A Network Information Theory for Wireless Communications

Liang-Liang Xie and P. R. Kumar

Dept. of Electrical and Computer Engineering, and
Coordinated Science Lab
University of Illinois, Urbana-Champaign

Phone 217-333-7476
Email prkumar@uiuc.edu
Web http://black.csl.uiuc.edu/~prkumar

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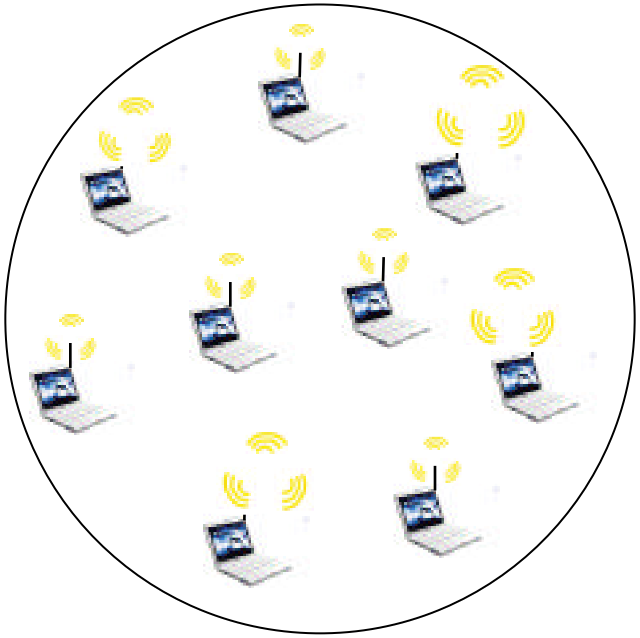
Wireless networks

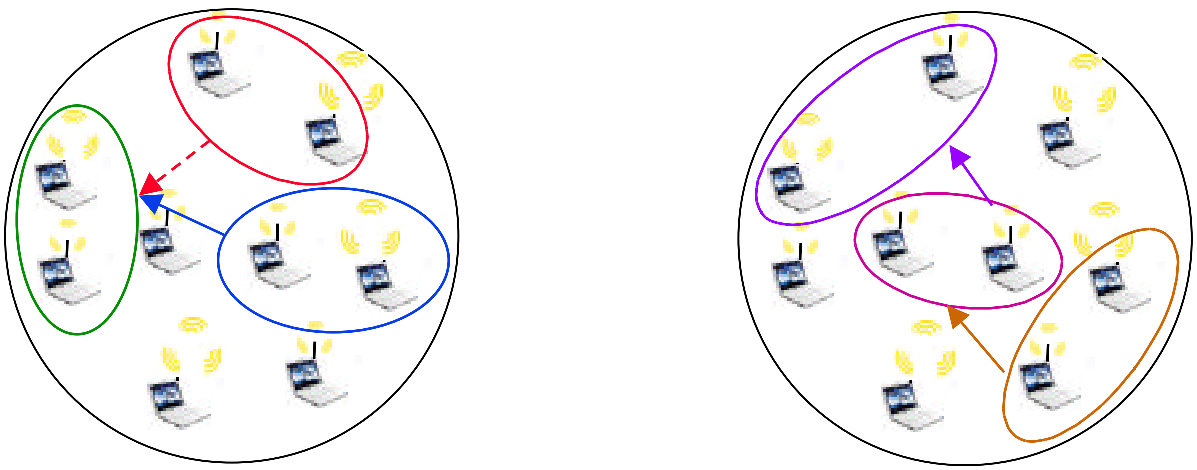
- Communication networks formed by nodes with radios



  - There is no *a priori* notion of “links”
  - All nodes simply radiate energy

How should nodes cooperate?

- Nodes can cooperate in complex ways



Nodes in Group A can help cancel the interference of nodes in Group B at nodes in Group C **while** Nodes in Group D coherently transmit to relay packets from Group E to Group F **while ….. etc**

  - Very complicated feedback strategies are possible
    - » Even notions such as "coherent transmission," "interference cancellation," "relaying," etc., may be too simplistic
    - » The problem has all the complexities of team theory, partially observed systems,etc

Two fundamental questions

- How should nodes cooperate in maximizing information transfer in a wireless network?
  - The strategy space is infinite dimensional
  - If Information Theory can tell us what the basic strategy should be then we can develop *protocols* to realize the strategy

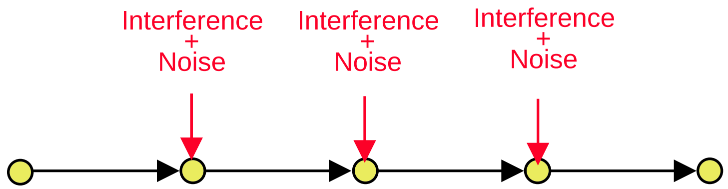
- How much information can be transported in a wireless network?
  - What are the fundamental limits to information theory?
  - How far is current technology from the optimal?
  - When can we quit trying to do better?
    - » E.g.. If "Telephone modems are near the Shannon capacity" then we can stop trying to build better telephone modems

Key Main Results

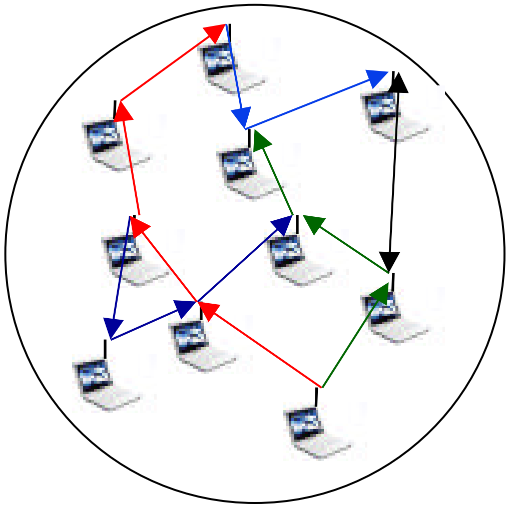
- If there is absorption in the medium
  - Transport capacity grows like $\Theta(n)$ where n = number of nodes
  - Multi-hop operation is optimal
- If there is no absorption, and attenuation is very small
  - Transport capacity can grow like $\Theta(n^{\theta})$ for $\theta > 1$
  - Coherent multi-stage relaying with interference cancellation can be optimal
- Along the way
  - Total power used by a network bounds the transport capacity
    - » or not
  - A feasible rate for a Gaussian multiple relay channel
  - A max-flow min-cut bound

Current proposal for ad hoc networks

- Multi-hop transport
  - Packets are relayed from node to node
  - A packet is fully decoded at each hop
  - All interference from all other nodes is simply treated as noise

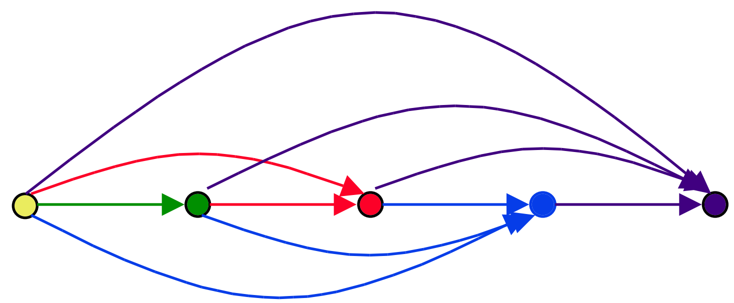


- This choice for the mode of operation gives rise to
  - Routing problem
  - Media access control problem
  - Power control problem



Another strategy

- Coherent multi-stage relaying with interference cancellation (COMSRIC)



- All upstream nodes coherently cooperate to send a packet to the next node
- A node cancels all the interference caused by all transmissions to its downstream nodes

The Transport Capacity: Definition

- Source-Destination pairs
  - $(s_1, d_1), (s_2, d_2), (s_3, d_3), \ldots, (s_l, d_l)$
- Distances
  - $\rho_1, \rho_2, \rho_3, \ldots, \rho_l$ distances between the sources and destinations
- Feasible Rates
  - $(R_1, R_2, R_3, \ldots, R_l)$ feasible rates for these source-destination pairs
- Distance-weighted sum of rates
  - $\Sigma_i R_i \rho_i$
- <u>Transport Capacity</u>
  - $C_T = \sup \Sigma_i R_i \rho_i$
    - » Supremum is taken over all feasible rates $(R_1, R_2, R_3, \ldots, R_l)$

The Transport Capacity

- $C_T = \sup \Sigma_i R_i \rho_i$

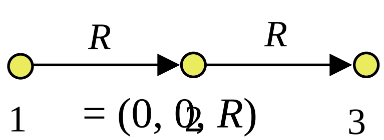
  – Measured in bit-meters/second or bit-meters/slot
  – Analogous to man-miles/year considered by airlines

  – Upper bound to what network can carry
    » irrespective of which sources, destinations and their rates

  – Satisfies a scaling law
    » Conservation law which restricts what network can provide
    » Irrespective of whether it is of prima facie interest

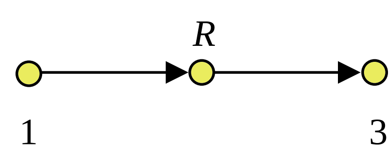
  – However it is also of natural interest

  – Allows us to compare apples with apples



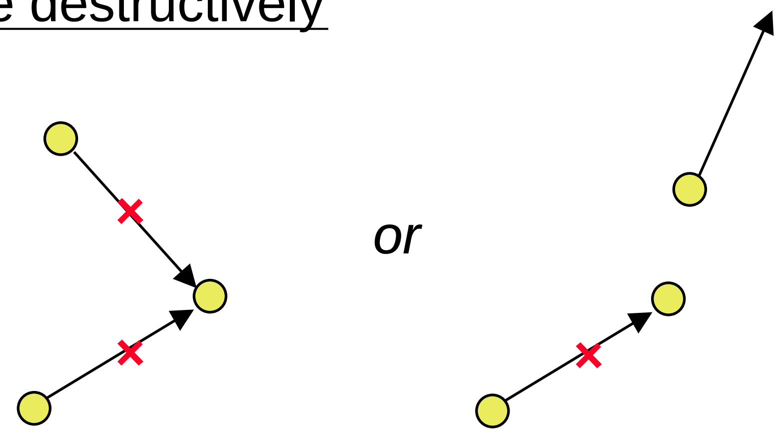
$= (R, R, 0)$

or

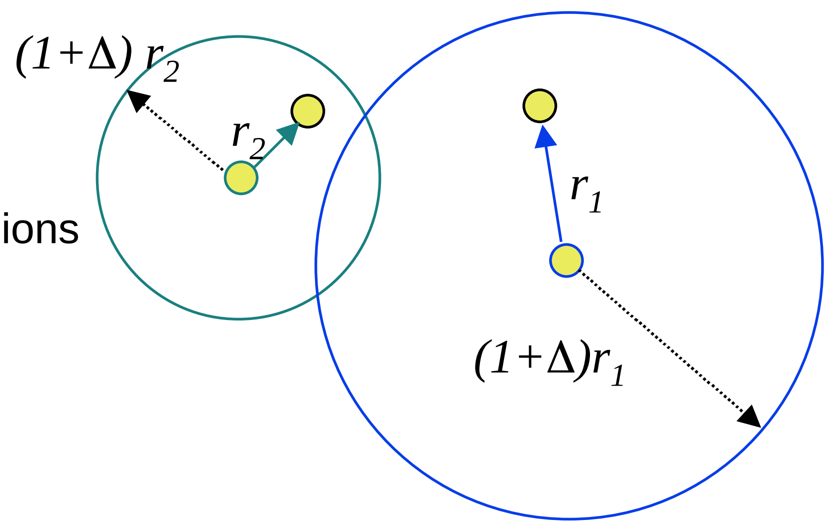


Models where packets "collide"

- In some technologies - Packets collide destructively
  - Example: If all interference is regarded as noise

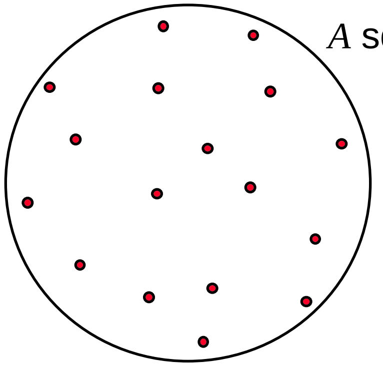


- Gupta-Kumar model
  - Reception is successful if
    - » Receiver not in vicinity of two transmissions
    - » Or SINR > β
    - » Or Rate depends on SINR



Scaling laws under "collision" model

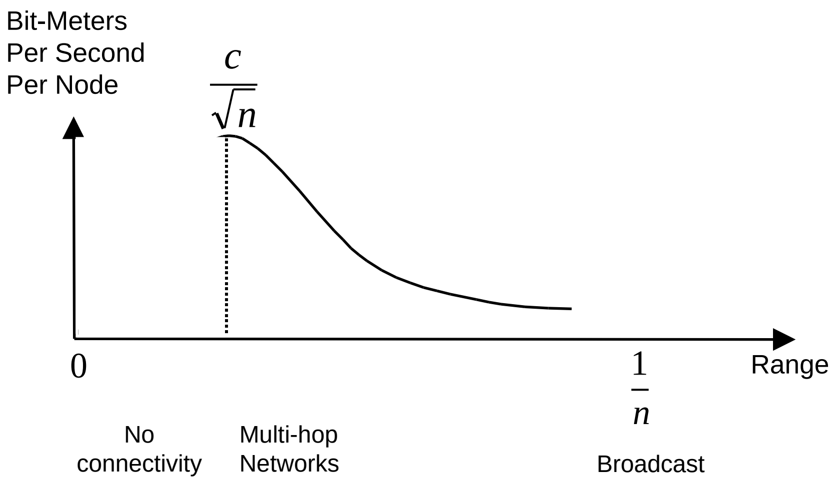
- Theorem (Gupta-Kumar 2000)
  - Disk of area A square meters
  - n nodes
  - Each can transmit at W bits/sec

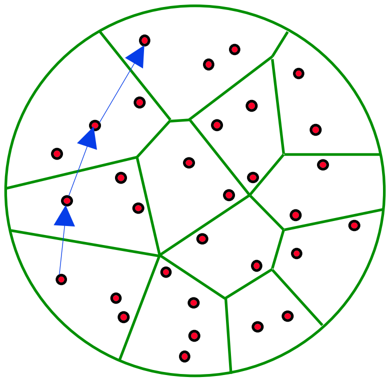


- Best Case: Network can transport $\Theta\left(W\sqrt{An}\right)$ bit-meters/second
- Random case: Each node can obtain a throughput $\Theta\left(\frac{1}{\sqrt{n\log n}}\right)$ bits/second
- Square root law
  - Transport capacity doesn't increase linearly, but only like square-root
  - Each node gets $\frac{c}{\sqrt{n}}$ bit-meters/second

Optimal operation under "collision" model

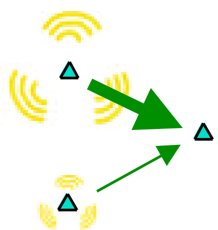
- Optimal operation
  - Multi-hop is optimal
- Optimal multi-hop architecture
  - Group nodes into cells of size log n
  - Common power level for all nodes is nearly optimal
  - Power should be as small as possible subject to network connectivity
  - Just enough power to reach all points in neighboring cell
  - Can route packets along nearly straight line path from cell to cell





But interference is not interference

- Interference is information!

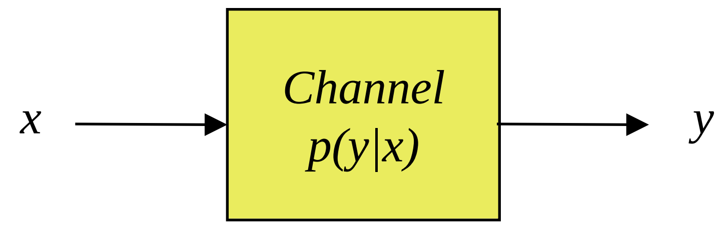


  - Receiver can first decode loud signal perfectly
  - Then subtract the loud signal
  - Then decode the soft signal perfectly
  - So excessive interference can be very good
  - Packets do not destructively collide

- So we need an information theory for networks to determine
  - How to operate wireless networks
  - How much information wireless networks can transport

Shannon's Information Theory

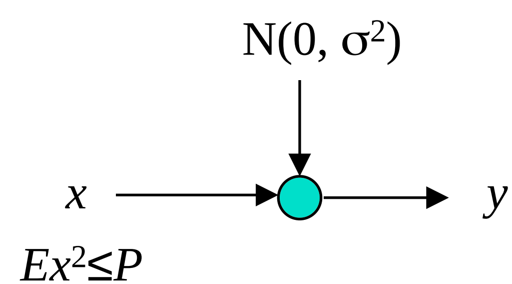
- <u>Shannon's Capacity Theorem</u>
  - Channel Model $p(y|x)$
    - » Discrete Memoryless Channel



  - Capacity = $\text{Max}_{p(x)} I(X;Y)$ bits/channel use

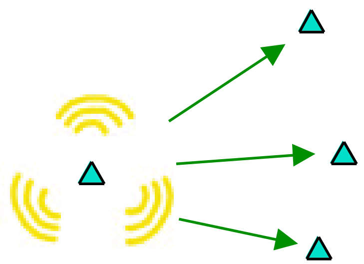
$$I(X;Y) = \sum_{x,y} p(x,y) \log\left(\frac{p(X,Y)}{p(X)p(Y)} \right)$$

- <u>Additive White Gaussian Noise (AWGN) Channel</u>
  - Capacity = $S\left(\frac{P}{\sigma^2}\right)$ where $S(z) = \frac{1}{2}\log(1+z)$

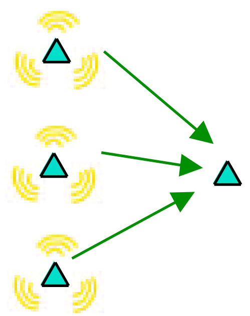


Network information theory: The triumphs

- Gaussian broadcast channel

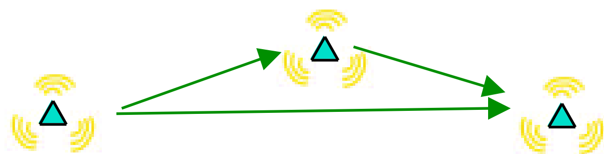


- Gaussian multiple access channel

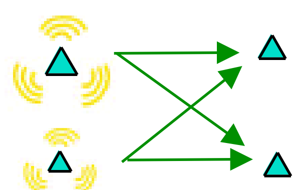


Network information theory: The unknowns

- The simplest relay channel



- The simplest interference channel

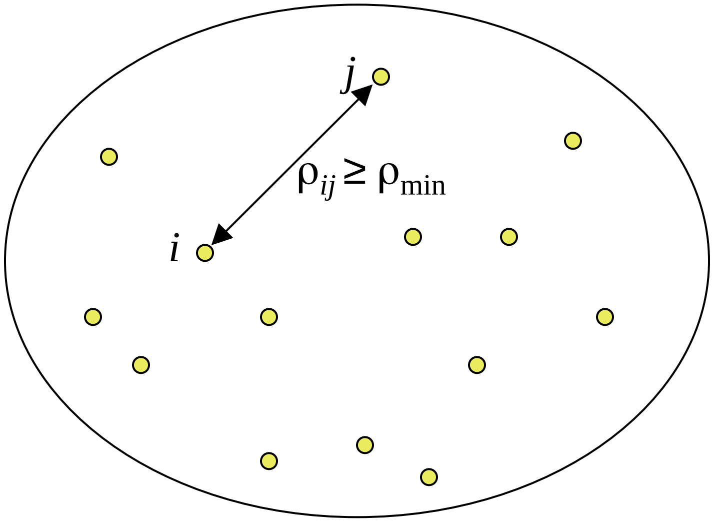


- Systems being built are much more complicated and the possible modes of cooperation can be much more sophisticated
  - How to analyze?
  - Need a large scale information theory

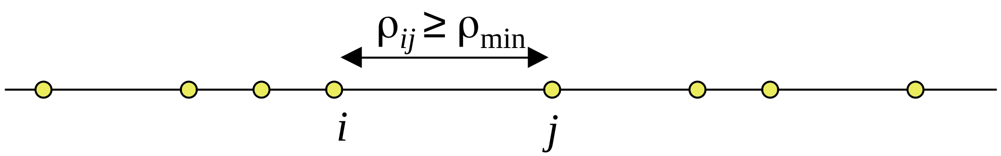
Model of system: A planar network

- n nodes in a plane
- ρ_{ij} = distance between nodes i and j
- Minimum distance between nodes

$$\rho_{ij} \geq \rho_{min} > 0$$



- Or a linear network



Model of signal attenuation

- Signal path loss attenuation with distance:

$$\text{Attenuation over a distance } \rho = \frac{e^{-\gamma\rho}}{\rho^{\delta}}$$

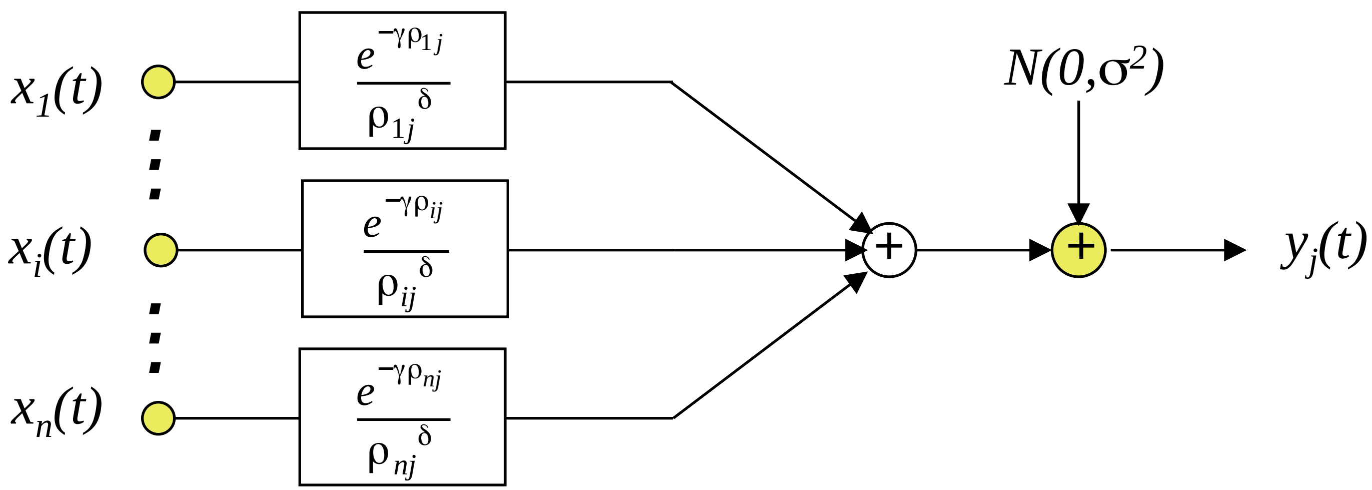
  - ρ = distance between transmitter and receiver
  - $\gamma \geq 0$ is the absorption constant
    - » Loss of $20\gamma \log_{10} e$ db per meter
  - Generally $\gamma > 0$ since the medium is absorptive unless over a vacuum
  - $\delta > 0$ is the path loss exponent

Transmitted and received signals

- $x_i(t)$ = signal transmitted by node i time t
- $y_j(t)$ = signal received by node j at time t

$$y_j(t) = \sum_{i=1}^{n} \frac{e^{-\gamma \rho_{ij}}}{\rho_{ij}^{\delta}} x_i(t) + z_j(t)$$

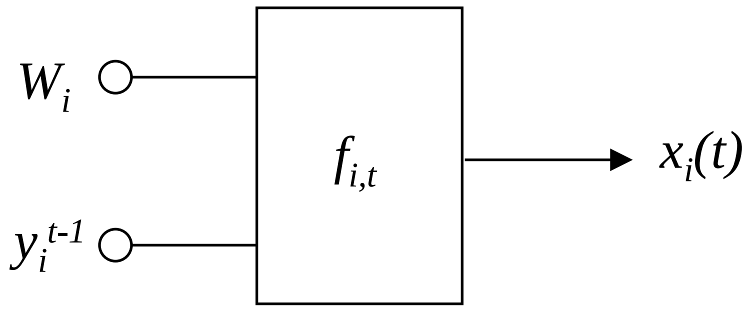
- Additive white Gaussian noise $z_j(t)$ of variance σ^2 at all receivers



Feedback allowed

- W_i = symbol from some alphabet $\{1, 2, 3, \ldots, 2^{TR_{ik}}\}$
  - to be sent by node i to its destination node k in T transmissions

- $x_i(t) = f_{i,t}(y_i^{t-1}, W_i)$
  - can depend on
  - past observations
  - and any symbol(s) W_i that node i wants to send



- Power constraints on $P_i = \frac{1}{T}\sum_{t=1}^{T} x_i^2(t)$
  - Individual power constraint $P_i \le P_{ind}$ for all nodes i
  - or Total power constraint $\sum_{i=1}^{n} P_i \le P_{total}$

Definitions of feasible rate vector

- Source i chooses a symbol W_i from $\{1, 2, 3, \ldots, 2^{TR_i}\}$
- Transmits $x_i(t) = f_{i,t}(y_i^{t-1}, W_i)$ for $t = 1, 2, \ldots, T$
- Destination j receives $y_j(t) = \sum_{k=1}^{n} \frac{e^{-\gamma \rho_{kj}}}{\rho_{kj}^{\delta}} x_k(t) + w_j(t)$
- Destination j uses the decoder $\hat{W}_i = g_j(y_j^T, W_j)$
- Error if $\hat{W}_i \neq W_i$
- $(R_1, R_2, \ldots, R_l)$ is feasible rate vector if there is a sequence of codes wi

$$\underset{W_1, W_2, \ldots, W_l}{Max} \Pr(\hat{W}_i \neq W_i \text{ for some } i \mid W_1, W_2, \ldots, W_l) \to 0 \text{ as } T \to \infty$$

- Transport Capacity $C_T = \sup_{l \text{ and } (R_1, R_2, \ldots, R_l)} \sum_{i=1}^{l} R_i \rho_i$

The Results

When there is absorption or a large path loss

The total power bounds the transport capacity

- Theorem

  – Suppose $\gamma > 0$, there is some absorption,

  or $\delta > 3$, if there is no absorption at all

  – Then for all Planar Networks

$$C_T \le \frac{c_1(\gamma, \delta, \rho_{\min})}{\sigma^2} P_{total}$$

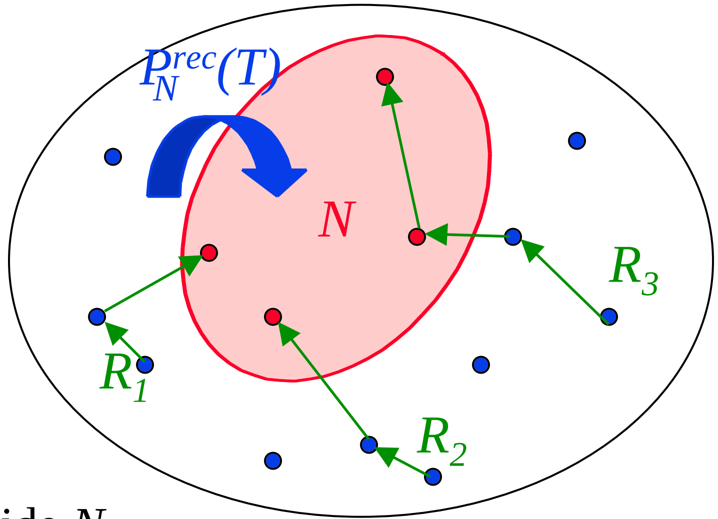
where

$$c_1(\gamma, \delta, \rho_{\min}) = \frac{2^{2\delta+7}}{\gamma^2 \rho_{\min}^{2\delta+1}} \frac{e^{-\gamma\rho_{\min}/2}(2 - e^{-\gamma\rho_{\min}/2})}{(1 - e^{-\gamma\rho_{\min}/2})} \quad \text{if } \gamma > 0$$

$$= \frac{2^{2\delta+5}(3\delta - 8)}{(\delta-2)^2(\delta-3)\rho_{\min}^{2\delta-1}} \quad \text{if } \gamma = 0 \text{ and } \delta > 3$$

Idea behind proof

- A Max-flow Min-cut Lemma



  – N = subset of nodes

  – $P_N^{rec}(T) =$ Power received by nodes in N from outside N

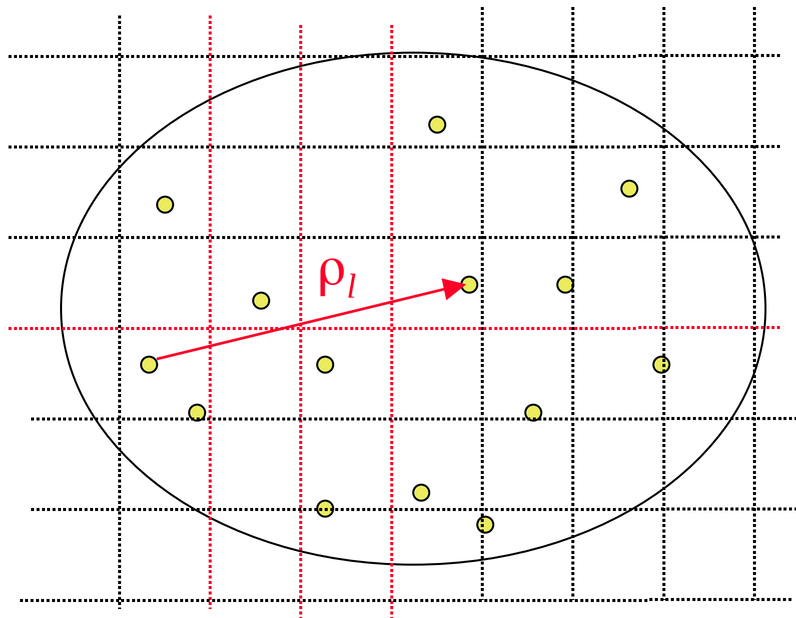
$$= \frac{1}{T}\sum_{t=1}^{T}\sum_{j \notin N} E\left(\sum_{i \in N} \frac{x_i(t)}{\rho_{ij}^{\delta}}\right)^2$$

  – Then

$$\sum_{\{l: d_l \in N \text{ but } s_l \notin N\}} R_l \le \frac{1}{2\sigma^2} \liminf_{T \to \infty} P_N^{rec}(T)$$

To obtain power bound on transport capacity

- Idea of proof
- Consider a number of cuts one meter apart
- Every source-destination pair (s_l, d_l) with source at a distance ρ_l is cut by about ρ_l cuts

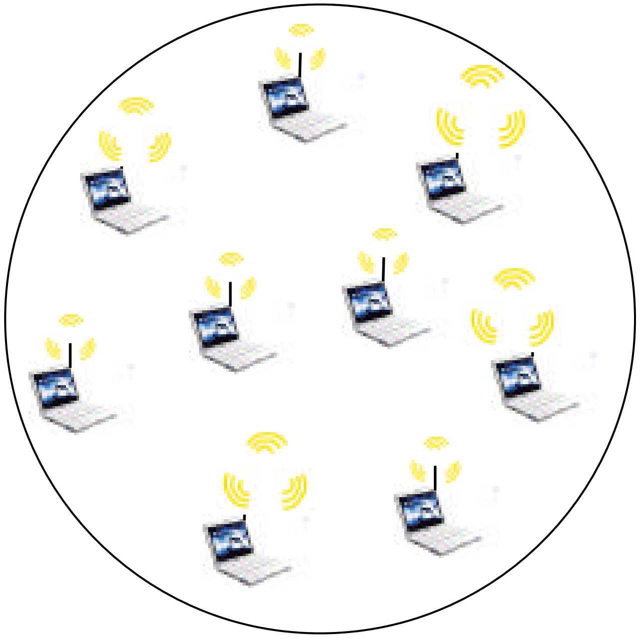


- Thus

$$\sum_l R_l \rho_l \le c \sum_{N_k} \sum_{\{l \text{ is cut by } N_k\}} R_l \le \frac{c}{2\sigma^2} \sum_{N_k} \liminf_{T\to\infty} P_{N_k}^{rec}(T) \le c P_{total}$$

O(n) upper bound on Transport Capacity

- Theorem
  - Suppose $\gamma > 0$, there is some absorption,
  - Or $\delta > 3$, if there is no absorption at all
  - Then for all Planar Networks



$$C_T \leq \frac{c_1(\gamma, \delta, \rho_{\min}) P_{ind}}{\sigma^2} \, n$$

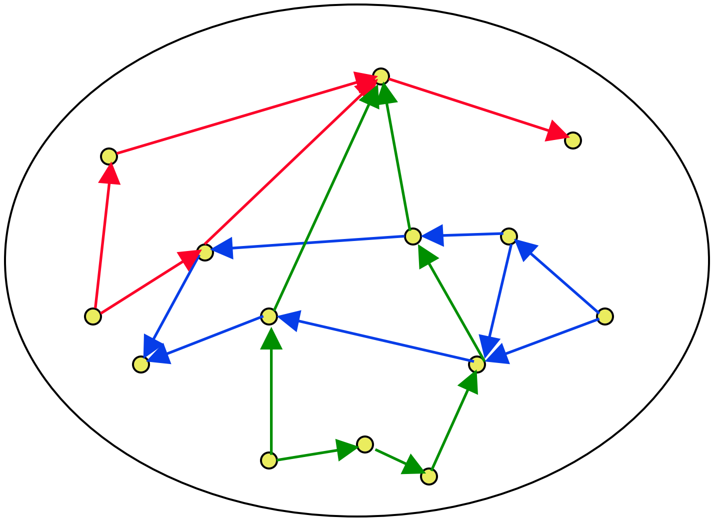
where

$$c_1(\gamma, \delta, \rho_{\min}) = \frac{2^{2\delta+7}}{\gamma^2 \rho_{\min}^{2\delta+1}} \frac{e^{-\gamma\rho_{\min}/2}(2 - e^{-\gamma\rho_{\min}/2})}{(1 - e^{-\gamma\rho_{\min}/2})} \qquad \text{if } \gamma > 0$$

$$= \frac{2^{2\delta+5}(3\delta - 8)}{(\delta-2)^2(\delta-3)\rho_{\min}^{2\delta-1}} \qquad \text{if } \gamma = 0 \text{ and } \delta > 3$$

What can multihop transport achieve?

- <u>Theorem</u>



  - A set of rates $(R_1, R_2, \ldots, R_l)$ can be supported by multi-hop transport if
  - Traffic can be routed, possibly over many paths, such that
  - No node has to relay more than $S\left(\dfrac{e^{-2\gamma\bar{\rho}}P_{ind}/\bar{\rho}^{2\delta}}{c_3(\gamma,\delta,\rho_{\min})P_{ind}+\sigma^2}\right)$
  - where $\bar{\rho}$ is the longest distance of a hop

    and

$$c_3(\gamma,\delta,\rho_{\min}) = \frac{2^{3+2\delta}e^{-\gamma\rho_{\min}}}{\gamma\rho_{\min}^{1+2\delta}} \quad \text{if } \gamma > 0$$

$$= \frac{2^{2+2\delta}}{\rho_{\min}^{2\delta}(\delta-1)} \quad \text{if } \gamma = 0 \text{ and } \delta > 1$$

Multihop transport can achieve $\Theta(n)$

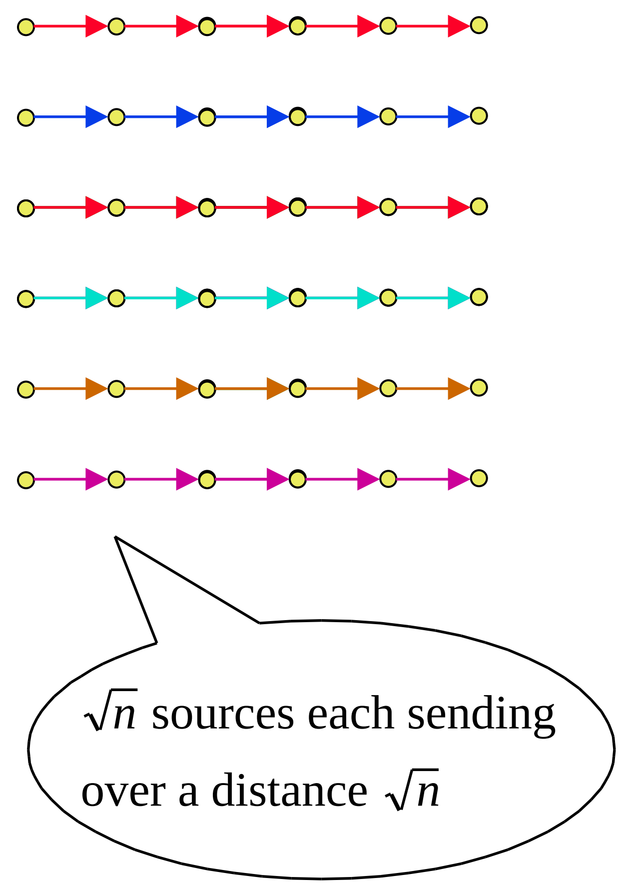
- Theorem
  - Suppose $\gamma > 0$, there is some absorption,
  - Or $\delta > 1$, if there is no absorption at all
  - Then in a regular planar network

$$C_T \geq S \left(\frac{e^{-2\gamma} P_{ind}}{c_2(\gamma,\delta) P_{ind} + \sigma^2} \right) n$$

where

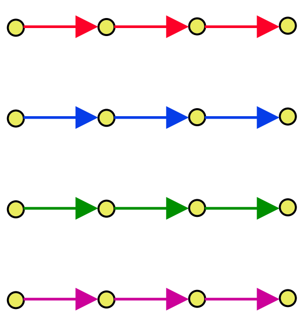
$$c_2(\gamma,\delta) = \frac{4(1+4\gamma)e^{-2\gamma} - 4e^{-4\gamma}}{2\gamma(1-e^{-2\gamma})} \quad \text{if } \gamma > 0$$

$$= \frac{16\delta^2 + (2\pi - 16)\delta - \pi}{(\delta-1)(2\delta-1)} \quad \text{if } \gamma = 0 \text{ and } \delta > 1$$

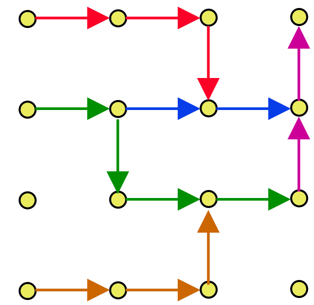


Optimality of multi-hop transport

- Corollary
  - So if $\gamma > 0$ or $\delta > 3$
  - And multi-hop achieves $\Theta(n)$
  - Then it is optimal with respect to the transport capacity - at least up to order
- Example



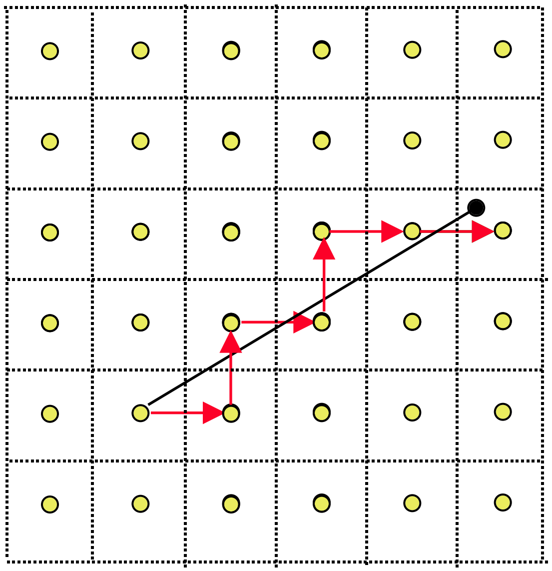
Multi-hop is almost optimal in a random network



- Theorem
  - Consider a regular planar network
  - Suppose each node randomly chooses a destination
    - » Choose a node nearest to a random point in the square
  - Suppose $\gamma > 0$ or $\delta > 1$
  - Then multihop can provide $\Omega\left(\frac{1}{\sqrt{n \log n}}\right)$ bits/time-unit for every source with probability $\rightarrow 1$ as the number of nodes n $\rightarrow \infty$
- Corollary
  - Nearly optimal since transport achieved is $\Omega\left(\frac{n}{\sqrt{\log n}}\right)$

Idea of proof for random source - destination pairs

- Simpler than Gupta-Kumar since cells are square and contain one node each
- A cell has to relay traffic if a random straight line passes through it
- How many random straight lines pass through cell?
- Use Vapnik-Chervonenkis theory to guarantee that no cell is overloaded



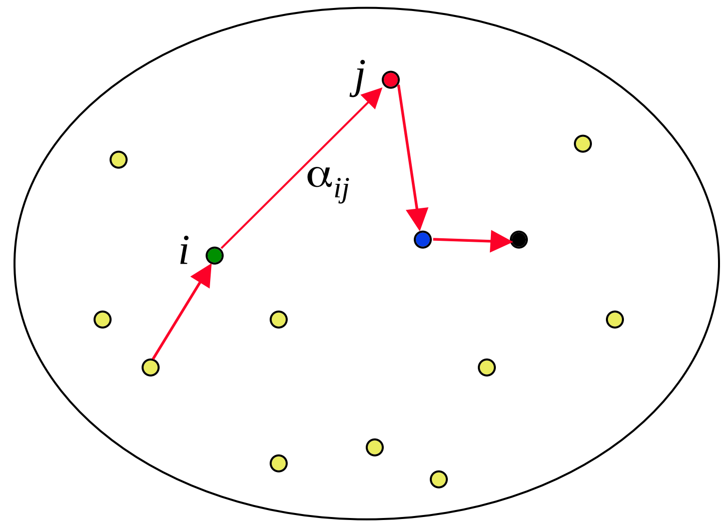
What happens when the attenuation is very low?

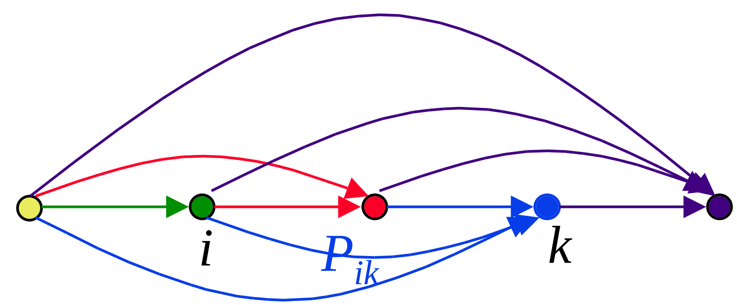
A feasible rate for the Gaussian multiple-relay channel

- Theorem
  - Suppose α_{ij} = attenuation from i to j
  - Choose power P_{ik} = power used by i intended directly for node k
  - where $\sum_{k=i}^{M} P_{ik} \le P_i$
  - Then

$$R < \min_{1\le j\le n} S\left(\frac{1}{\sigma^2}\sum_{k=1}^{j}\left(\sum_{i=0}^{k-1}\alpha_{ij}\sqrt{P_{ik}}\right)^2\right)$$

    is feasible



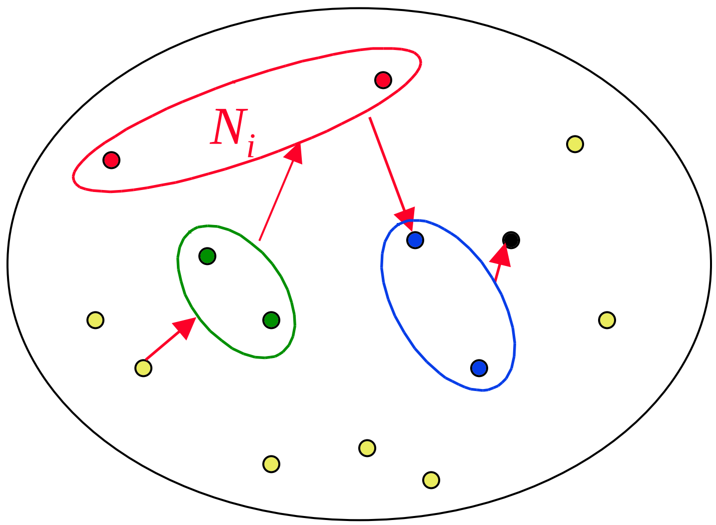


- Proof based on coding

A group relaying version

- <u>Theorem</u>

  – A feasible rate for group relaying



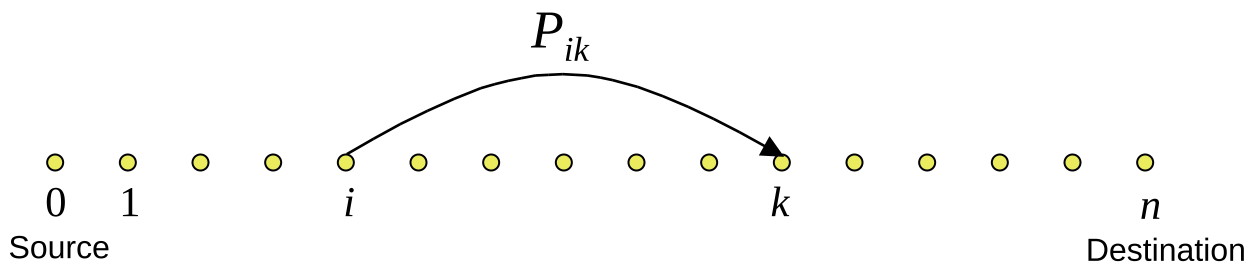
– $$R < \min_{1 \le j \le M} S\left(\frac{1}{\sigma^2} \sum_{k=1}^{j} \left(\sum_{i=0}^{k-1} \alpha_{N_i N_j} \sqrt{P_{ik} / n_i} \; n_i \right)^2 \right)$$

Unbounded transport capacity can be obtained for fixed total power

- Theorem
  - If $\gamma = 0$, there is no absorption at all,
  - And $\delta < 3/2$
  - Then C_T can be unbounded in regular planar networks even for fixed P_{total}
- Theorem
  - If $\gamma > 0$ and $\delta < 1$ in regular planar networks
  - Then no matter how many many nodes there are
  - No matter how far apart the source and destination are chosen
  - A fixed rate R_{min} can be provided for the single-source destination pair

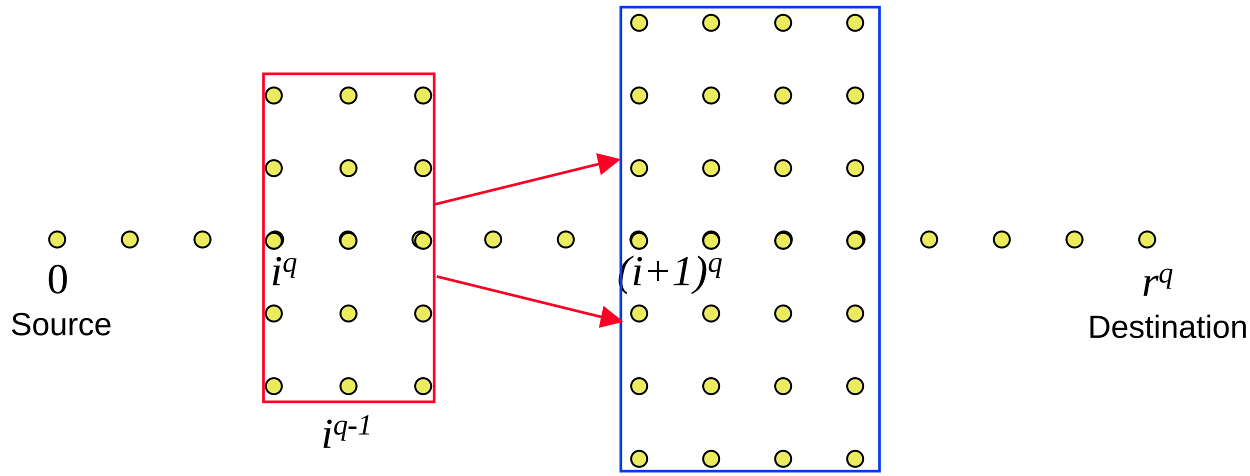
Idea of proof of unboundedness

- Linear case: Source at 0, destination at n



- Choose $P_{ik} = \dfrac{P}{(k-i)^{\alpha} k^{\beta}}$

- Planar case



Networks with transport capacity $\Theta(n^{\theta})$

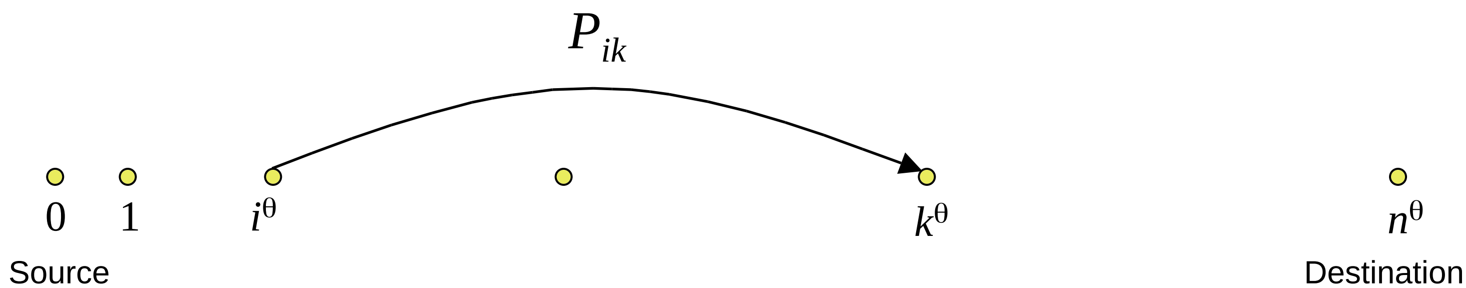
- Theorem
  - Suppose $\gamma = 0$
  - For every $1/2 < \delta < 1$, and $1 < \theta < 1/\delta$
  - There is a family of linear networks with

    $C_T = \Theta(n^{\theta})$

  - The optimal strategy is coherent multi-stage relaying with interference cancellation

Idea of proof

- Consider a linear network



- Choose $P_{ik} = \dfrac{P}{(k-i)^\alpha}$ where $1 < \alpha < 3 - 2\theta\delta$

- A positive rate is feasible from source to destination for all n
  – By using coherent multi-stage relaying with interference cancellation

- To show upper bound
  – Sum of power received by all other nodes from any node j is bounded
  – Source destination distance is at most n^θ

Concluding Remarks

- Studied networks with arbitrary numbers of nodes
  - Explicitly incorporated distance in model
    - » Distances between nodes
    - » Attenuation as a function of distance
- Make progress by asking for less
  - Instead of studying capacity region, study the transport capacity
  - Instead of asking for exact results, study the scaling laws
    - » The exponent is more important
    - » The preconstant is also important but is secondary - so bound it
  - Draw some broad conclusions
    - » Optimality of multi-hop when absorption or large path loss
    - » Optimality of coherent multi-stage relaying with interference cancellation when no absorption and very low path loss
- Open problems abound
  - What happens for intermediate path loss when there is no absorption
  - The channel model is simplistic

To obtain paper

- Papers can be downloaded from

  http://black.csl.uiuc.edu/~prkumar

- For hard copy send email to

  prkumar@uiuc.edu