Visualizing level lines and curvature in images

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Introduction: What is an image?

Computer scientist’s point of view: array of pixel values: 0=black, 255=white

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Mathematician’s point of view: \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \) (but only samples are known). Which regularity?
Introduction: description of a shape by its curvatures

Idea: a shape is well described by its extrema of curvature:

Question: how to compute these curvatures in the presence of pixelization?
Answer: We need to smooth the lines
Introduction: images, contrast, level lines
Introduction: topographic map
Introduction: importance of invariance

- There is a necessity to smooth the image.
- The curvature is a Euclidean invariant, so our smoothing process should not disturb that invariance.
- Commutation of the smoothing process with a rotation is a nice property and will be satisfied.
- Actually, we will get even stronger invariance: commutation with special affine transforms.
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Theorem (Alvarez, Guichard, Lions, Morel 93)

*Causal, local, Euclidean and contrast covariant scale-spaces are all governed by a curvature equation of type:*

\[
\frac{\partial u}{\partial t} = |Du| G(\text{curv}(u), t).
\]

Two of these are the most interesting for image processing:
- the simplest one, the **mean curvature motion**:
  \[
  \frac{\partial u}{\partial t} = |Du| \text{curv}(u) \tag{MCM}
  \]
- the unique special affine covariant one, the **affine curvature motion**:
  \[
  \frac{\partial u}{\partial t} = |Du| \text{curv}(u)^{1/3} \tag{ACM}
  \]
Curvatures

We can define the curvature in two different ways:

- If $u$ is $C^2$ and $Du(x_0) \neq 0$, the scalar curvature at $x_0$ is
  \[
  \text{curv}(u)(x_0) = \frac{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}(x_0)
  \] (1)

- If $x(s)$ is a $C^2$ curve parameterized by length $s$ ($|x'(s)| = 1$), the vector curvature at $x_0 = x(s_0)$ is
  \[
  \kappa(x_0) = x''(s_0)
  \] (2)

Link: denote by $x(s)$ the level line of $u$ passing by $x_0$ ($u(x(s)) = u(x_0)$, $x(s_0) = x_0$), then

\[
\kappa(x_0) = -\text{curv}(u)(x_0) \frac{Du}{|Du|}(x_0).
\]

This suggests two ways to compute curvature: 2D differential operator (1) or curvature of level line (2).
Curve shortening

[Mackworth Mokhtarian 92]: The curvature motion \( \frac{\partial x}{\partial t} = \kappa(x) \) (CS) can be implemented by applying the heat equation to each coordinate independently:

**Theorem (Grayson 87)**

If \( x(s, 0) \) is a \( C^2 \) Jordan curve, then applying the intrinsic heat equation:

- For \( t > 0 \), \( x(s, t) \) is \( C^\infty \) and satisfies (CS).
- For \( t > 0 \), \( x(., t) \) has a finite and non-increasing number of inflection points and curvature extrema.
- For \( t \geq t_o \), \( x(., t) \) is convex, and for \( t \geq t_1 \), \( x(., t) \) is a point.

**Algorithm:** Input: polygon \( P \). Output: smoothed polygon \( P \). Iteratively:

- Sample \( P \) uniformly by length.
- Convolve coordinates of \( P \) by Gaussian kernel \( G_\sigma \).
Affine shortening

Algorithm [Moisan 98]

- Break $P$ into convex and concave parts
- Replace each part by the middle points of $\sigma$-chords originating from vertices of $P$.
- Concatenate the pieces of curves

Properties:

- Fast and simple algorithm (but numerically delicate)
- Special affine covariance (only inflection points, areas and middle points are involved)
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Algorithm [Koenderink and van Doorn 86]
Input: a closed subset $X$ of $\mathbb{R}^N$

- Compute $u(x, t) = G_t \ast 1_X(x)$
- Threshold at $1/2$: $X_t = [u(., t) \geq 1/2]$.

Problem: fusion of shapes that are close.

An improvement is the threshold dynamic shape:
Algorithm [Merriman Bence Osher 92]
Iteratively:

- Convolve $1_X$ with $G_\sigma$
- Threshold at $1/2$: $X \leftarrow [G_\sigma \ast 1_X \geq 1/2]$

This is a Gaussian-weighted median filter applied to a binary image.
Median filters

Algorithm: Iterated median filter. Iteratively, for every point $x$:
- Gather points $y$ in a discrete neighborhood of $x$
- Put at $x$ the median value of the discrete neighborhood

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Problem: the algorithm is blind to small curvatures, due to its discrete nature.

Theorem (Ishii 95)

*Iterated weighted median filters on images converge to the MCM.*
Numerical schemes

Finite difference schemes

- [Guichard Morel 97] uses the second derivative in the direction orthogonal to the gradient: $|Du| \text{curv}(u) = u_{\xi\xi}$ with $\xi = Du^\perp / |Du|$

$$ (u_{\xi\xi})_{ij} = \begin{pmatrix} \lambda_3 & \lambda_2 & \lambda_4 \\ \lambda_1 & -4\lambda_0 & \lambda_1 \\ \lambda_4 & \lambda_2 & \lambda_3 \end{pmatrix} (\theta) \star u_{i,j} \text{ with } Du = |Du| \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} $$

and all $\lambda_i = 1/2$ when $|Du| < \epsilon$ (half Laplacian)

- [Crandall Lions 96] Explicit scheme

$$ u^{n+1} = u^n + \frac{dt}{h^2} \sum_{i=1}^{N} (u^n(x + ha(Du^n)e_i) + u^n(x - ha(Du^n)e_i) - 2u^n(x)) $$

with $(e_i)$ basis of $\mathbb{R}^N$ and $a(p) = I - \frac{pp^T}{|p|^2 + \epsilon}$
Stack filters

Finite difference schemes do not commute with contrast changes. To recover the contrast covariance, we can “stack” the results on level sets:

**Algorithm [Stack filter]**

- Extract upper level sets \( X_\lambda = [u \geq \lambda] \)
- Apply FDS to \( 1_{X_\lambda} \)
- Set \( X'_\lambda \) by threshold at 1/2 of the results
- Reconstruct by superposition: \( u(x) \leftarrow \max\{\lambda : x \in X'_\lambda\} \)

Anyway, other problems of FDSs persist, including lack of:

- Monotonicity, we always have slightly oscillatory solutions
- Euclidean (or affine) covariance, since they are grid dependent
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Inclusion tree

Morse functions and critical points

Tree of inclusion of Jordan curves

Level lines at different levels:
Left: pixels still visible. Center: Jordan curves. Right: saddle points
Bilinear interpolated images

- Convolution of a lattice of Dirac masses by separable triangle function
  \[ \varphi(x) \varphi(y) \text{ with } \varphi(x) = (1 - |x|)^+ \]

- Between a square \(ABCD\) of data values, we have the equation
  \[ u(x, y) = axy + bx + cy + d \]

- Level line is intersection of the square with hyperbola
  \[ a(x - x_s)(y - y_s) = \lambda - \lambda_s \]

- If \(\lambda_s \in [\min_{P=ABCD} u(P), \max_{P=ABCD} u(P)]\), we have a saddle point
  and the piece of level line at \(\lambda_s\) is two orthogonal segments
Tree extraction algorithm

- Choose a number of levels, avoiding:
  1. Initial values
  2. Saddle values
- Two steps: follow level lines, then recover inclusion structure
- **Step 1:** Follow the level line inside square: given entry point, find exit point and sample the curve in between
  - Store intersections with horizontal edges
- **Step 2:** Order intersection points at each horizontal line
- Set inclusion by parity argument:
  1. Odd: get inside
  2. Even: go outside
Sampling in a dual pixel

- We can write

\[ f(x, y) = a(x - x_S)(y - y_S) + \lambda_S \quad (a = u_{00} + u_{11} - u_{01} - u_{10}). \]

When \( a \neq 0 \), level lines are equilateral hyperbola branches.
- Maximum curvature point is at \( |x - x_S| = |y - y_S| \), we add it if inside the dual pixel.
- We can sample the hyperbola branch by writing \( y(x) \) (if \( |y'| \leq 1 \)) or \( x(y) \) (if \( |x'| \leq 1 \)) and sampling uniformly along \( x \) or \( y \).
Following the level line

- Between two adjacent pixels, linear variation, thus a single point at level $\lambda$
- We follow the level line from dual pixel to dual pixel.
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Particular cases

- Saddle point inside the dual pixel

\[ \lambda < \lambda < \lambda < \lambda \]

\[ \lambda > \lambda > \lambda > \lambda \]

Initial image level
Particular cases

- Saddle point inside the dual pixel

\[
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\]
Particular cases

- Saddle point inside the dual pixel

- Initial image level
Defect at saddle point

$B$ goes through two saddle points: by a local decision, it is not possible to treat both according to formal definition.
Reconstruction from the tree

Walk of the tree in preorder, and for each node representing a level line do (noting $\lambda$ its level):

Two steps: intersection with horizontal lines, then filling

Step 1: similar to step 2 of extraction, intersection with each horizontal line and ordering inside each horizontal line, $(x_1^i \ldots x_{2N_i}^i)$

Step 2: for all $i$ and all $k = 0, \ldots, N_i - 1$ do

for all $j \in \mathbb{N} \cap [x_{2k+1}^i, x_{2k+2}^i]$:

\[
\text{pixel } (j, i) \leftarrow \lambda
\]
Geometric scheme curvature motion

Proceed in 3 steps:

1. Extract tree of bilinear level lines
2. Let each level line evolve by curve shortening or affine shortening
3. Reconstruct image from the tree of (shortened) level lines

\[
\begin{aligned}
\mathbf{u}_0(\cdot) &\quad \xrightarrow{\text{level lines extraction}} &\quad \sum_{\lambda,i}^{0} \lambda,i \\
MCM/ACM &\quad \Downarrow &\quad CS/AS \\
\mathbf{u}(\cdot, t) &\quad \xleftarrow{\text{reconstruction}} &\quad \sum_{\lambda,i}^{t} \lambda,i
\end{aligned}
\]
Curvature map

- Each level line is represented by a polygon.
- For each vertex $P_i$ of the polygon, compute curvature as the inverse of the radius of the circumscribed circle of $P_{i-1}P_iP_{i+1}$.
- Inside a pixel, average the curvatures of level lines passing through.
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Fattening effect

(a) Original and interpolation
(b) Level lines shortening
(c) Finite difference scheme
(d) Same with stack filter
JPEG artifacts

- Original image
- Original level lines
- Curvature map
- LLS image
- Shortened LL
- After filtering
Accurate mean curvature evolution

Left: detail of painting. Center: diff with LLS. Right: diff with stack FDS
Junctions
Curvature microscope
Topographic map of a digital elevation model
Textures
The best scheme for motion by curvature of images proceeds in 3 steps:

1. Decompose the image into its level lines
2. Smooth each line independently
3. Reconstruct from shortened level lines

Such a scheme satisfies the covariance requirements (geometric and contrast)

This is necessary to estimate reliably the curvature and avoid pixel artifacts

This can be used as a microscope on the image: look at fine structures at any scale
Further reading

- On inclusion tree and applications: Vicent Caselles and PM, *Geometric description of images as topographic maps* (Springer Lecture Notes in Mathematics) 2010
- FDSs in IPOL:
  - http://www.ipol.im/pub/algo/cm_fds_mcm_amss/
  - http://dev.ipol.im/~monasse/ipol_demo/cmmm_image_curvature_microscope/