Quantum Walks

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Classical Random Walk:
1. Start at origin
2. Toss a coin
3. Move left or right accordingly
4. Repeat

Quantum Walk:
1. Start at origin
2. Toss a quantum coin, e.g.
3. Move left and right simultaneously according to coin state
4. Repeat
**Artist Impression – classical random walk**

**Artist Impression – quantum random walk**

Entangled Quantum Walk:
1. Start at origin
2. Apply quantum coins
3. Move left and right simultaneously according to coin state
4. Repeat

\[ \psi_0 = \frac{1}{\sqrt{2}} (|0, \uparrow, 0, \uparrow \rangle - |0, \downarrow, 0, \downarrow \rangle) \]

Walking on a line

\[ T_{12} = i+1, \uparrow; j+1, \uparrow \]
\[ + i-1, \downarrow; j+1, \uparrow \]
\[ - i-1, \downarrow; j-1, \downarrow \]
\[ + i+1, \downarrow; j-1, \downarrow \]
Entangled Quantum Walk:
1. Start at origin
2. Apply quantum coins
3. Move left and right simultaneously according to coin state
4. Repeat (with interaction)

Classical: coin flip and move (stochastic)

Quantum: apply coin operator and evolve (unitary)
Classical: stochastic Markov chains
\[ P(t) = e^{-H t} P(0) \] probability

Quantum: unitary evolution
\[ \psi(t) = e^{-i \hat{H} t} \psi(0) \] amplitude

Walking on a directed/weighted graph

Discrete-time quantum walk
\[ \psi_{n+1} = \hat{C} \hat{T} \psi_n = \hat{A} \psi_n \quad (\hat{A}: \text{non-unitary}) \]

Continuous-time quantum walk
\[ \psi(t) = e^{-i \hat{H} t} \psi(0) \quad (\hat{H}: \text{Hermitian}) \]

Classical random walk applications
Examples: DNA synopsis, animal foraging strategies, diffusion and mobility in materials, exchange rate forecast, stock market analysis, solving differential equations, quantum monte carlo, optimization, clustering and classification, graph connectivity, fractal theory, structure analysis of facebook, Google, MSN and Yahoo search engines, etc.

Quantum walk applications?
Quantum Algorithms for the NISQ Era
(https://quisa.tech/publications/)

- Combinatorial optimization via highly efficient quantum walks
- Quantum optimisation of financial portfolios
- Quantum optimisation of capacitated vehicle routing
- Quantum algorithm for network analysis and centrality ranking
- Quantum walk based algorithms for graph similarity, isomorphism, and other graph-theoretic quantum algorithms
- Quantum algorithm for video visual tracking
- Quantum informatics: protein sequence engineering
- Quantum data compression by principal component analysis
- Gibbs partition function using quantum Clifford sampling
- Quantum predictive algorithms on phase transition and criticality
- …
Non-Unitary

\[ \hat{U} = \hat{C} \hat{T} e^{-i \hat{H} t} \]

\( \hat{C} \)

1. Graphs with certain symmetry

- Complete 8-graph with Grover Coin

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}; \quad \hat{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]

- 2D hypercycle

\[ H = \begin{pmatrix} 0 & 1 \end{pmatrix}; \quad \hat{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \mathcal{T} = H \hat{T} H \]

\[ |0\rangle \]

\[ \mathcal{T} = \frac{1}{\sqrt{3}} e^{\frac{i\pi}{3}(j-k)(k+1)} \]

Qutrit equivalent of Hadamard
1. Graphs with certain symmetry

"Twisted" toroidal lattice

\[
\Hat{\mathcal{T}}
\]

3CT-2

\((t, n, s)\)

\[
\begin{pmatrix}
G_1 \\
G_2 \\
G_3 \\
G_4
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{pmatrix}
\]

3CT-3

dCT-n

1. Graphs with certain symmetry

"Twisted" star

\[
\begin{cases}
0 \\
\hat{T}_{ij}, & i \leq n, j \leq n, \\
\hat{T}_{ij}, & \text{otherwise,}
\end{cases}
\]

\[
\begin{pmatrix}
\hat{T}_1 \\
\hat{T}_2 \\
\hat{T}_3 \\
\hat{T}_4
\end{pmatrix}
\]

Glued-trees

sparse

\[
\begin{cases}
R_1 \\
R_2
\end{cases}
\]

Entrance

Exit

node

subnode

level

side of tree

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{pmatrix}
\]

1. Graphs with certain symmetry
2. Sparse graphs

\[
\hat{C} T
\]

\[
H = \begin{pmatrix}
0 & U \\
U^\dagger & 0
\end{pmatrix}; \quad U: \text{unitary}
\]
\[
H: \text{Hermitian}; \quad H^2 = I; \quad ||H|| = 1
\]

\[
e^{-iHt} = \cos(\theta)I - i\sin(\theta)H
\]

\[
e^{-iH\pi/2} = -iH = -i\begin{pmatrix}
0 & U \\
U^\dagger & 0
\end{pmatrix}
\]

\[
\therefore e^{-iH\pi/2} \begin{pmatrix}
0 \\
\psi
\end{pmatrix} = -i\begin{pmatrix}
U\psi \\
0
\end{pmatrix}
\]

\[
\therefore e^{-iH\pi/2} |\psi\rangle = -i|0\rangle U|\psi\rangle
\]

\[
\text{Jordan and Wocjan PRA 80, 062301 (2009)}
\]

1. Sparse Hamiltonian with efficiently computable entries

- Lloyd (1996)
- Aharonov and Ta-Shma (2003)
- Berry, Ahokas, Cleve, Sanders (2007)
- Childs (2008)
- Wiebe, Berry, Hoyer, Sanders (2010)
- Childs (2010)
- Poulin, Qarry, Somma, Verstraete (2011)
- Childs and Wiebe (2012)
- Berry and Childs (2012)
- Berry, Childs, Kothari (2015)
- Berry and Novo (2016)
- Low and Chuang (2017, 2019)
- Chen, Dalzell, Berta, Brandão, Tropp (2023)

2. Efficiently diagonalizable dense graph

Spectral theorem \( e^{-iHt} = Q^\dagger e^{-i\Lambda t} Q \)

\( \Lambda \): diagonal matrix of eigenvalues of \( H \)

\( Q \): matrix of column eigenvectors of \( H \)

\[
C = \begin{pmatrix}
c_1 & c_2 & \cdots & c_n \\
c_n & c_1 & \cdots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_2 & c_3 & \cdots & c_1
\end{pmatrix}
\]

\[
Q = F_{jk} = \omega^{jk}/\sqrt{n}, \quad \omega = e^{2\pi i/n}
\]

\[
\Lambda = \text{diag}(\sqrt{n} F\{c_1, c_2, \ldots, c_n\})
\]

\[
\text{DFT: } O(n \log n) \quad \text{QFT: } O(\log^2 n)
\]

\[
\text{Nature Communication 7:11511 (2016)}
\]
Circulant graphs

\[ e^{-iH_t} = F^\dagger e^{-i\Lambda t} F \]

(1) \(O(\text{poly log}(n))\) distinct eigenvalues

\[ \Lambda = \text{diag}\left\{ \{m,0,0,0,0,0,0\}\right\} \]

Complete \(K_m\)

\[ \text{# gates: } 2\log^2(n) + 1 \]

Paley Graph (SRG)

\[ \Lambda = \text{diag}\left\{ \{a,b,b,b,b,b,c,c,c,c,c,c\}\right\} \]

(all SRGs have at most three distinct eigenvalues)

Nature Communication 7:11511 (2016)

(2) eigenvalues can be computed efficiently

\[ \Lambda = \text{diag}\left\{ \{2\cos\left(\frac{2k\pi}{n}\right)\} | k = 0,1,\ldots,n-1\} \right\} \]

Cycles

\[ \Lambda = \text{diag}\left\{ \{4\sin^2\left(\frac{k\pi}{n}\right) + (1 - (-1)^k)\} | k = 0,1,\ldots,n-1\} \right\} \]

Möbius

Nature Communication 7:11511 (2016)

(2) eigenvalues can be computed efficiently

\[ \sum \beta_n|n\rangle = F^\dagger e^{-i\Lambda t} F \sum \beta_n|n\rangle \]

\[ \Lambda = \text{diag}\left\{ \{4\sin^2\left(\frac{k\pi}{n}\right) + (1 - (-1)^k)\} | k = 0,1,\ldots,n-1\} \right\} \]

Childs PhD thesis

Möbius

Nature Communication 7:11511 (2016)
Circulant graphs: $e^{-iH_c t} = F\dagger e^{-i\Lambda t} F$

3. Composite graphs

(1) COMMUTING GRAPHS

$e^{-i\gamma (A+B)t} \equiv e^{-i\gamma A t} e^{-i\gamma B t}$

(identity interconnection between two complete graph)
2. Composite graphs

(1) COMMUTING GRAPHS

\[ e^{-i\gamma(A+B)t} \equiv e^{-i\gamma At} e^{-i\gamma Bt} \]

complete interconnection between two disjoint degree-regular graphs

(2) CARTESIAN PRODUCT OF GRAPHS

\[ H_1 \oplus H_2 = H_1 \otimes I_{n_2} + I_{n_1} \otimes H_2 \]

\[ e^{-i(H_1 \oplus H_2)t} \equiv e^{-iH_1 t} \otimes e^{-iH_2 t} \]

hypercube graph \( Q_n = K_2^{\oplus n} \)

Quantum Circuit Implementation
Can we implement Non-Unitary operations efficiently?

1. Arbitrary sparse matrix and inverse

Harrow, Hassidim, Lloyd (2009)
Cai, Weedbrook, Su, Chen, Gu, Zhu, Li, Liu, Lu, Pan (2013)

$$|0\rangle \rightarrow H|0\rangle \rightarrow FT^2 |0\rangle \rightarrow FT \rightarrow H |1\rangle$$

$$|0\rangle^{	ext{def}} \rightarrow H|0\rangle \rightarrow FT^2 |0\rangle \rightarrow FT \rightarrow H |0\rangle^{	ext{def}}$$

$$U = \sum_{k=0}^{n-1} |k\rangle \langle k| \otimes e^{iA_{k\bar{k}} / T}$$

$$|x\rangle = c A^{-1} |\beta\rangle$$

2. Dense Circulant, Toeplize & Hankel

Circulant Matrices

$$C = \left( \begin{array}{ccccc} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{array} \right) = c_0 \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right) + c_1 \left( \begin{array}{cccc} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{array} \right) + \cdots + \sum_{j=0}^{n-1} c_j V_j$$

$$|0^L\rangle \rightarrow O_c |\psi\rangle \rightarrow O_{c1} |\psi\rangle$$

$$|\psi\rangle \rightarrow V_j |\psi\rangle \rightarrow C |\psi\rangle$$

where $$V_j = \sum_{k=0}^{n-1} (k - j) \text{ mod } n \langle k|$$

$$O_c |0^L\rangle = \sum_{j=0}^{n-1} \sqrt{c_j} |j\rangle$$

Every Matrix is a Product of Toeplitiz Matrices

Ke Ye · Lek-Heng Lim (2015)

The choice of Toeplitiz factors is natural for two reasons. Firstly, Toeplitiz matrices are ubiquitous and are one of the most well-studied and understood classes of structured matrices. They arise in pure mathematics: algebra [6], algebraic geometry [46], analysis [31], combinatorics [35], differential geometry [40], graph theory [26], integral equations [5], operator algebra [23], partial differential equations [50], probability [45], representation theory [25], topology [42], as well as in applied mathematics: approximation theory [54], compressive sensing [32], numerical integral equations [39], numerical integration [51], numerical partial differential equations [52], image processing [19], optimal control [44], quantum mechanics [24], queueing networks [7], signal processing [53], statistics [22], time series analysis [18], and among other areas.
Non-Unitary

2. Dense Circulant, Toeplize & Hankel

\[ T = \begin{pmatrix}
T_0 & T_{-1} & T_{-2} & \cdots & T_{-(n-1)} \\
T_{-1} & T_0 & T_{-1} & \cdots & T_{-(n-2)} \\
T_{-2} & T_{-1} & T_0 & \cdots & T_{-(n-3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{n-1} & T_{n-2} & T_{n-3} & \cdots & T_0
\end{pmatrix} \]

\[ B_T = \begin{pmatrix}
0 & T_{n-1} & \cdots & T_2 & T_1 \\
T_{n-1} & 0 & \cdots & T_3 & T_2 \\
T_{n-2} & T_{n-1} & \cdots & T_4 & T_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
T_1 & T_2 & \cdots & T_0 & 0
\end{pmatrix} \]

\[ C_T = \begin{pmatrix} T & B_T \\ B_T & T \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi \end{pmatrix} = \begin{pmatrix} T\psi \\ B_T\psi \end{pmatrix} \]

3. PT-symmetric Quantum Walk

**PHYSICAL REVIEW LETTERS** 125, 240501 (2020)

**Experimental Parity-Time Symmetric Quantum Walks for Centrality Ranking on Directed Graphs**


4. Szegedy quantum walk

**NATURE PHOTONICS** | VOL 12 | SEPTEMBER 2018 | 534–539 |

Large-scale silicon quantum photonics implementing arbitrary two-qubit processing


PRL 125, 240501 (2020)
5. Unitary dilation

\[ U_A(t) = \begin{pmatrix} \cos A(t) & -e^{-iA(t)} \sin A(t) \\ e^{iA(t)} \sin A(t) & \cos A(t) \end{pmatrix} \]

**Experimental realization of continuous-time quantum walks on directed graphs and their application in PageRank**

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**Fig. 1.** Directed graphs with three nodes. Both graphs, \( A_1 \) and \( A_2 \), have two directed edges. Graph \( A_1 \) is a smooth directed graph, and \( A_2 \) is triangle.

**Fig. 2.** (a) The circuit diagram for the 4 × 4 unitary transformation \( U_\theta \). (b) The circuit diagram for the 6 × 4 unitary transformation \( U \).

**Summary**

- high degree symmetry
- sparse unitary
- sparse
diagonalizable
- circulant, Toeplize, Hankle
- COMPOSITE
  - commuting graphs
  - cartesian product