Gabor phase retrieval: robustness and generative priors

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Workshop IV: Multi-Modal Imaging with Deep Learning and Modeling

Example: diffraction imaging

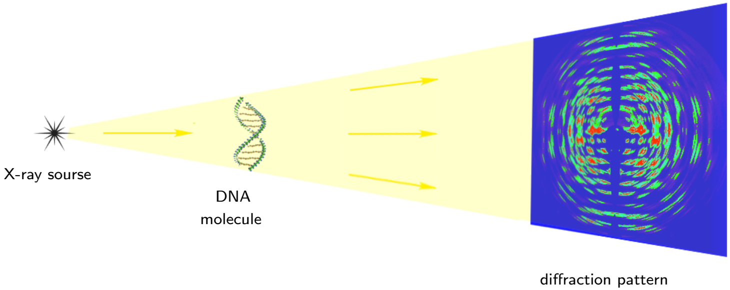

Figure: Structured illuminations in *diffraction imaging*.

Discretization: assume the unknown signal is *finite dimensional* $x \in \mathbb{C}^M$.

Measurement map: $\mathcal{A} : x \mapsto \{|\mathcal{F}(x)(\ell)|^2\}_{\ell \in \Omega}$.

Fourier phase retrieval

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$$\boxed{\begin{array}{ll} \text{find} & x \\ \text{subject to} & \mathcal{A}(x) = b. \end{array}}$$

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Trivial ambiguities: for $x, y \in \mathbb{C}^M$, $\mathcal{A}(x) = \mathcal{A}(y)$ if

- $y = e^{i\theta}x$, for some $\theta \in [0, 2\pi)$;
- $y(m) = x(m - t)$, for some $t \in \mathbb{Z}_M$ and all $m \in \mathbb{Z}_M$;
- $y(m) = \overline{x(M - m)}$ for all all $m \in \mathbb{Z}_M$.

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Idea: we need to generate more measurements, to introduce redundancy.

Example: diffraction imaging with masks

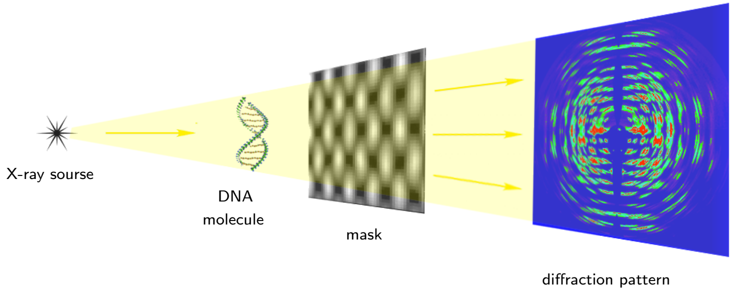

Figure: Structured illuminations in *diffraction imaging* using a *mask*.

Discretization: assume the unknown signal is *finite dimensional* $x \in \mathbb{C}^M$.

Measurement map: $\mathcal{A} : x \mapsto \{|\mathcal{F}(x \odot g_k)(\ell)|^2\}_{\substack{\ell \in \mathbb{Z}_M \\ k \in K}}$, where $g_k \in \mathbb{C}^M$ are known masks and $\odot$ denotes pointwise product.

Masked Fourier phase retrieval

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Questions

- *For which $\{g_k\}_{k\in K} \subset \mathbb{C}^M$ the measurement map $\mathcal{A}$ is injective (up to the minimal ambiguity)?*
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Idea: consider a more general setup!

Frames: definition

Definition

A set of vectors $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is called a frame with frame bounds $0 < A \leq B$ if, for any $x \in \mathbb{C}^M$,

$$A||x||_2^2 \leq \sum_{j=1}^N |\langle x, \varphi_j \rangle|^2 \leq B||x||_2^2.$$

Note: $\Phi \subset \mathbb{C}^M$ is a frame iff $span(\Phi) = \mathbb{C}^M$.

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- The vector $\Phi^* x = (\langle x, \varphi_j\rangle)_{j=1}^N$ is called the vector of frame coefficients of x.
- The signal x can be reconstructed from the vector of its frame coefficients using a dual frame $\widetilde{\Phi} = \{\widetilde{\varphi}_j\}_{j=1}^N$ as $x = \sum_{j=1}^N \langle x, \varphi_j\rangle \widetilde{\varphi}_j = \widetilde{\Phi}\Phi^* x$.
- The standard dual frame $\widetilde{\Phi} = (\Phi\Phi^*)^{-1}\Phi$.

Phase retrieval with frames

Measurement map: $\mathcal{A}_\Phi : x \mapsto \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$, for a frame $\Phi = \{\varphi_j\}_{j=1}^N$

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Measurement map: $\mathcal{A}_\Phi : [x] \mapsto \{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$, for a frame $\Phi = \{\varphi_j\}_{j=1}^N$, where $[x] \in \mathbb{C}^M/_\sim$ is an equivalence class for $x \sim e^{i\theta} y$, $\theta \in [0, 2\pi)$.

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Examples

1. *Fourier phase retrieval.* $\Phi = \{\varphi_\omega\}_{\omega\in\Omega}$, *with* $\varphi_\omega(m) = \frac{1}{\sqrt{M}} e^{-i\omega m/M}$, $m \in \mathbb{Z}_M$
2. *Masked Fourier phase retrieval. For a set of masks* $\{g_k\}_{k\in K} \subset \mathbb{C}^M$, $\Phi = \{\varphi_{\omega,k}\}_{\substack{\omega\in\Omega \\ k\in K}}$, *with* $\varphi_{\omega,k}(m) = \frac{1}{\sqrt{M}} e^{-i\omega m/M} g_k(m)$, $m \in \mathbb{Z}_M$

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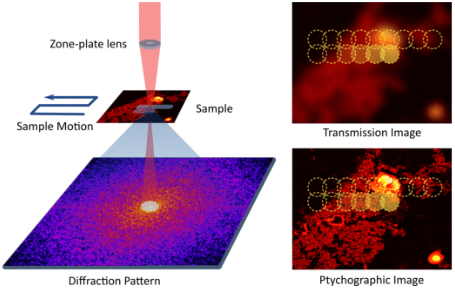
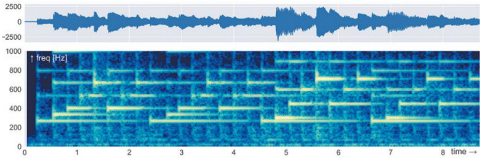
   - *Special case: shifting mask.* $g_k(m) = g(m-k)$, $m \in \mathbb{Z}_M$, *for a window* $g \in \mathbb{C}^M$. *In this case,* Φ *is a Gabor frame.*

Phase retrieval with shifting window: application examples



Ptychography. A short support mask is shifted across the sample, so that different (overlapping) regions are illuminated at different times (*figure courtesy: Iwen, Viswanathan, and Wang, 2016*).



Audio processing. Spectrograms are a common tool used for speech recognition and source separation tasks.

Gabor frames and phase retrieval applications

Definition (Gabor frames)

For a window $g \in \mathbb{C}^M \setminus \{0\}$ and $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, the Gabor frame is given by $(g, \Lambda) = \{\pi(k,\ell)g\}_{(k,\ell)\in\Lambda}$, where

1. $\pi(k,\ell) = M_\ell T_k$ is a time-frequency shift operator;
2. $T_k x = (x(m-k))_{m\in\mathbb{Z}_M}$ is translation operator;
3. $M_\ell x = \left(e^{2\pi i \ell m/M} x(m)\right)_{m\in\mathbb{Z}_M}$ is modulation operator.

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In phase retrieval:

- mostly studied for random Gaussian frames: $\varphi_j(m) \sim i.i.d.\ \mathcal{CN}(0, 1/n)$.
- much less is known for structured, application relevant frames, such as Gabor frames.

Injectivity of phase retrieval

Question

For which frames Φ *is the associated phaseless measurement map* $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$ *injective and stable?*

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Stability of phase retrieval

Definition

For a frame Φ, the associated measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ is called stable with a constant C in a set $T \subset \mathbb{C}^M$ if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C \min_{\theta \in [0,2\pi)} ||x - e^{i\theta} y||_2 ||x + e^{i\theta} y||_2.$$

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Random frames: various stability results with different assumptions on the distribution of $\varphi_j(m)$, admissible set T, and frame cardinality.

Yonina C. Eldar and Shahar Mendelson

Phase retrieval: Stability and recovery guarantees, *Applied and Computational Harmonic Analysis* 36(3), 2014.

Felix Krahmer and Yi-Kai Liu

Phase retrieval without small-ball probability assumptions, *IEEE Transactions on Information Theory* 64(1), 2017.

Maryia Kabanava, Richard Kueng, Holger Rauhut and Ulrich Terstiege

Stable low-rank matrix recovery via null space properties, *Information and Inference: A Journal of the IMA* 5(4), 2016.

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Stability of phase retrieval problem. *13th International Conference on Sampling Theory and Applications*, IEEE, 2019.

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Common condition in the mentioned results: independent frame vectors.

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How to exploit structure and geometric properties of frames instead of probabilistic characteristics of the individual frame vectors?

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- (Bojarovska, Flinth, 2016) $\mathcal{A}_{(g,\mathbb{Z}_M\times\mathbb{Z}_M)}$ is injective up to a global phase factor if $\mathcal{A}_{(g,\mathbb{Z}_M\times\mathbb{Z}_M)}(g)(\lambda) \neq 0$ for all $\lambda \in \mathbb{Z}_M \times \mathbb{Z}_M$.
- (Alaifari, Wellershoff, 2021) $\mathcal{A}_{(g,\mathbb{Z}_M\times\mathbb{Z}_M)}$ is stable on $T_\delta = \{|x(m)| \geq \delta\}$ if $\mathcal{A}_{(g,\mathbb{Z}_M\times\mathbb{Z}_M)}(g)(\lambda) \geq \varepsilon$ for all $\lambda \in \mathbb{Z}_M \times \mathbb{Z}_M$.

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Phase retrieval from Gabor measurements, *Journal of Fourier Analysis and Applications* 22(3), 2016.

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- (S.) $\mathcal{A}_{(g,\mathbb{Z}_M\times\mathbb{Z}_M)}$ is stable on $\mathbb{C}^M$ for $g\sim\text{Unif.}(\mathbb{S}^{M-1})$ with high probability.

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Stability using frame order statistics

Phase retrieval with general frames: stability depends on the geometric properties (frame vectors are well spread in $\mathbb{C}^M \leftrightarrow$ for each direction, there are not too many frame vectors that are almost collinear or almost orthogonal to it.

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Definition (Frame order statistics)

Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a frame. For $0 < \alpha, \beta \leq N$ and $x \in \mathbb{S}^{M-1}$, define

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) = \max_{\substack{J \subseteq \{1,\dots,N\}, \\ |J| \geq \alpha}} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$

$$\mathcal{L}_{FOS}(\Phi, \beta, x) = \min_{\substack{J \subseteq \{1,\dots,N\}, \\ |J| \geq \beta}} \max_{j \in J} |\langle x, \varphi_j \rangle|.$$

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Theorem (S., 2019)

Let $\Phi \subset \mathbb{C}^M$ be a frame. Fix $\alpha < 1 - \frac{1}{2C_0}$. Then $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ with constant $C \geq (2\alpha - 1)|\Phi| \min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha|\Phi|, x)^2$.

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2. Counterexamples: $\mathcal{A}_{(g,\Lambda)}$ is not always injective, even for large $|\Lambda|$.
3. Geometric properties approach leads to a non-uniform stability result in the case or random window g:
   For any pair of signals $x, y \in \mathbb{C}^M$, with high probability

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C\frac{|\Lambda|}{M} \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2||x + e^{i\theta}y||_2.$$

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Idea: consider several masks - back to masked Fourier transform measurements!

Masked Fourier phase retrieval: $\mathcal{A}(x) = \{|\mathcal{F}(x \odot g_k)(\ell)|^2\}_{k \in K, \ell \in \mathbb{Z}_M}$

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- Mostly studied for Gaussian frames: PhaseLift, Wirtinger flow, ...
- Recovery guarantees for structured, application relevant frames are rare.
- Gabor frames: lack of injectivity results for $|\Lambda| < M^2$.

Idea: consider several masks - back to masked Fourier transform measurements!

Masked Fourier phase retrieval: $\mathcal{A}(x) = \{|\mathcal{F}(x \odot g_k)(\ell)|^2\}_{k\in K, \ell\in\mathbb{Z}_M}$

Consider a set of masks $\{g_k\}_{k\in K}$ constructed in one of the two ways:

1. $g_k \sim$ i.i.d. Unif.$(\mathbb{S}^{M-1})$;
2. $g_k = T_k g$, where $g \sim$ Unif.$(\mathbb{S}^{M-1})$.

Phase retrieval algorithm for structured frames

Theorem (Pfander, S., 2019)

Fix $x \in \mathbb{C}^M$, and let $\{g_k\}_{k\in K}$ be as above. Then there exist a set of additional masks $\{g_t\}_{t\in T}$ with $|T| = O(\log(M))$, and a reconstruction algorithm, such that the estimate $\tilde{x}$ produced by it from the measurements with masks $\{g_t\}_{t\in K\cup T}$ satisfies

$$\min_{\theta\in[0,2\pi)} ||\tilde{x} - e^{i\theta}x||_2^2 \leq C\sqrt{M}||\nu||_2,$$

with overwhelming probability, provided the noise vector ν satisfies $\frac{||\nu||_2}{||x||_2^2} \leq \frac{c}{M}$.

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1. Not just an existence result: the algorithm and the set of additional masks $\{g_t\}_{t\in T}$ construction are provided.
2. In the case when $g_k = T_k g$, where $g \sim \mathrm{Unif.}(\mathbb{S}^{M-1})$, the set of additional masks is also formed as time shifts of a modified window. In this case, the measurement frame is a union of two Gabor frames.

Idea of the polarization approach

Let $\Phi_\Lambda = (g, \Lambda)$ with g *uniformly distributed* on the unit sphere $\mathbb{S}^{M-1} \subset \mathbb{C}^M$ and $\Lambda = F \times \mathbb{Z}_M$, $F \subset \mathbb{Z}_M$ with $|F|$ being a constant not depending on M.

Given: phaseless measurements $b_\lambda = |\langle x, \pi(\lambda) g\rangle|^2$, $\lambda \in \Lambda$.

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For this in addition to phaseless measurements b, we need to know *relative phases* between frame coefficients, defined for $b_{\lambda_1}, b_{\lambda_2} \neq 0$:

$$\omega_{\lambda_1 \lambda_2} = u_{\lambda_1}^{-1} u_{\lambda_2} = \frac{\overline{\langle x, \pi(\lambda_1) g\rangle} \langle x, \pi(\lambda_2) g\rangle}{|\langle x, \pi(\lambda_1) g\rangle| |\langle x, \pi(\lambda_2) g\rangle|}, \ (\lambda_1, \lambda_2) \in E,$$

Here $E \subset \Lambda \times \Lambda$ to be chosen later.

Polarization identity

Lemma (Polarization identity)

Let $\omega = e^{2\pi i/3}$. *For any* $\lambda_1, \lambda_2 \in \Lambda$, *such that* $b_{\lambda_1}, b_{\lambda_2} \neq 0$,

$$\omega_{\lambda_1\lambda_2} = \frac{1}{3|\langle x, \pi(\lambda_1)g\rangle||\langle x, \pi(\lambda_2)g\rangle|} \sum_{t=0}^{2} \omega^t \left|\langle x, \pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\rangle\right|^2.$$

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We take measurements with respect to the union of two frames:

$$\Phi_\Lambda \cup \Phi_E = (g, \Lambda) \cup \{\pi(\lambda_1)g + \omega^t \pi(\lambda_2)g\}_{t \in \{0,1,2\},\ (\lambda_1, \lambda_2) \in E}.$$

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The additional measurements are masked Fourier transform coefficients:

$$b_{\lambda_1\lambda_2 t} = |\langle x, \pi(k_1,\ell_1)g + \omega^t \pi(k_2,\ell_2)g\rangle|^2 = \left|\mathcal{F}\left(x \odot \bar{p}_{\ell_2-\ell_1,k_1,k_2}(t) \odot T_{k_1}\bar{g}\right)(\ell_1)\right|^2,$$

where $p_{c,k_1,k_2}(t)(m) = 1 + e^{2\pi i\left(\frac{cm}{M}+\frac{t}{3}\right)} \frac{g(m-k_2)}{g(m-k_1)},\ m \in \mathbb{Z}_M$.

Polarization approach: phase propagation algorithm

Algorithm 1: Phase propagation algorithm

Input : for given $\Lambda \subset \mathbb{Z}_M \times \mathbb{Z}_M$, $E \subset \Lambda \times \Lambda$, and $g \in \mathbb{C}^M$, measurements of the form $\{b_\lambda = |\langle x, \pi(\lambda) g\rangle|^2\}_{\lambda \in \Lambda}$, $\left\{\omega_{\lambda_1 \lambda_2} = \frac{\overline{\langle x, \pi(\lambda_1) g\rangle} \langle x, \pi(\lambda_2) g\rangle}{|\langle x, \pi(\lambda_1) g\rangle| |\langle x, \pi(\lambda_2) g\rangle|}\right\}_{(\lambda_1, \lambda_2) \in E}$.

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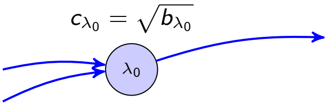

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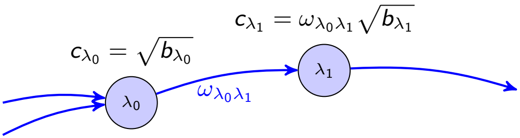
while not all c_λ are set **do**
    choose $c_{\lambda_1} \neq 0$ already known;

end while



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    for λ_2, s.t. $(\lambda_1, \lambda_2) \in E$ and c_{λ_2} is not set, **do**
        set $c_{\lambda_2} = \omega_{\lambda_1\lambda_2}\frac{c_{\lambda_1}}{|c_{\lambda_1}|}\sqrt{b_{\lambda_2}}$.
    end for
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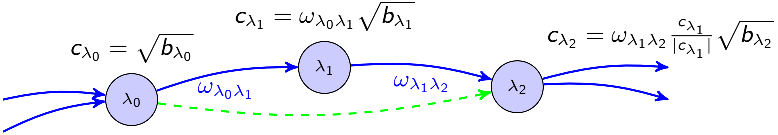

How large should set E be?

Noiseless case: cannot propagate through zero measurements!

Define a weighted graph $G = (\Lambda, E)$ with

- $w(\lambda) = b_\lambda$;
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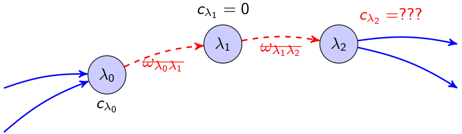

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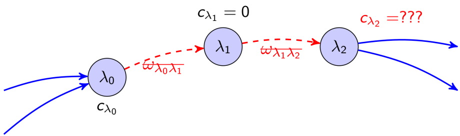

To enable recovery: after deleting all "zero" vertices, the graph should have a connected component $G' = (\Lambda', E')$ such that (g, Λ') is a frame.

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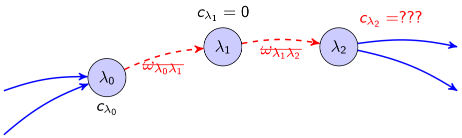

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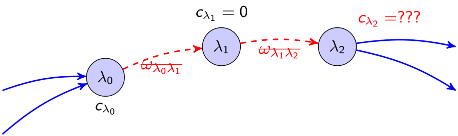

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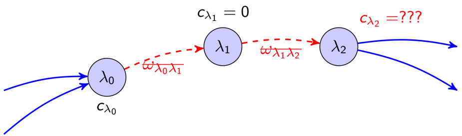

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Idea: Choose E so that G is an expander graph (has large spectral gap)

- $|E| = O(\log(M))$

Reconstruction in the case of noisy measurements

Algorithm 2: Phaseless reconstruction in the noisy case

Input : phaseless measurements $b = \mathcal{A}_{\Phi_\Lambda \cup \Phi_E}(x)$;
parameters $\tau_0, \alpha, \beta \in (0,1)$.

Output: $\tilde{x} \in [x]$, initial signal up to a global phase.

1: construct the graph of measurements $G = (\Lambda, E)$;

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Adaptive measurement design

Question

Can we further reduce the number of measurements needed for robust phaseless reconstruction of x*?*

Adaptive measurement design

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Can we further reduce the number of measurements needed for robust phaseless reconstruction of x?

Idea: Construct the additional set of masks adaptively

Theorem (S.)

Fix $x \in \mathbb{C}^M$, and let $\{g_k\}_{k \in K}$ be as before. Then one can construct the set of additional measurement vectors Φ_E with $|\Phi_E| = O(M)$, such that the estimate $\tilde{x}$ produced by polarization algorithm satisfies

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with overwhelming probability, provided the noise vector ν satisfies $\frac{||\nu||_2}{||x||_2^2} \leq \frac{c}{M}$.

Prior information

Question

Can we use the available prior information about x to make phase retrieval easier?

Approaches to formalize prior knowledge:

1. $x \in T$, for a known $T \subset \mathbb{C}^M$
   - Important example: sparse signals $T_s = \{x \in \mathbb{C}^M, \text{ s.t. } |\mathrm{supp}(x)| \leq s\}$.
2. **Generative priors:** $x = G(h)$ for a known $G\colon \mathbb{C}^d \to \mathbb{C}^M$ with $d < M$
   - Deep generative priors: $G(h) = \rho(W_n \dots \rho(W_2\rho(W_1 h)) \dots)$.
   - "Toy problem": Linear generative model $G \in \mathbb{C}^{M \times D}$ learned from data (PCA, NMF, ...)

Sparse phase retrieval

Gaussian frames:

- For $x \in T_s$, robust reconstruction from $O\left(s^2 \log\left(M/s\right)\right)$ measurements
- Two-step reconstruction procedure with $O\left(s \log\left(M/s\right)\right)$ measurements

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- If $s + 2 * \mathrm{supp}(g) < M - 1$, $\mathcal{A}_{(g, \mathbb{Z}_M \times \mathbb{Z}_M)}$ is not injective!
- Numerical results for sparse phase retrieval with a Gabor frame

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Polarization approach:

- Measurement design with $|(g, \Lambda) \cup \Phi_E| = O(s^2 \log^3(s) \log^2(M))$ and Steinhaus window $g(m) = \frac{1}{\sqrt{M}} e^{i\theta_m}$, $\theta_m \sim$ i.i.d. Unif $[0, 2\pi)$.

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Gaussian frames:

- Untrained deep generative priors.
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Paul Hand, Oscar Leong, and Vlad Voroninski
Phase retrieval under a generative prior, *Advances in Neural Information Processing Systems* 31, 2018.

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  $x = G(h) = \rho(W_n \ldots \rho(W_2\rho(W_1 h)) \ldots)$ with $W_k(i,j) \sim$ i.i.d. $\mathcal{CN}(0, 1/m_k)$ can be reconstructed from $O\left(sn \log\left(m_1 \ldots m_n\right)\right)$ measurements.

Polarization approach:

- "Toy problem": Linear generative model $G \in \mathbb{C}^{M\times d}$, $\mathrm{rank}(G) = d$.

Theorem (S.)

Fix $x = Gh$ for some $h \in \mathbb{C}^d$. Then one can construct a measurement setup with $|(g,\Lambda) \cup \Phi_E| = O(d\log(d))$, such that the estimate $\tilde{x}$ produced by polarization algorithm with high probability satisfies

$$\min_{\theta\in[0,2\pi)} ||\tilde{x} - e^{i\theta}x||_2^2 \leq C\sqrt{d}||\nu||_2.$$

Paul Hand, Oscar Leong, and Vlad Voroninski

Phase retrieval under a generative prior, *Advances in Neural Information Processing Systems* 31, 2018.

Conclusions

Challenges of Gabor phase retrieval:

1. lack of injectivity and stability results
2. reducing the number of measurements

Possible approaches:

1. use of geometric properties of Gabor frames
2. adaptive selection of mask shifts
3. incorporating priors and using them in measurement design construction

Thank You for Your Attention!