Learning Beyond Stabilizer States

Based on: arXiv:2305.13409

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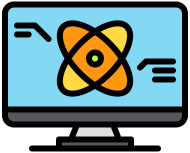
Quantum State Tomography

 $\rightarrow |\psi\rangle$

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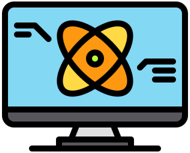 $\rightarrow |\psi\rangle$

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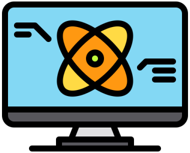

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Provably hard!

Efficient Quantum State Tomography

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The same way we do in classical learning!

Solution

- Move the goal post:
  - PAC learning [Aar07]
  - Shadow tomography [Aar18, HKP20]
  - Distinguishing/property testing [GNW21, GIKL23d]

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- Move the goal post:
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  - Shadow tomography [Aar18, HKP20]
  - Distinguishing/property testing [GNW21, GIKL23d]
- Restrict the states:
  - Free-fermion states [AG23]
  - Low-Degree Phase states [ABDY23]
  - **Stabilizer States** [Mon17]

Stabilizer States & Clifford Unitaries

Definition

A *Clifford unitary* is any unitary generated by H, S, and CNOT.

$$H := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad \mathrm{CNOT} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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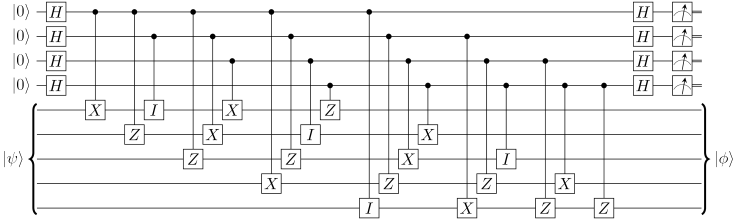
A *stabilizer state* is a state generated by a Clifford unitary on $|0^n\rangle$.

Not a universal gate set!

Applications

Error-correcting Codes



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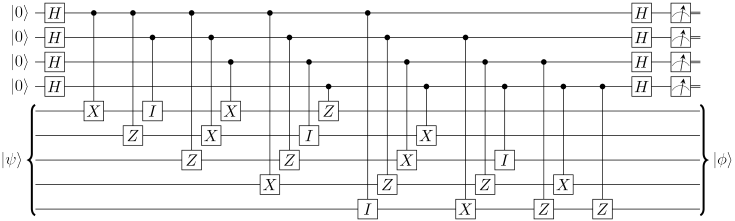
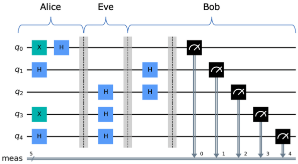


Quantum Key Distribution



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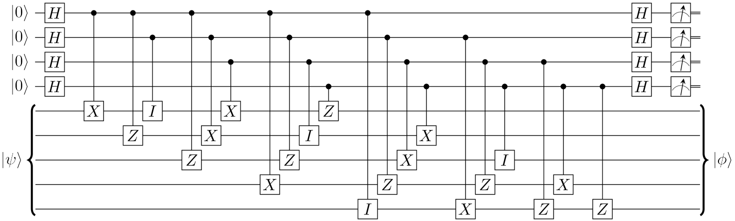

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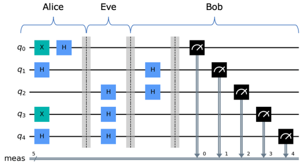
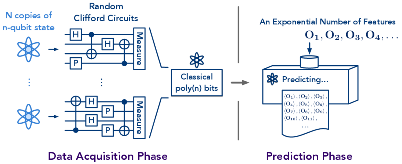


Learning Algorithms



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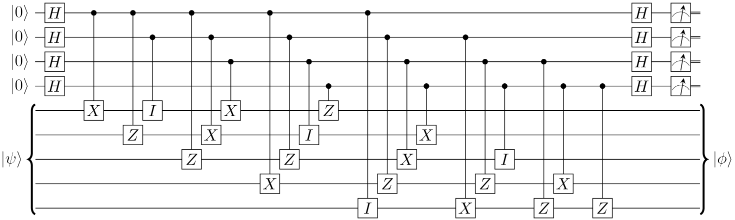

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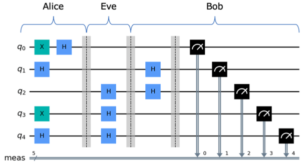

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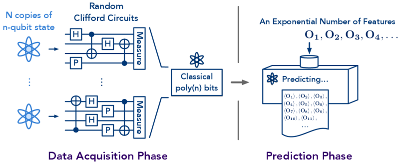

And more! Unitary Designs, Quantum Money, Classical Simulation, ...

Algebraic Structure of Stabilizer States

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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$$\text{Stab}(|\psi\rangle) := \left\{ W \in \mathcal{P}_n : |\langle\psi|W|\psi\rangle|^2 = 1 \right\}.$$

Fact: $|\text{Stab}(|\psi\rangle)| = 2^n$ if and only if $|\psi\rangle$ is a stabilizer state.

Learning Stabilizer States: A Warmup

Lemma ([AG04])

Given Stab$(|\varphi\rangle)$*, there exists a Clifford circuit* C *such that*

$$C|\varphi\rangle = |x\rangle$$

for some $x \in \{0,1\}^n$*.*

Moreover, C can be computed in time $O(n^2)$.

Learning Stabilizer States: A Warmup (cont.)

Lemma ([Mon17])

Given copies of a stabilizer state $|\varphi\rangle$, there exists a measurement to efficiently sample from the uniform distribution over Stab$(|\varphi\rangle)$.

Algorithm: Sample $O(n)$ times and output the group generated by the samples.

Learning Beyond Stabilizer States

Question

Various generalizations of stabilizer states:

- Low-stabilizer-rank states
- Low-degree phase states
- **Clifford + T states**

Can we learn any of them efficiently?

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T gates take us further and further from the nice algebraic properties of stabilizer states:

- Classical simulation algorithms run in time $\mathrm{poly}(n)\exp(k)$.

Our Work

Theorem ([GIKL23a, LOH23, HG23])

Can learn any state produced by k T gates in time $\mathrm{poly}(n)\exp(k)$.

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- **Single-copy learning algorithm:** [GIKL23b, CLL23].

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Learning States with Many Stabilizers

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Let $|\psi\rangle$ be produced by Clifford gates and at most k T gates. Then $|Stab(|\psi\rangle)| \geq 2^{n-k}$.

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Theorem ([GIKL23a])

Can learn any state such that $|\mathit{Stab}(|\psi\rangle)| \geq 2^{n-k}$ *in time* $\mathrm{poly}(n)\exp(k)$.

A Compression Scheme

Critical Observation

Let $|\text{Stab}(|\psi\rangle)| \geq 2^{n-k}$. Then learning $\text{Stab}(|\psi\rangle)$ is enough to learn $|\psi\rangle$ in time $\text{poly}(n)\exp(k)$.

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Given $\text{Stab}(|\psi\rangle)$, there exists a Clifford circuit C such that

$$C\,|\psi\rangle = |x\rangle \otimes \underbrace{|\varphi\rangle}_{k \text{ qubits}}$$

for some $x \in \{0,1\}^{n-k}$.

Moreover, C can be computed in time $O(n^2)$.

Initial Algorithm

Algorithm 1: First Approach

Input: Copies of $|\psi\rangle$ and description of Stab($|\psi\rangle$)

Promise: $|\text{Stab}(|\psi\rangle)| \geq 2^{n-k}$

Output: $|\widehat{\psi}\rangle \approx |\psi\rangle$

1. Find C such that $C\,|\psi\rangle = |x\rangle\,|\varphi\rangle$.
2. Measure first register of $C\,|\psi\rangle$ to learn x.
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How do we find Stab$(|\psi\rangle)$?

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- p_ψ is a distribution [Mon17]
- Can sample from $q_\psi = p_\psi * p_\psi$ via Bell difference sampling [GNW21]

A Fourier Duality Theorem

Theorem ([GIKL23c])

Given a subgroup $G \subseteq \{I, X, Y, Z\}^{\otimes n}$*:*

$$\sum_{W \in G} p_\psi(W) = \frac{|G|}{2^n} \sum_{W \in G^\perp} p_\psi(W)$$

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Definition

$G^\perp \subseteq \{I, X, Y, Z\}^{\otimes n}$ is the set of Pauli matrices that commutes with all of G.

Claim: $\left(G^\perp\right)^\perp = G$.

Corollary

The support of p_ψ lies in Stab$(|\psi\rangle)^\perp$.

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By definition of Stab$(|\psi\rangle)$:

$$\sum_{W \in \text{Stab}(|\psi\rangle)} p_\psi(W) = \sum_{W \in \text{Stab}(|\psi\rangle)} \frac{1}{2^n} \langle\psi|W|\psi\rangle^2 = \frac{|\text{Stab}(|\psi\rangle)|}{2^n}.$$

□

How to learn Stab($|\psi\rangle$)

Algorithm 2: Learning Algorithm v2

Input: Copies of $|\psi\rangle$
Promise: $|\text{Stab}(|\psi\rangle)| \geq 2^{n-k}$
Output: $|\widehat{\psi}\rangle \approx |\psi\rangle$

Draw $m = O(n)$ samples: $W_1, W_2, \cdots W_m \sim p_\psi$.
Compute $\widehat{\text{Stab}(|\psi\rangle)} := \langle W_1, W_2, \cdots, W_m \rangle^\perp$.

Claim: Given G, $G^\perp$ can be computed in time $O(n^3)$.

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3 Run compression scheme from previous algorithm.

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Can learn such a subgroup with $O(n/\varepsilon^2)$ samples.

The Learning Algorithm

Algorithm 3: Tomography of States with many Stabilizers

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Return $C^{\dagger} |0^{n-k'}\rangle \otimes |\widehat{\varphi}\rangle$.

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Goal:

$$|\psi\rangle \coloneqq C^{\dagger} \sum_{x \in \{0,1\}^{n-l}} \alpha_x |x\rangle \otimes |\varphi_x\rangle .$$

$$\max_{x \in \{0,1\}^{n-l}} |\alpha_x|^2 \geq 1 - \varepsilon^2$$

Collision Probability and p_ψ

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Proof (cont.)

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$$|\psi\rangle \approx_\varepsilon |x_{\text{max}}\rangle \, |\varphi_{x_{\text{max}}}\rangle$$

Finding C

Lemma ([GIKL23a])

Let $G \subseteq \{I, X, Y, Z\}^{\otimes n}$ *such that*

$$\sum_{W \in G} p_\psi(W) > \frac{3}{4}.$$

Then there exists a Clifford circuit C *such that* $C(G^\perp) = \mathcal{Z}^{n-l}$.

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$$\sum_{x \in G} p_\psi(x) = \sum_{x \in C(G)} p_{\psi'}(x)$$

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   - $G^\perp \supseteq \text{Stab}(|\psi\rangle)$.
3. Apply C such that $C(G^\perp) = \{I, Z\}^{\otimes n-l} \otimes I^{\otimes l}$.
   - $C\,|\psi\rangle = \sum_{x \in \{0,1\}^{n-l}} \alpha_x \,|x\rangle\,|\varphi_x\rangle$ such that

$$\max_{x \in \{0,1\}^{n-l}} |\alpha_x|^2 \geq 1 - \varepsilon^2/4.$$

Algorithm Overview (cont.)

4. Measure first register of $C|\psi\rangle$ $O(1)$ times, to learn x_{max} w.h.p.
   - Probability of measuring x_{max} is $|\alpha_{x_{\text{max}}}|^2 > 3/4$.

Algorithm Overview (cont.)

4. Measure first register of $C \left|\psi\right\rangle$ $O(1)$ times, to learn x_{max} w.h.p.
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5. Post-select on measuring $\left|x_{\text{max}}\right\rangle$.
   - Left with $\left|\eta\right\rangle := \left|x_{\text{max}}\right\rangle \left|\varphi\right\rangle$.
   - $d_{\text{Tr}}\left(C\left|\psi\right\rangle, \left|\eta\right\rangle\right) \leq \varepsilon/2$.

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6. Run pure state tomography on second register such that $d_{\text{Tr}}\left(\left|\varphi\right\rangle, \left|\widehat{\varphi}\right\rangle\right) \leq \varepsilon/2$.
   - Total trace distance is at most $\varepsilon/2 + \varepsilon/2$ via triangle inequality.

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   - Probability of measuring x_{max} is $|\alpha_{x_{\text{max}}}|^2 > 3/4$.
5. Post-select on measuring $|x_{\text{max}}\rangle$.
   - Left with $|\eta\rangle := |x_{\text{max}}\rangle|\varphi\rangle$.
   - $d_{\text{Tr}}(C|\psi\rangle, |\eta\rangle) \leq \varepsilon/2$.
6. Run pure state tomography on second register such that $d_{\text{Tr}}(|\varphi\rangle, |\widehat{\varphi}\rangle) \leq \varepsilon/2$.
   - Total trace distance is at most $\varepsilon/2 + \varepsilon/2$ via triangle inequality.
7. Output $C^\dagger|x_{\text{max}}\rangle|\widehat{\varphi}\rangle$.
   - Trace distance preserved by unitaries.

Open Questions

Lower Bounds

Current best-known lower bounds are $\approx \Omega(\sqrt[4]{k})$, due to unitary t-designs [HMMH$^+$23].

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Proper Learning

Output state is not necessarily produced by $O(\log n)$ T-gates, can be as many as $\mathrm{poly}(n)$.

Thank You!