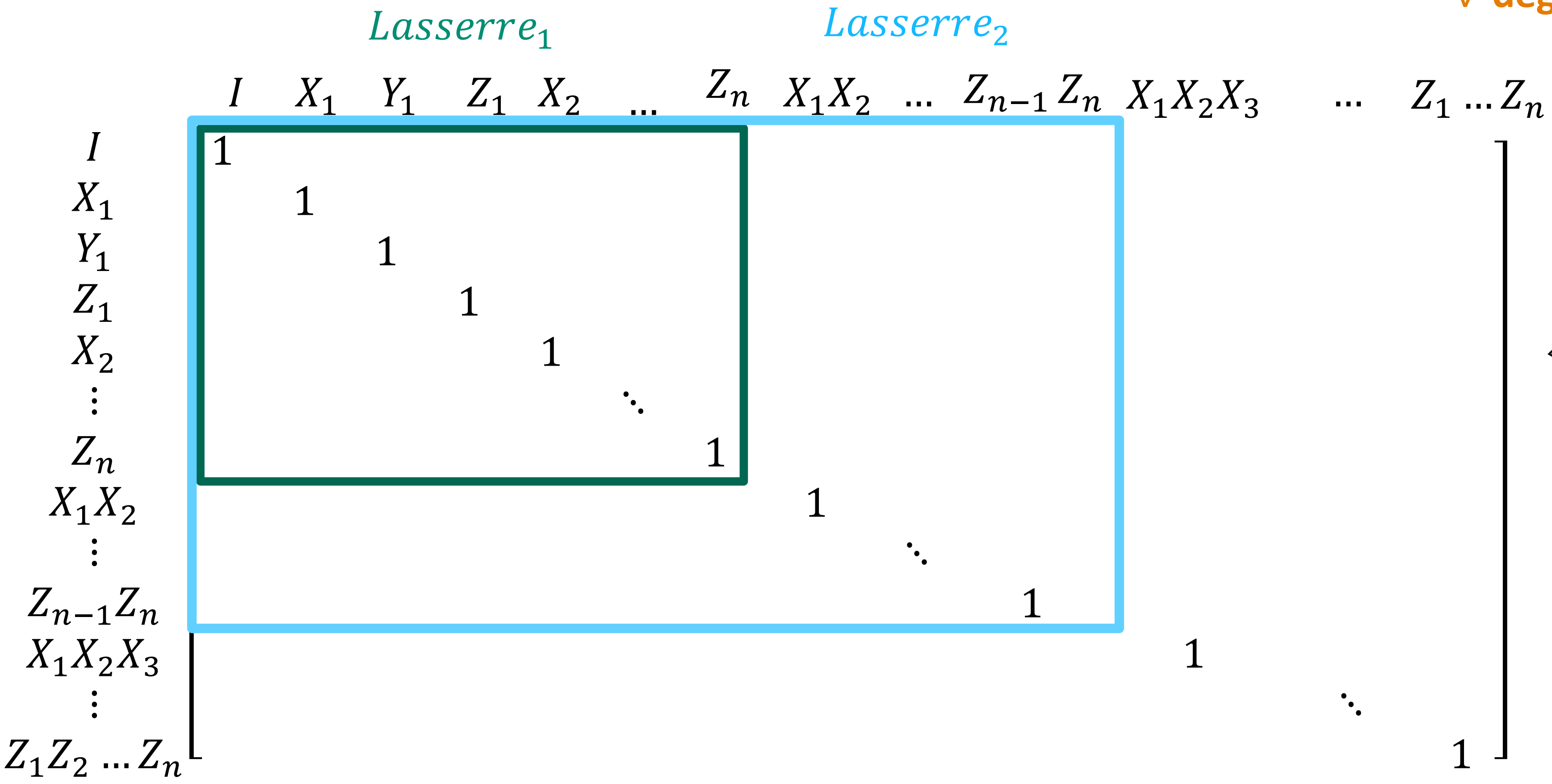
Max Cut vector relaxation

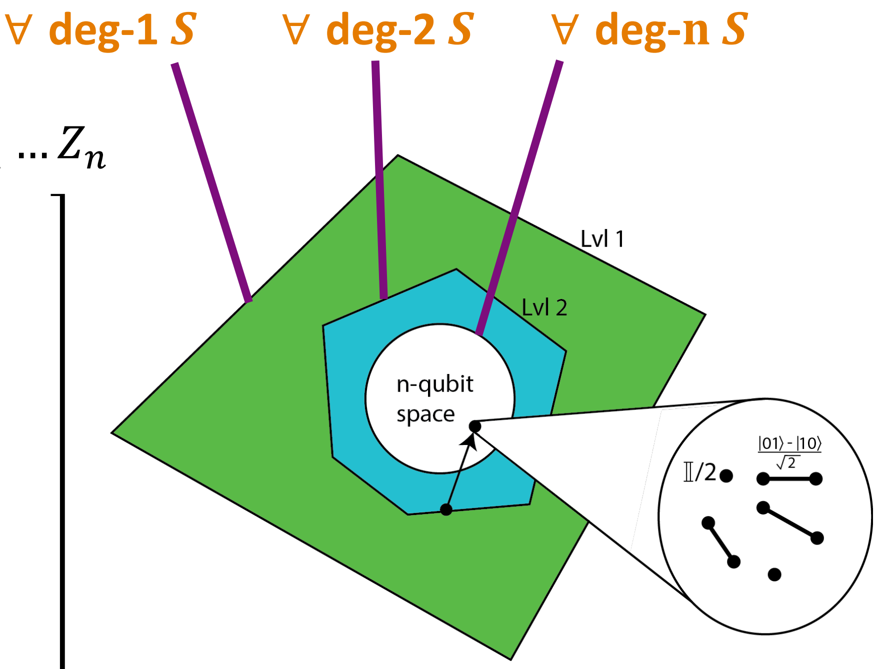
$$\text{Max } \sum_{ij \in E} (1 - v_i \cdot v_j)$$

$\|v_i\| = 1$, for all $i \in V$

$(v_i \in \mathbb{R}^n)$

Quantum Lasserre Hierachy





$$\mathit{Max}\ Tr[H\tilde{\rho}]$$
$$Tr[\tilde{\rho}] = 1$$
$$Tr[\tilde{\rho}\, S^\dagger S] \geq 0, \forall \text{ deg-}k\ S$$

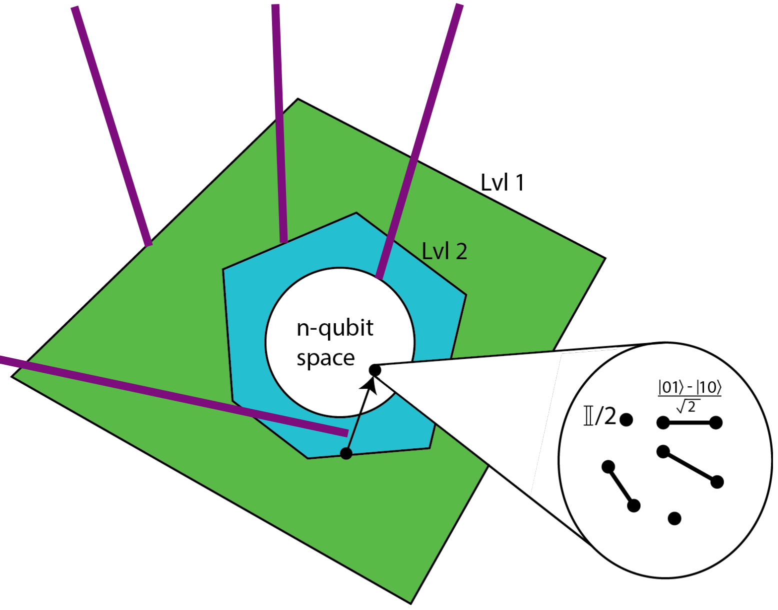
$\tilde{\rho}$ is called degree-k pseudo density

Classical
[Lasserre 2001]
[Parillo 2003]

Non-commutative/Quantum
[Navascués, Pironio, Acìn 2009 (2010 SIAM J Opt)]

Rounding Infeasible Solutions

$\forall$ **deg-1** S $\forall$ **deg-2** S $\forall$ **deg-n** S



α-Approximation Algorithm

Round optimal non-positive pseudo-density $\tilde{\rho}$ to sub-optimal positive density ρ so that:

$$Tr[H\rho] \geq \alpha\, Tr[H\tilde{\rho}] \geq \alpha\, \lambda_{max}(H)$$

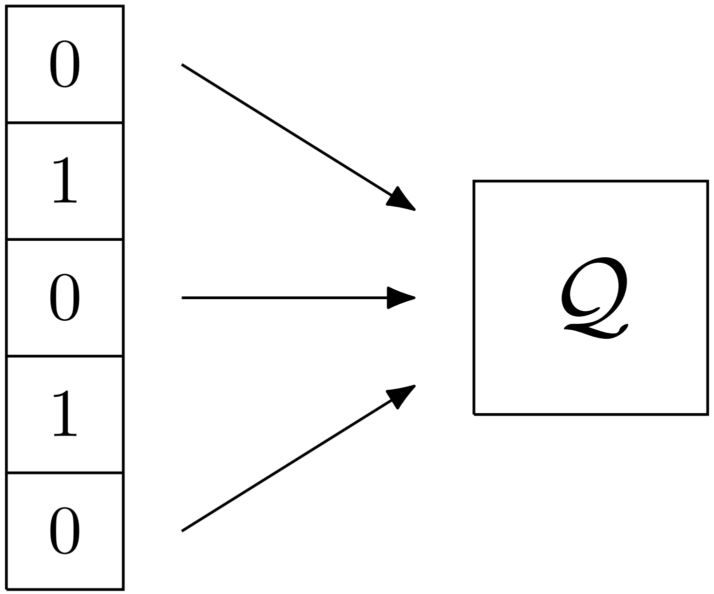
$$Max\ Tr[H\tilde{\rho}]$$
$$Tr[\tilde{\rho}] = 1$$
$$Tr[\tilde{\rho}\, S^{\dagger} S] \geq 0, \forall \text{ deg-}k\ S$$

$\tilde{\rho}$ is called degree-k pseudo density

QUANTUM STREAMING ADVANTAGES

Space Efficiency

We would like algorithms that need very few bits/qubits



Ideally a number *sublinear* in the size of the input, e.g. $O(\sqrt{n})$ or $O(\log(n))$ for a size-n input

Why Space-Efficient Algorithms?

Two reasons, pointing to different kinds of algorithm:

Qubits are expensive

- Even under the most optimistic assumptions, qubits will continue to be much more expensive than classical bits
- Motivates algorithms that use very few *qubits*, but maybe many classical bits

Qubits can be exponentially more powerful than classical bits

- We know there are problems that require exponentially fewer qubits than bits
- This is provable! (unlike with time complexity)
- Motivates looking at algorithms that use very little total space (bits + qubits) (and impossibility results)

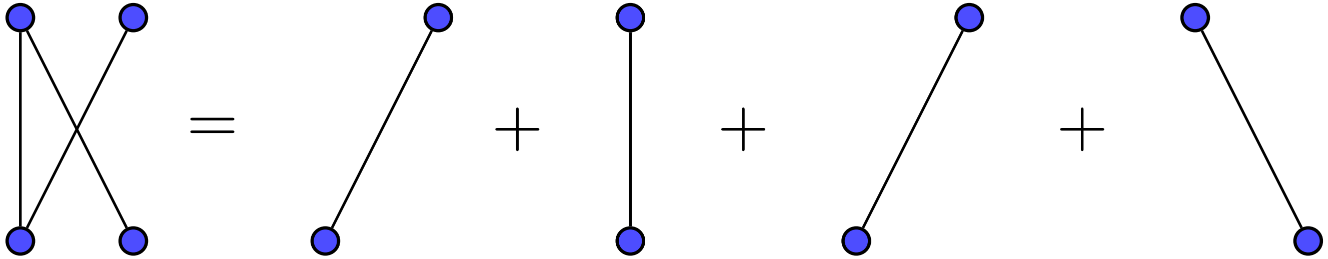
Our focus has been on the second case

Streaming Algorithms

When dealing with very small space algorithms, it matters *how* you receive the input dataset

Streaming

- Dataset is built up by a “stream” of small updates
- Answer is expected at the end of the stream



Examples

- Calculating traffic statistics on a router
- Estimating properties of a large social networking graph given as a sequence of friendships

QUANTUM STREAMING ADVANTAGES FOR GRAPH PROBLEMS

Exponential advantage for Boolean Hidden Matching
[Gavinsky, Kempe, Kerenidis, Raz, and de Wolf 2008]

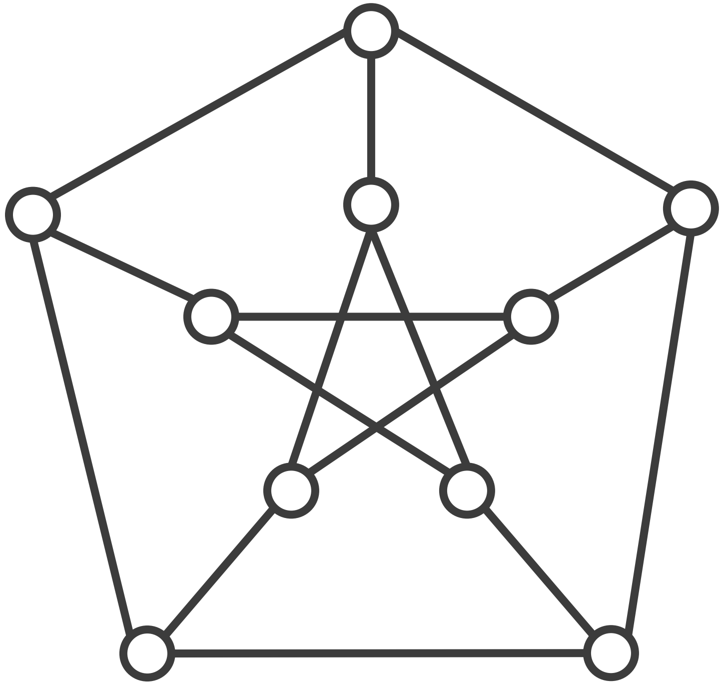
First natural problem: polynomial advantage for triangle counting
[Kallaugher 2021]

No quantum advantage possible: Max Cut or Quantum Max Cut
[Kallaugher, P 2022]

Exponential advantage for natural problem: Directed Max Cut
[Kallaugher, P, Voronova 2023]

QUANTUM GENERALIZATIONS OF VERTEX COVER

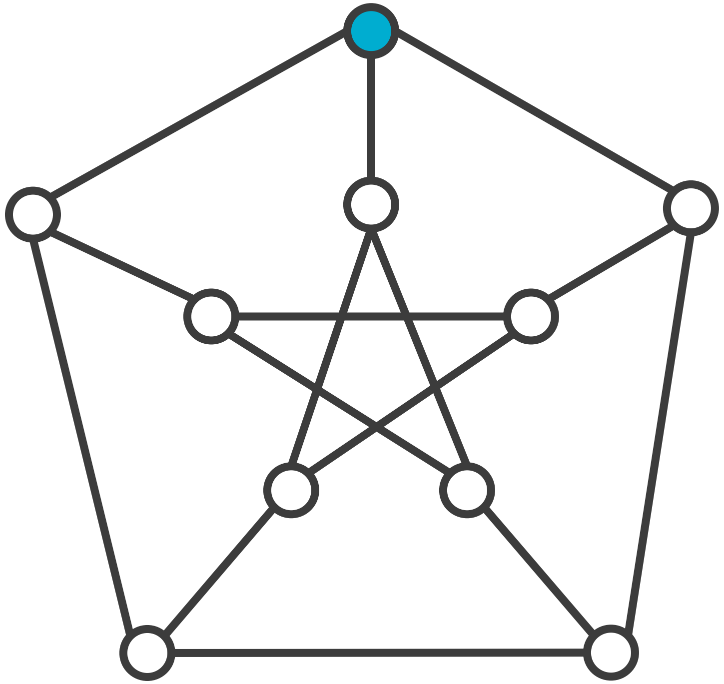
VERTEX COVER



$\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$

Goal: **color** minimum number of vertices, so each edge has at least 1 colored endpoint

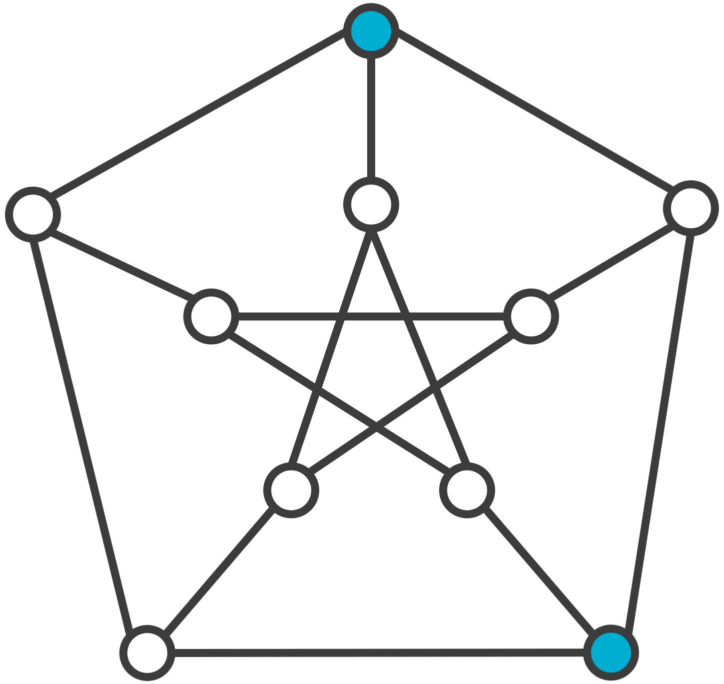
VERTEX COVER



$G = (V, E)$

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

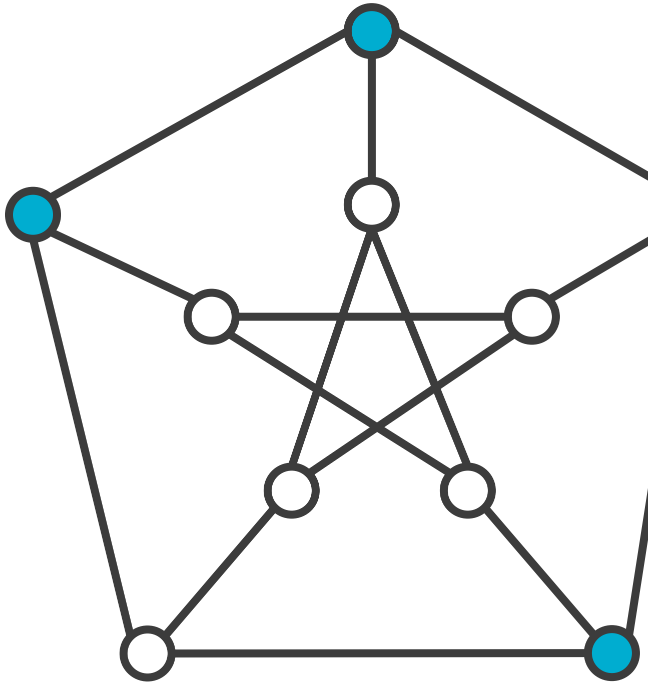
VERTEX COVER



$G = (V, E)$

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

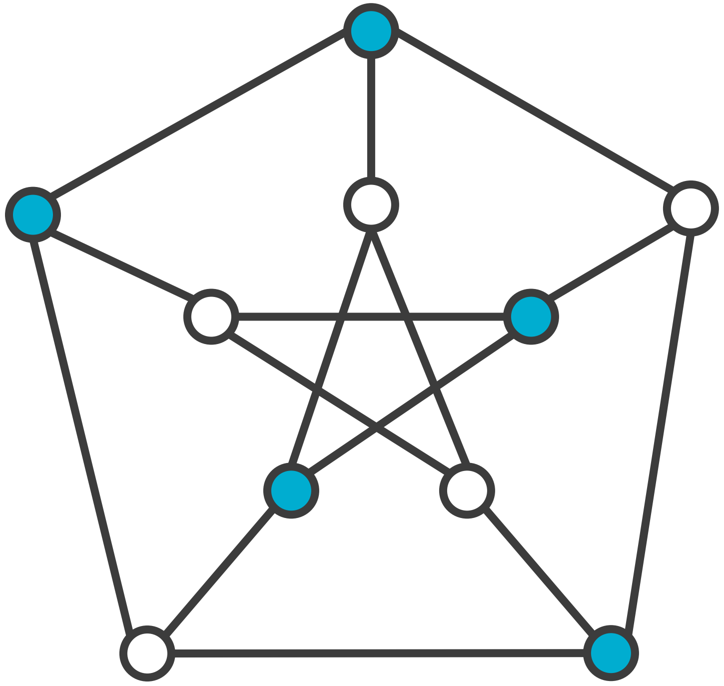
VERTEX COVER



$G = (V, E)$

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

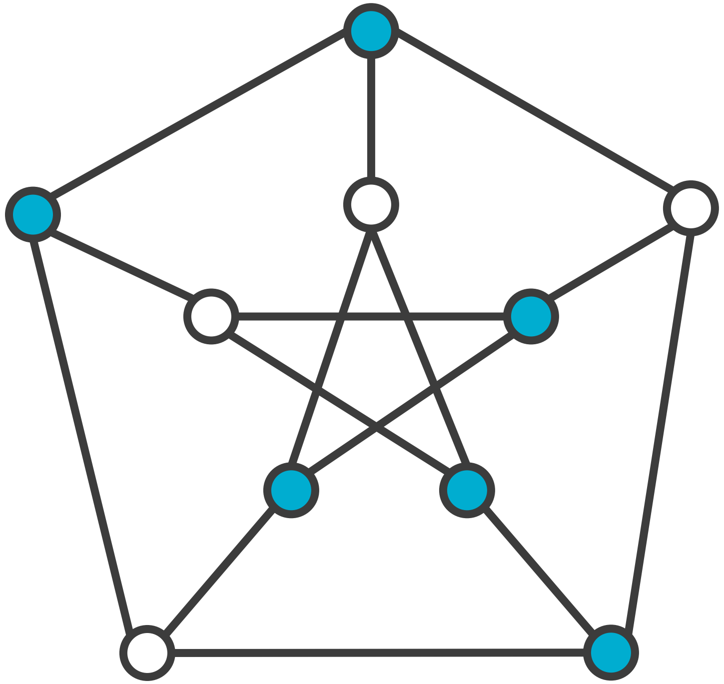
VERTEX COVER



$G = (V, E)$

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

VERTEX COVER



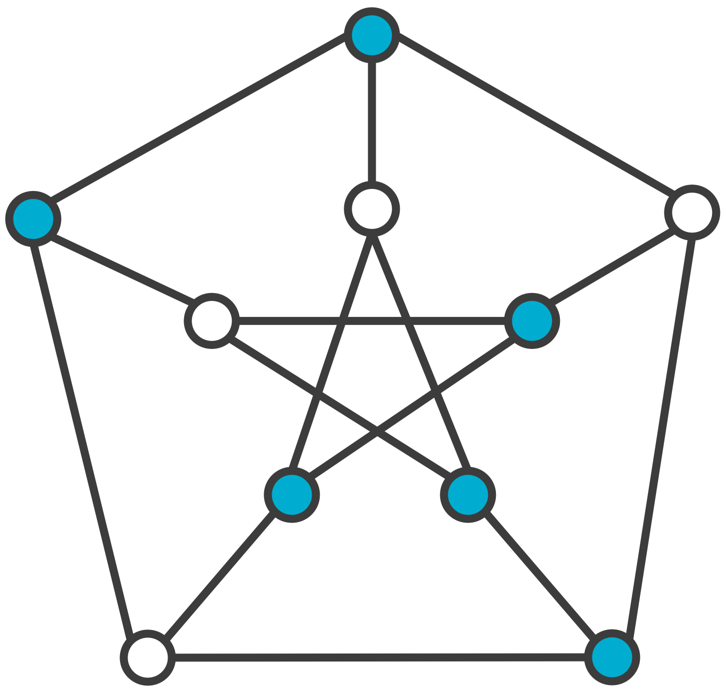
$G = (V, E)$

6 vertices colored

Optimal since each 5-cycle needed 3!

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

VERTEX COVER



$G = (V, E)$

Goal: color minimum number of vertices, so each edge has at least 1 colored endpoint

NP-hard: one of Karp's original 21 problems

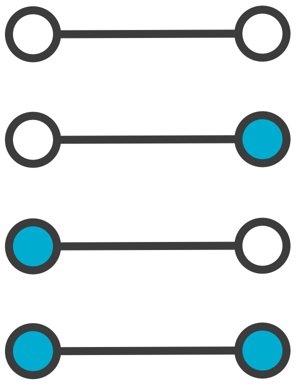
Several 2-approximations known
e.g. [Bar-Yehuda, Bendel, Freund, Rawitz 2004]

Best possible under Unique Games Conjecture
[Khot, Regev 2008]

VERTEX COVER AS CONSTRAINED LOCAL HAMILTONIAN

$$\min_{|\psi\rangle} \langle\psi| \textstyle\sum_u |1\rangle\langle 1|_u |\psi\rangle$$

$\langle\psi||00\rangle\langle 00|_{uv}|\psi\rangle = 0$ for all edges (u,v)



$|00\rangle$ Unhappy edge

$|01\rangle$

$|10\rangle$ Happy edge

$|11\rangle$

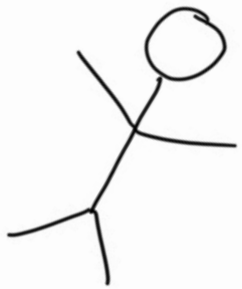
PUT A TRANSVERSE FIELD ON IT

$$\min_{|\psi\rangle} \langle\psi| \textstyle\sum_u (I - Z_u)/2 \,|\psi\rangle \quad \sum_u X_u$$

+Secret Sauce

Solution: x4

$$\langle\psi|(I - Z_u)(I - Z_v)/4|\psi\rangle = 0$$ for all edges (u,v)



PUT A TRANSVERSE FIELD ON IT

$$\min_{|\psi\rangle} \langle\psi| \textstyle\sum_u (I - Z_u)/2 \,|\psi\rangle + \displaystyle\sum_u X_u$$

$\langle\psi|(I + Z_u)(I + Z_v)/4|\psi\rangle = 0$ for all edges (u,v)

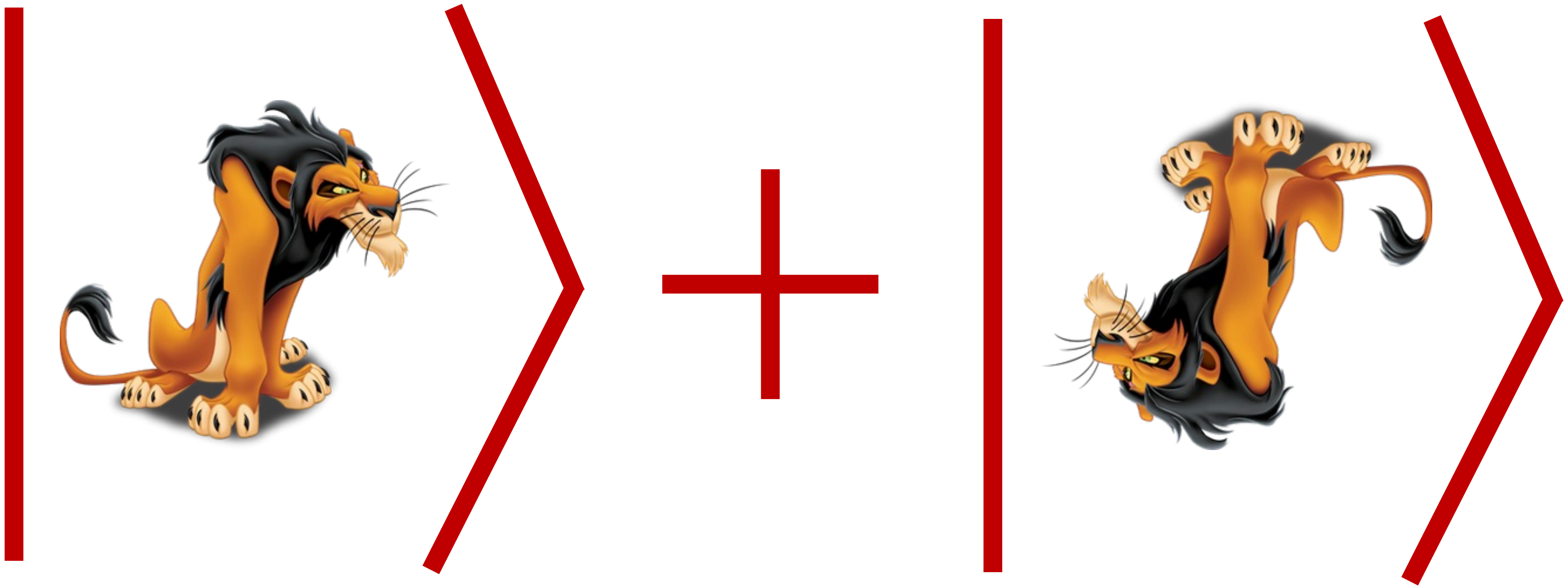
Equivalent to PXP model (Rydberg blockade interactions)

We show Transverse Vertex Cover/PXP are StoqMA-complete

Simple $(2 + \sqrt{2})$-approximation via quantum version of local ratio

[P, Rayudu, Thompson 2023]

QUANTUM SCARS



Thanks for staying awake to read this!