Hamiltonian learning: recent progress and open problems

Yu Tong

Institute for Quantum Information and Matter, Caltech

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This talk is based on

- Hsin-Yuan Huang, **Yu Tong**, Di Fang, Yuan Su, 2022, *Learning many-body Hamiltonians with Heisenberg-limited scaling*.

- Haoya Li, **Yu Tong**, Hongkang Ni, Tuvia Gefen, Lexing Ying, 2023, *Heisenberg-limited Hamiltonian learning for interacting bosons*.

Scope of the talk

- Learning the Hamiltonian from time-evolution, and with focus on the Heisenberg limit and the role of quantum control.

[1] Anshu, Arunachalam, Kuwahara, Soleimanifar, 2020, *Sample-efficient learning of interacting quantum systems*.

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- Learning the Hamiltonian from time-evolution, and with focus on the Heisenberg limit and the role of quantum control.
- Not covered: learning the Hamiltonian from the Gibbs state or the ground state.[1,2,3,4]

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- Restriction: we cannot apply control-e^{-iHt} or e^{iHt}.

Measuring the cost

- We can get the Hamiltonian by learning the unitary $e^{-iH\tau}$ for a small τ. Requires $e^{\mathcal{O}(N)}\epsilon^{-1}$ queries to $e^{-iH\tau}$.[5]

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  - We use total evolution time: if we use e^{-iHt_1}, e^{-iHt_2}, ..., $e^{-iHt_{N_{\mathrm{exp}}}}$, then the total evolution time is $t_1 + t_2 + \cdots t_{N_{\mathrm{exp}}}$.

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  - We also need to make sure that the number of experiments N_{exp} and the number of unitaries are not too large.

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Connection with quantum metrology

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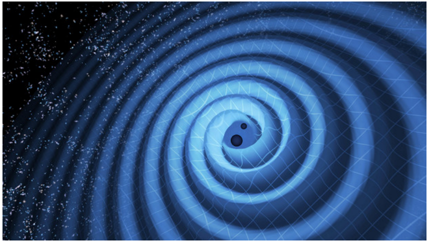

Figure: Image credit: LIGO/T. Pyle

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- **Quantum metrology:** high-precision estimation of a few physical parameters. Asymptotic convergence governed by the quantum Fisher information.

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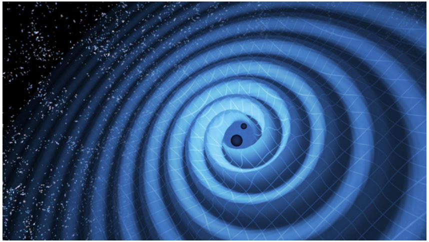
- **Hamiltonian learning:** Estimation of many parameters. Non-asymptotic (without good prior information).



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A brief history

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- **Experimental implementation:** single spin (NV center),[8] non-interacting boson (superconducting qubits).[9]

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  - Pauli channel estimation ($\mathcal{O}(\epsilon^{-4}\log(N/\delta))$, SPAM-robust).[13]

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}[\rho e^{iHt} O e^{-iHt}]|_{t=0} = i\mathrm{Tr}[\rho[H,O]] = i\mathrm{Tr}[H[O,\rho]].$$

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- Choose ρ (Pauli eigenstate) and O (Pauli) so that $[O,\rho] = \frac{i}{2^{N-1}} P$.

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}[\rho e^{iHt} O e^{-iHt}]|_{t=0} = -2\lambda_P.$$

- Derivatives can be estimated accurately using polynomial interpolation. Many derivatives can be estimated simultaneously using classical shadows.[17,18,19]

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- **The Heisenberg limit:** $T = \epsilon^{-1}$, and N_s can be $\mathcal{O}(\log(\epsilon^{-1}))$.

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- The proof can be extended to the adaptive and biased case.
- Reaching the Heisenberg limit requires something **qualitatively different**.

The Heisenberg limit: an example

Consider time-dependent signal $S(t)$, $t \geq 0$

$$S(t) = e^{i\theta t} + g, \quad g \sim \mathcal{N}(\mu, \sigma^2 I).$$

We want to estimate $\theta \in (-1, 1]$ to precision ϵ.

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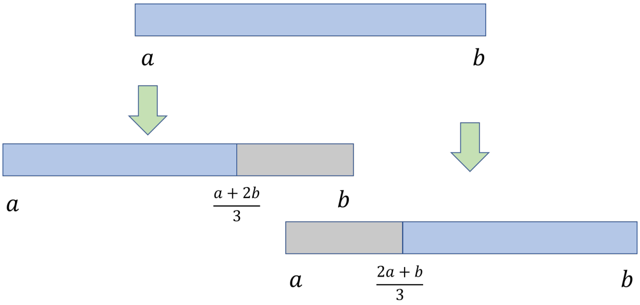

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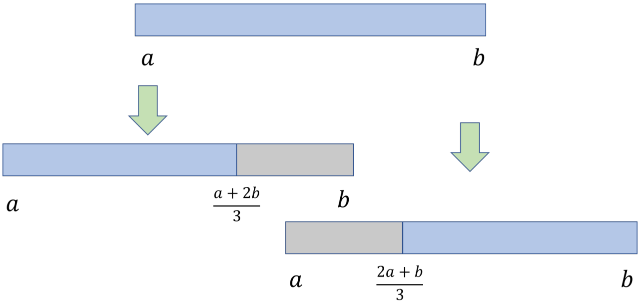

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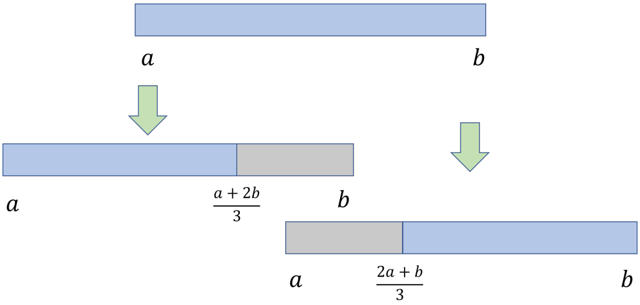

- We can then update $a \leftarrow a$, $b \leftarrow (1/3)a + (2/3)b$, or $a \leftarrow (2/3)a + (1/3)b$, $b \leftarrow b$.
- We can **reduce the uncertainty by** $1/3$ at each step. $\mathcal{O}(\log(\epsilon^{-1}))$ steps are needed for ϵ precision.

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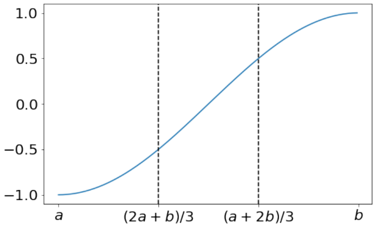

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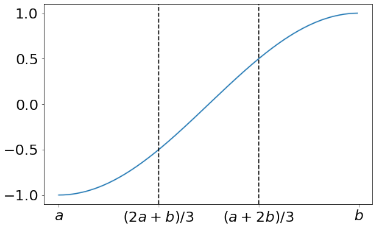

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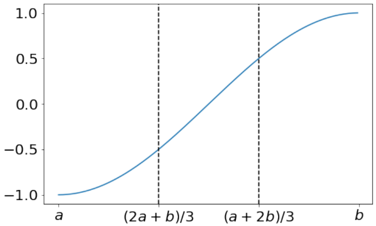

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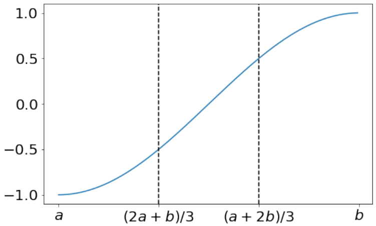

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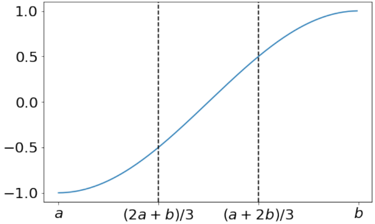

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- Need $\delta' = \mathcal{O}(\delta / \log(\epsilon^{-1}))$ to ensure that all steps are successful with probability $1 - \delta$.
- Total evolution time is $\mathcal{O}(\epsilon^{-1} \log(\delta^{-1}))$ and the number of samples is $\mathcal{O}(\log(\epsilon^{-1}))$.
- Robust to noise ($|\mu| + \sigma = \mathcal{O}(1)$).

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- Combine to get a signal $e^{2i\theta t}+$ noise.

The difficulties of reaching the Heisenberg limit

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- Using non-local observables does not help either (under the eigenstate thermalization hypothesis and learning many parameters).[21]

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- If we can artificially create conservation laws we may use it to get coherent signal at late times.

Hamiltonian reshaping

- Inserting random Pauli operators.[22]

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$$e^{-iHt} = e^{-iH\tau} \cdots e^{-iH\tau} e^{-iH\tau} \to P_r e^{-iH\tau} P_r \cdots P_2 e^{-iH\tau} P_2 P_1 e^{-iH\tau} P_1,$$

where P_j are uniformly randomly drawn from a Pauli subgroup $K \leq G_N$.

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- Because ${P_j}^2 = I$,

$$P_r e^{-iH\tau} P_r \cdots P_2 e^{-iH\tau} P_2 P_1 e^{-iH\tau} P_1 = e^{-iP_r H P_r \tau} \cdots e^{-iP_2 H P_2 \tau} e^{-iP_1 H P_1 \tau}.$$

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In one time step

$$\rho \mapsto \rho - i\mathbb{E}_{P\sim\mathcal{U}(K)}[PHP, \rho]\tau + \mathcal{O}(\tau^2)$$
$$= \rho - i[H_{\text{effective}}, \rho]\tau + \mathcal{O}(\tau^2),$$

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- This is the same idea underlying the qDRIFT algorithm.[23]

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- **The coefficients we want to learn are preserved.** For any Pauli operator $P' \in G_N$,

$$\frac{1}{|K|} \sum_{P \in K} P P' P = \begin{cases} P', & P' \in C_{G_N}(K), \\ 0, & P' \notin C_{G_N}(K). \end{cases} \implies H_{\text{effective}} = \sum_{P \in C_{G_N}(K)} \lambda_P P.$$

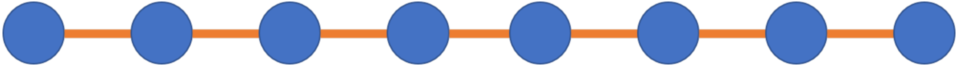
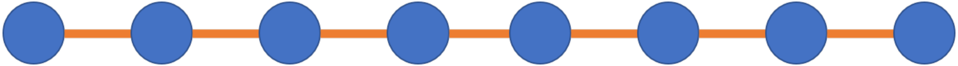


Figure: Every qubit interacts with its neighbors.

- Choose $K = \langle Z_3, Z_6, Z_9, \cdots, X_3, X_6, X_9, \cdots \rangle$.



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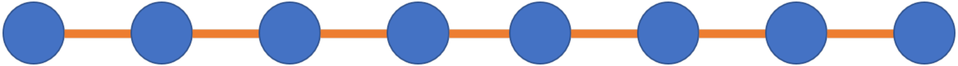

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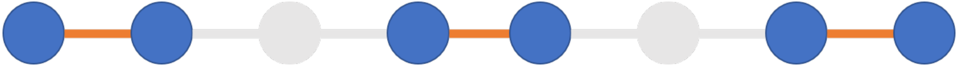

Figure: Suppressing qubits so that the rest are isolated.

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- Close connection to dynamical decoupling, but more versatile.
- Similar subgroup-based strategy can be used to suppress coherent errors in quantum circuits.[24]

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Based on the Hamiltonian reshaping technique, we propose a Hamiltonian learning protocol that

- Achieves the Heisenberg scaling with $\mathcal{O}(\epsilon^{-1}\log(N/\delta))$ total evolution time;
- Uses $\mathcal{O}(\text{polylog}(\epsilon^{-1})\log(N/\delta))$ experiments;
- Uses only single-qubit Pauli eigenstates, Pauli gates, and single-qubit measurements;
- Is robust against state preparation and measurement (SPAM) error.

Hamiltonian reshaping for bosons

- Let H be a bosonic Hamiltonian, e.g.

$$H = \sum_{\langle i,j \rangle} h_{ij} b_i^\dagger b_j + \sum_i \omega_i n_i + \frac{1}{2} \sum_i \xi_i n_i (n_i - 1),$$

where $b_i^\dagger$ (b_i) are the bosonic creation (annihilation) operators.

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- We can apply $e^{i\theta n_i}$ (phase shifter) for $\theta \sim \mathcal{U}([0, 2\pi])$ to enforce local particle number conservation (U(1) symmetry).[25]

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- This can be used to isolate parts of the quantum system (no particle can hop to or from site i).

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- Learning off-diagonal terms: apply $e^{i\theta(b_i^\dagger b_j + b_j^\dagger b_i)/2}$ (beam splitter) for $\theta \sim \mathcal{U}([0, 2\pi])$ to conserve $b_i^\dagger b_j + b_j^\dagger b_i$ (similarly for $ib_i^\dagger b_j - ib_j^\dagger b_i$).

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Open problems

- Quantum control is necessary, but "how much" control do we need?
  - $e^{-iH\tau} \approx I - iH\tau \mapsto I - iH_{\text{effective}}\tau$: error of order $\mathcal{O}(\tau^2)$.

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  - For terms not in the Lindblad span, can we design a non-asymptotic protocol to learn all of them scalably in the presence of quantum noise?

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- Open questions remain as to how fast and strong the control needs to be and tolerance of quantum noise.