Deterministic Fluid Models for Internet Congestion Control

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An Optimization Based Framework (Kelly et al.)

- n users, single resource with capacity nC
- x_r: Transmission rate of user r
- $U_r(\cdot)$: Utility function of user r

  $\rightarrow$ Concave increasing function

  Assume $U_r(x) = \log(x)$ (Proportionally Fair Controller)

SYSTEM PROBLEM

$$\text{Maximize} \sum_{r=1}^{n} \frac{\Delta_r}{\beta} \log(x_r)$$

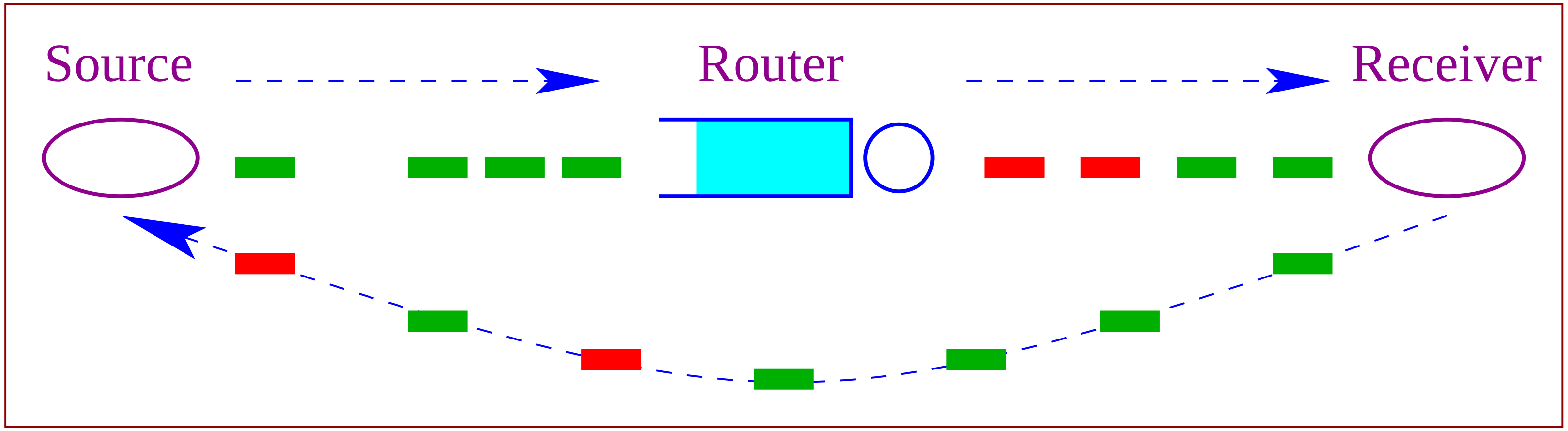
subject to

$$\sum_r x_r \leq nC$$

$$x_r \geq 0 \ \ \forall r = 1, 2, \ldots, n$$

Is there a decentralized means of achieving this allocation?

A Decentralized Solution



- Router reacts to the aggregate flow passing through it
- Marks packets during congestion

  $\rightarrow$ Explicit Congestion notification (ECN)

- Users adapt their transmission rate
- Round trip delay d

A Decentralized Solution: The Router Behavior

- Router unaware of individual flow rates

  $\rightarrow$ Reacts to the aggregate flow passing through it

- Marks packets during congestion

- Has a *Marking Function* $p(x)$

  $\rightarrow$ Determines the fraction of flow to be marked

  $\rightarrow$ $0 \leq p(x) \leq 1$, "smooth", increasing function

  $\rightarrow$ Example: Mark the fraction of flow which exceeds a threshold $n\tilde{c}$

$$p(x) \quad = \quad \frac{(x - n\tilde{c})^+}{x}$$

A Decentralized Solution: User Behavior

- User r receives marks at rate $m_r(t)$

$$m_r(t) \quad = \quad x_r(t - \frac{d}{2})\, p\left(\sum_{j=1}^{n} x_j(t - \frac{d}{2})\right)$$

- User r adapts rate $x_r(t)$ at time t according to

$$\dot{x}_r(t) \quad = \quad \Delta_r - \beta m_r(t - \frac{d}{2})$$

- Additive Increase Multiplicative Decrease (AIMD) algorithm
- Increases rate linearly in time if no packets are marked
- When a mark is received, multiplicatively decrease rate
- Without delays, the above system converges and the equilibrium rates solve SYSTEM PROBLEM

Issues in Networks

(i) Short flows which **do not** react to congestion control

$\rightarrow$ example: web traffic

(ii) Probabilistic marking functions

$\rightarrow$ May decide to mark flows with '1' or '0' with probability $p(x)$

(iii) Router measures rates by averaging over a small time interval

$\rightarrow$ Implication: System operates in "discrete" time

Question: Is a delay-differential equation the appropriate model?

Key Idea

- Dynamics of a single flow in *discrete-time*

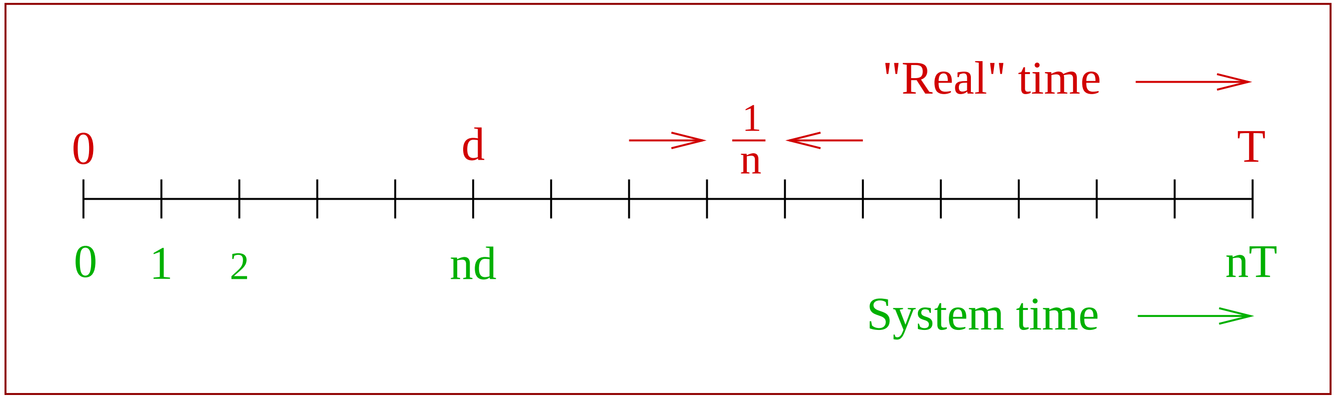
  $\rightarrow$ time-step = processing time of a fixed size packet

$$\boxed{x_{i+1} = (x_i + \Delta - \beta x_{i-d}\, p(x_{i-d} + \hat{e}_{i-d}))^+}$$

- $\hat{e}_i$: Randomness due to web mice, probabilistic marking

- Many such flows $\Longrightarrow$ "average" rate will "appear" as a differential equation

Modeling a Large System



- n flows, each with round trip delay d
- Capacity proportional to the number of users (nC)

  $\rightarrow$ to have reasonable bandwidth per flow
- Amount of time required to process a packet goes down as $\frac{1}{n}$
- Unit of Time: Amount of time required to process a packet
- Time measured on a “finer” scale as n increases

... (contd.) Large System

- $y_i^k, \{k = 1, \ldots, n\}, \{i = 0, \ldots, nT\}$: transmission rate of user k at time i
- Noise process: $(a + \tilde{e}_i^k)$, $\{k = 1, 2, \ldots, n\}$

  $\rightarrow \text{Mean}(\tilde{e}_i^k) = 0$

- Evolution of user rate

$$y_{i+1}^k = \left[y_i^k + \frac{\Delta}{n} - \frac{\beta}{n} y_{i-nd}^k p \left(\frac{1}{n} \sum_{j=1}^{n} (y_{i-nd}^j + \tilde{e}_{i-nd}^j + a) \right) \right]^+$$

- Round trip delay (in time-steps) scaled by n
- Gains (Δ, β) scaled by $\frac{1}{n}$

$$x_i^n = \frac{1}{n} \sum_{k=1}^{n} y_i^k \quad \text{average rate at time } i$$

$$e_i^n = \frac{1}{n} \sum_{k=1}^{n} \tilde{e}_i^k \quad \text{average (centered) noise at time } i$$

Evolution of Average Rate

$$x^n_{i+1} \quad = \quad x^n_i + \frac{\Delta}{n} - \frac{\beta}{n} x^n_{i-nd}\; p\left(x^n_{i-nd} + a + e^n_{i-nd}\right)$$

- We have neglected non-negativity constraints

  $\rightarrow$ We can show that this constraint is asymptotically redundant

- How does it look in "Real" time?

- For $t \in [0,T]$ and $nt \in \{0,1,\ldots,nT\}$,

$$x^n(t) = x^n_{nt}, \qquad e^n(t) = e^n_{nt},$$

- Straight line approximation to interpolate between times $t = \frac{i}{n}$

- Average rate described by

$$\dot{x}^n(t) \quad = \quad \Delta - \beta x^n\left(\frac{\lfloor n(t-d)\rfloor}{n}\right)\left(x^n\left(\frac{\lfloor n(t-d)\rfloor}{n}\right) + a + e^n\left(\frac{\lfloor n(t-d)\rfloor}{n}\right)\right)$$

...(contd.) Evolution of Average Rate

- Assumptions on “noise” imply $\sup_{t\in[0,T]} |e^n(t)| \to 0$ a.s.
- Define $x(t)$: $\dot{x}(t) = \Delta - \beta x(t-d)p(x(t-d)+a)$

Claim: Almost surely, $\sup_{t\in[0,T]} |x^n(t) - x(t)| \to 0$ as $n \to \infty$

- Implication: Deterministic delay-differential equation model valid in the Many-Flows regime

TCP like Controllers

Jointly with Supratim Deb and Prof. R. Srikant

What about TCP like Controllers?

- Each user has utility function $U(x) = \frac{-1}{x}$

  $\rightarrow$ n users with user rates $\{y^{j,n}(.), j = 1, \ldots, n\}$

- User flow adapts according to

$$\dot{y}^{j,n}(t) \quad = \quad \kappa(w - y^{j,n}(t) y^{j,n}(t-d) p(x^n(t-d) + a + e^n(t-d)))$$

- Candidate limit system described by

$$\dot{x}(t) \quad = \quad \kappa(w - x(t) x(t-d) p(x(t-d) + a))$$

Is the following statement true ? As $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{j=1}^{n} y^{j,n}(t) \quad \rightarrow \quad x(t)$$

- In case of $\log(.)$ utility function, the answer is *Yes*
- For TCP like controllers, the answer is *No*, but ...

Main Result

Deterministic system stable $\Longrightarrow$ stochastic system "stable"

Reason: $\left|\frac{1}{n}\sum_{j=1}^{n} y^{j,n}(t) - x(t)\right|$ becomes small for large n, t if deterministic system is globally asymptotically stable (κd is small)

Key Idea: Flow Coupling

- Let us define $r^n_{ij}(t) = y^{i,n}(t) - y^{j,n}(t)$
- We can show that

$$r^n_{ij}(t) \quad = \quad -\kappa p\left(x^n(t-d)+a+e^n(t-d)\right)\left[y^{i,n}(t)r^n_{ij}(t-d)+y^{j,n}(t-d)r^n_{ij}(t)\right]$$

- Observe that

$$\frac{1}{n}\sum_{j=1}^{n} y^{j,n}(t)y^{j,n}(t-d) \quad = \quad x(t)x(t-d)+\{r^n_{ij}(.)\} \text{ terms}$$

- $\left|\frac{1}{n}\sum_{j=1}^{n} y^{j,n}(t)-x(t)\right|$ small **if we have** $\sup_{i,j} r^n_{ij}(t) \approx 0$

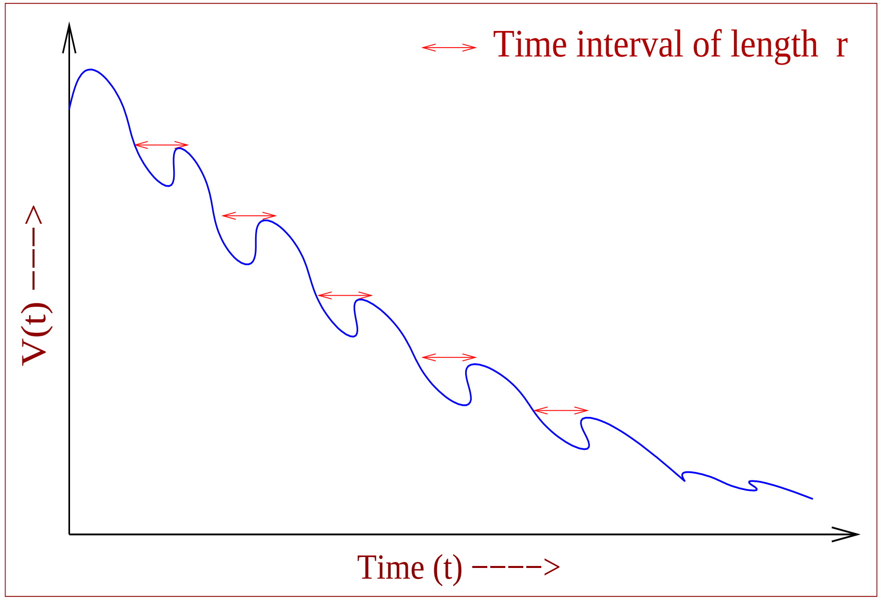
Why does $r^n_{ij}(.) \to 0$?

Razumikhin's Theorem

- Suppose we have a delay differential equation given by

$$\dot{z}(t) \quad = \quad a(t)z(t) + b(t)z(t-d)$$

- Consider a convex function $V(z(t))$ which behaves as shown



- $V(t)$ decreases whenever it reaches maximum over a past interval of length r

...(contd.) Razumikhin's Theorem

- The functional

$$W(z_t) = \sup_{t-r \le s \le t} V(z(s))$$

decreases with time and thus provides a Lyapunov functional for the system

- We choose $V(t) = (z(t) - z^*)^2$ and $r = 2d$.
- Exponential stability follows by showing that $(z(s) - z^*)^2$ decreases at least by a constant factor in some finite time

Result: If $a(t)$ and $b(t)$ are small, then $z(t) \to 0$ exponentially fast.

Back to TCP like Controllers

- We have

$$r^n_{ij}(t) \quad = \quad -\kappa p\left(x^n(t-d)+a+e^n(t-d)\right)\left[y^{i,n}(t)r^n_{ij}(t-d)+y^{j,n}(t-d)r^n_{ij}(t)\right]$$

- $a(t) = -\kappa p(.)y^{j,n}(t-d)$ and $b(t) = -\kappa p(.)y^{j,n}(t)$

  $\rightarrow$ Need to show these are small

  $\rightarrow$ To do so, we need to show $y^{j,n}(.)$ uniformly bounded in time

- (Shakkottai, Srikant and Meyn) We also know that $\exists M > 0$ such that

$$\frac{1}{n}\sum_{j=1}^{n} y^{j,n}(t) \quad < \quad M$$

- We jointly develop a bound on $y^{j,n}(.)$ as well as show $r^n_{ij}(.) \rightarrow 0$

  $\rightarrow$ (Deb and Srikant) Use Razumikhin's Theorem along with some bounding techniques

Discussion and Summary of Main Results

- Deterministic delay-differential models valid in a many-flows regime
- Models include proportionally fair controllers as well as TCP like controllers