Deterministic Fluid Models for Internet Congestion Control

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An Optimization Based Framework (Kelly et al.)

- $n$ users, single resource with capacity $nC$
- $x_r$: Transmission rate of user $r$
- $U_r(\cdot)$: Utility function of user $r$
  - Concave increasing function
  - Assume $U_r(x) = \log(x)$ (Proportionally Fair Controller)

SYSTEM PROBLEM

Maximize $\sum_{r=1}^{n} \frac{\Delta_r}{\beta} \log(x_r)$

subject to

$$\sum_{r} x_r \leq nC$$

$x_r \geq 0 \ \forall r = 1, 2, \ldots, n$

Is there a decentralized means of achieving this allocation?
A Decentralized Solution

- Router reacts to the aggregate flow passing through it
- Marks packets during congestion → Explicit Congestion notification (ECN)
- Users adapt their transmission rate
- Round trip delay $d$
A Decentralized Solution: The Router Behavior

- Router unaware of individual flow rates
  - Reacts to the aggregate flow passing through it
- Marks packets during congestion
- Has a **Marking Function** $p(x)$
  - Determines the fraction of flow to be marked
  - $0 \leq p(x) \leq 1$, “smooth”, increasing function
  - Example: Mark the fraction of flow which exceeds a threshold $n\tilde{c}$

$$p(x) = \frac{(x - n\tilde{c})^+}{x}$$
A Decentralized Solution: User Behavior

- User $r$ receives marks at rate $m_r(t)$

$$m_r(t) = x_r(t - \frac{d}{2}) p \left( \sum_{j=1}^{n} x_j(t - \frac{d}{2}) \right)$$

- User $r$ adapts rate $x_r(t)$ at time $t$ according to

$$\dot{x}_r(t) = \Delta_r - \beta m_r(t - \frac{d}{2})$$

- Additive Increase Multiplicative Decrease (AIMD) algorithm
  - Increases rate linearly in time if no packets are marked
  - When a mark is received, multiplicatively decrease rate

Without delays, the above system converges and the equilibrium rates solve

SYSTEM PROBLEM
Issues in Networks

(i) Short flows which do not react to congestion control
   → example: web traffic

(ii) Probabilistic marking functions
   → May decide to mark flows with ‘1’ or ‘0’ with probability $p(x)$

(iii) Router measures rates by averaging over a small time interval
   → Implication: System operates in “discrete” time

Question: Is a delay-differential equation the appropriate model?
Key Idea

- Dynamics of a single flow in *discrete-time*
  \[ x_{i+1} = (x_i + \Delta - \beta x_{i-d} p(x_{i-d} + \hat{e}_{i-d}))^+ \]

- \( \hat{e}_i \): Randomness due to web mice, probabilistic marking

- Many such flows \( \Rightarrow \) “average” rate will “appear” as a differential equation
Modeling a Large System

- $n$ flows, each with round trip delay $d$
- Capacity proportional to the number of users ($nC$) to have reasonable bandwidth per flow
- Amount of time required to process a packet goes down as $\frac{1}{n}$
- **Unit of Time**: Amount of time required to process a packet
- Time measured on a “finer” scale as $n$ increases
Large System

- \( y_i^k, \{k = 1, \ldots, n\}, \{i = 0, \ldots, nT\} \): transmission rate of user \( k \) at time \( i \)

- Noise process: \( (a + \tilde{e}_i^k), \{k = 1, 2, \ldots, n\} \)
  \( \rightarrow \text{Mean}(\tilde{e}_i^k) = 0 \)

- Evolution of user rate

\[
y_{i+1}^k = \left[ y_i^k + \frac{\Delta}{n} - \frac{\beta}{n} y_{i-nd}^k \left( \frac{1}{n} \sum_{j=1}^{n} (y_j^i + \tilde{e}_j^i + a) \right) \right]^+
\]

- Round trip delay (in time-steps) scaled by \( n \)

- Gains (\( \Delta, \beta \)) scaled by \( \frac{1}{n} \)

\[
x^n_i = \frac{1}{n} \sum_{k=1}^{n} y_i^k \quad \text{average rate at time } i
\]

\[
e^n_i = \frac{1}{n} \sum_{k=1}^{n} \tilde{e}_i^k \quad \text{average (centered) noise at time } i
\]
Evolution of Average Rate

\[ x_{i+1}^n = x_i^n + \frac{\Delta}{n} - \frac{\beta}{n} x_{i-nd}^n \ p (x_{i-nd}^n + a + e_{i-nd}^n) \]

- We have neglected non-negativity constraints
  → We can show that this constraint is asymptotically redundant

- How does it look in “Real” time?

  For \( t \in [0, T] \) and \( nt \in \{0, 1, \ldots, nT\} \),

  \[ x^n(t) = x_{nt}^n, \quad e^n(t) = e_{nt}^n, \]

- Straight line approximation to interpolate between times \( t = \frac{i}{n} \)

- Average rate described by

  \[ \dot{x}^n(t) = \Delta - \beta x^n(\frac{n(t-d)}{n}) \left( x^n(\frac{n(t-d)}{n}) + a + e^n(\frac{n(t-d)}{n}) \right) \]
Assumptions on “noise” imply \( \sup_{t \in [0, T]} |e^n(t)| \to 0 \) a.s.

Define \( x(t) \):

\[
\dot{x}(t) = \Delta - \beta x(t - d) p (x(t - d) + a)
\]

Claim: Almost surely, \( \sup_{t \in [0, T]} |x^n(t) - x(t)| \to 0 \) as \( n \to \infty \)

Implication: Deterministic delay-differential equation model valid in the Many-Flows regime
TCP like Controllers

Jointly with Supratim Deb and Prof. R. Srikant
What about TCP like Controllers?

- Each user has utility function $U(x) = \frac{1}{x}$
  $\rightarrow$ $n$ users with user rates \{\(y^{j,n}(.)\), $j = 1, \ldots, n\}$

- User flow adapts according to
  \[
  \dot{y}^{j,n}(t) = \kappa(w - y^{j,n}(t))y^{j,n}(t-d)p(x^{n}(t-d) + a + e^{n}(t-d))
  \]

- Candidate limit system described by
  \[
  \dot{x}(t) = \kappa(w - x(t)x(t-d)p(x(t-d) + a))
  \]

Is the following statement true? As $n \rightarrow \infty$,
\[
\frac{1}{n} \sum_{j=1}^{n} y^{j,n}(t) \rightarrow x(t)
\]

- In case of log(.) utility function, the answer is \textbf{Yes}

- For TCP like controllers, the answer is \textbf{No}, but ...
Main Result

Deterministic system stable $\implies$ stochastic system “stable”

**Reason:** $\frac{1}{n} \sum_{j=1}^{n} y_{j,n}(t) - x(t)$ becomes small for large $n, t$ if deterministic system is globally asymptotically stable ($\kappa d$ is small)
Key Idea: Flow Coupling

- Let us define $r_{ij}^n(t) = y_{i,n}^j(t) - y_{j,n}^i(t)$

- We can show that

$$r_{ij}^n(t) = -\kappa p \left( x^n(t-d) + a + e^n(t-d) \right) \left[ y_{i,n}^j(t) r_{ij}^n(t-d) + y_{j,n}^i(t-d) r_{ij}^n(t) \right]$$

- Observe that

$$\frac{1}{n} \sum_{j=1}^{n} y_{j,n}^i(t) y_{j,n}^j(t-d) = x(t)x(t-d) + \{r_{ij}^n(.)\} \text{ terms}$$

- $\left| \frac{1}{n} \sum_{j=1}^{n} y_{j,n}^i(t) - x(t) \right|$ small if we have $\sup_{i,j} r_{ij}^n(t) \approx 0$

Why does $r_{ij}^n(.) \to 0$?
Razumikhin’s Theorem

- Suppose we have a delay differential equation given by
  \[ \dot{z}(t) = a(t)z(t) + b(t)z(t - d) \]

- Consider a convex function \( V(z(t)) \) which behaves as shown

- \( V(t) \) decreases whenever it reaches maximum over a past interval of length \( r \)
...(contd.) Razumikhin’s Theorem

- The functional
  \[ W(z_t) = \sup_{t-r \leq s \leq t} V(z(s)) \]
  decreases with time and thus provides a Lyapunov functional for the system.

- We choose \( V(t) = (z(t) - z^*)^2 \) and \( r = 2d \).

- Exponential stability follows by showing that \( (z(s) - z^*)^2 \) decreases at least by a constant factor in some finite time.

**Result:** If \( a(t) \) and \( b(t) \) are small, then \( z(t) \to 0 \) exponentially fast.
Back to TCP like Controllers

- We have
  \[ r_{ij}^n(t) = -\kappa p(x^n(t-d) + a + e^n(t-d)) \left[ y_{i,n}^j(t) r_{ij}^n(t-d) + y_{j,n}^i(t-d) r_{ij}^n(t) \right] \]

- \( a(t) = -\kappa p(.) y_{j,n}^j(t-d) \) and \( b(t) = -\kappa p(.) y_{j,n}^j(t) \)
  
  \( \rightarrow \) Need to show these are small

  \( \rightarrow \) To do so, we need to show \( y_{j,n}(.) \) uniformly bounded in time

- (Shakkottai, Srikant and Meyn) We also know that \( \exists M > 0 \) such that
  \[ \frac{1}{n} \sum_{j=1}^{n} y_{j,n}^j(t) < M \]

- We jointly develop a bound on \( y_{j,n}(.) \) as well as show \( r_{ij}^n(.) \to 0 \)
  
  \( \rightarrow \) (Deb and Srikant) Use Razumikhin’s Theorem along with some bounding techniques
• Deterministic delay-differential models valid in a many-flows regime
• Models include proportionally fair controllers as well as TCP like controllers