An introduction to variational image processing

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RWTH AACHEN UNIVERSITY

IPAM Long Program on Computational Microscopy

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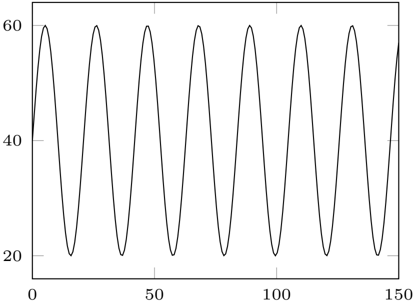
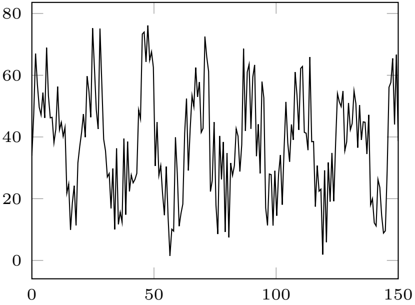
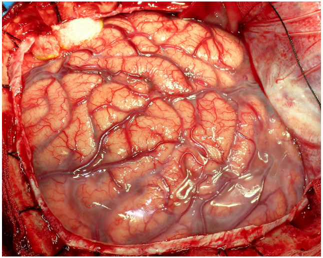
Image modalities





signals

Image modalities

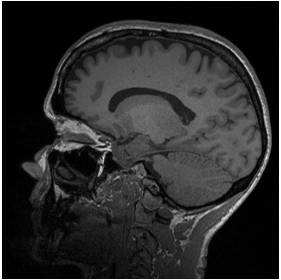
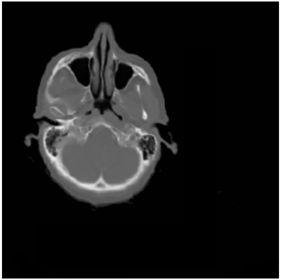


digital photographs

Image modalities

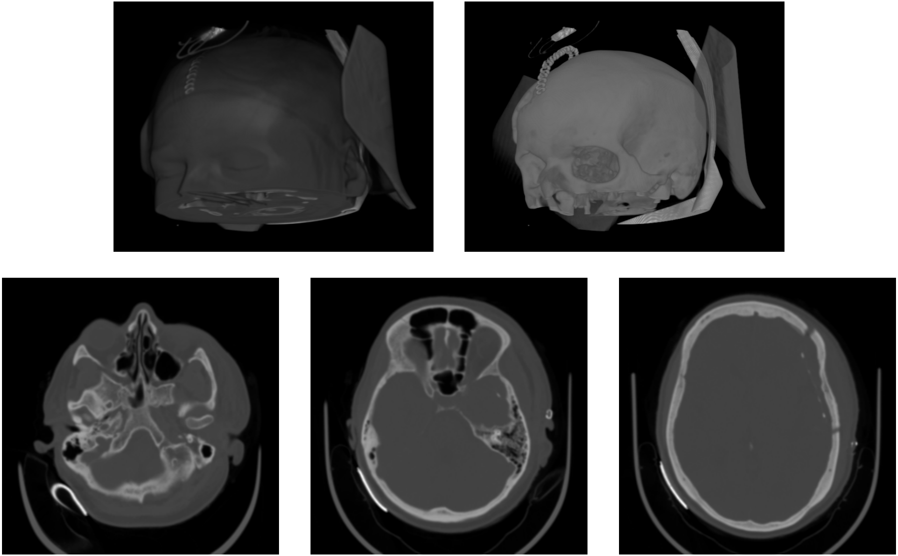
videos

Image modalities

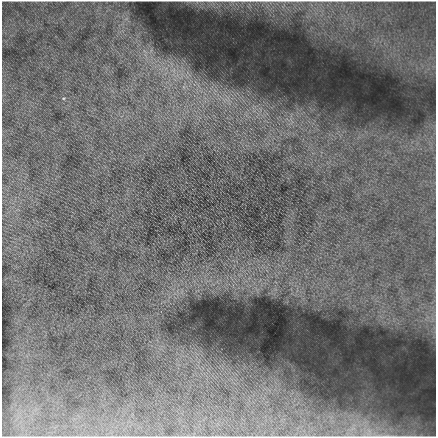

medical modalities (MR / CT / PET)

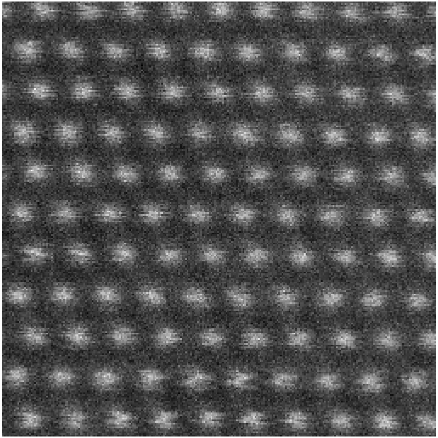
Image modalities



volumetric images

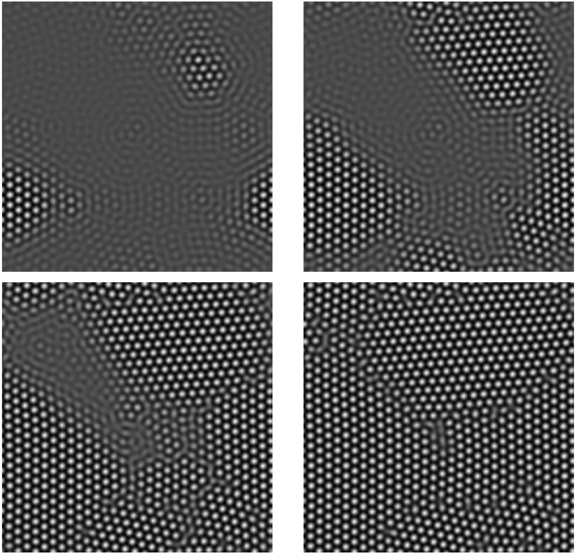
Image modalities





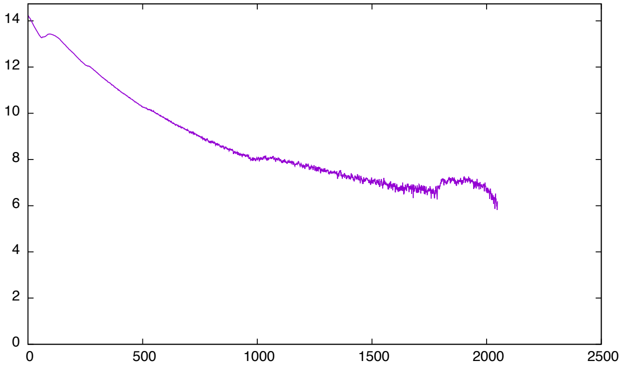
electron microscopy (TEM / STEM)

Image modalities



simulated / computer generated images

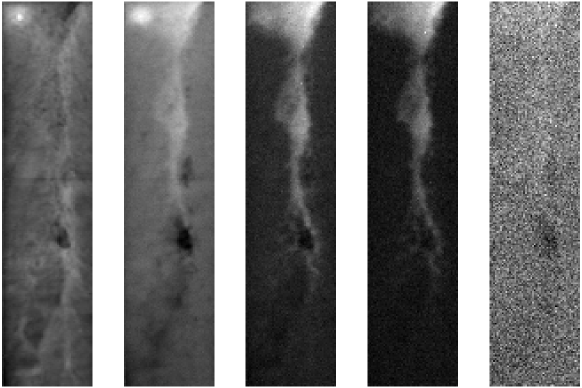
Image modalities



typical EELS spectrum (log scale)

Electron energy loss spectroscopy (EELS)

Image modalities



Electron energy loss spectroscopy (EELS)

Fundamental image processing tasks

Image denoising

Given: Noisy image $f = f_0 + n$.

Task: Recover f_0.

Image deblurring

Given: A blurred $f = Af_0$.

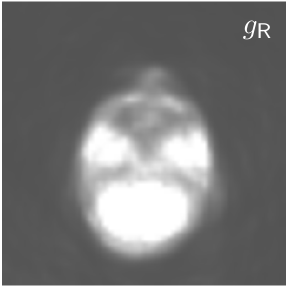
Task: Recover f_0.

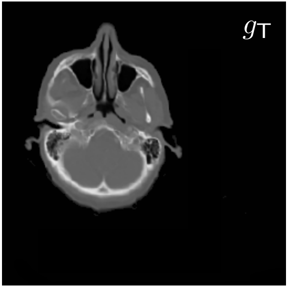
Given: Image f showing an object.

Task: Recover the object (as region).

Image registration

Given: Two images g_{T} (template) and g_{R} (reference).





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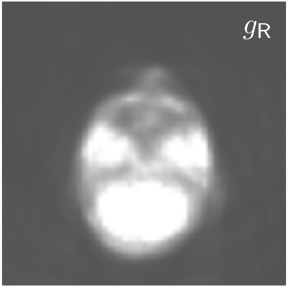

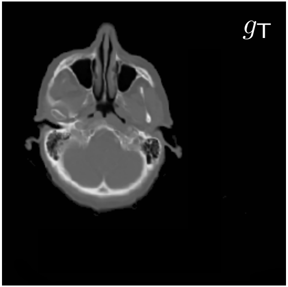

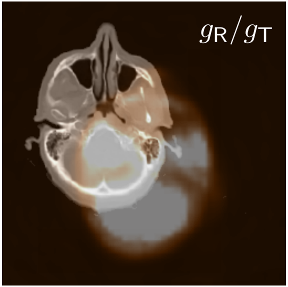

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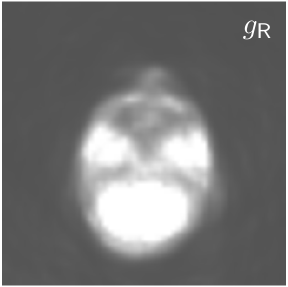

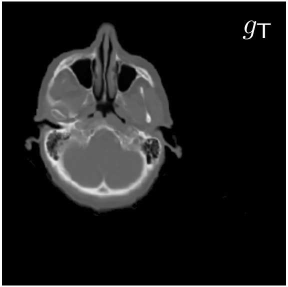

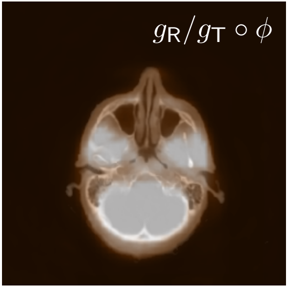

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Task: Find a deformation $\phi : \Omega \to \Omega$ (here $\Omega = [0,1]^2$) such that

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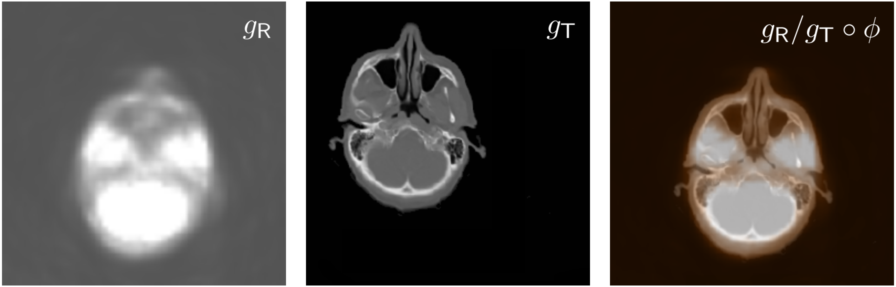

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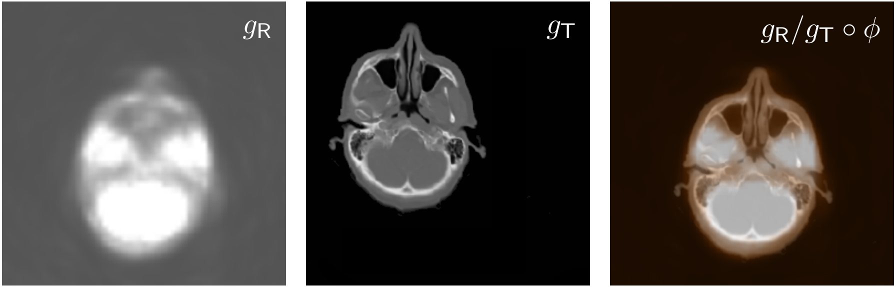

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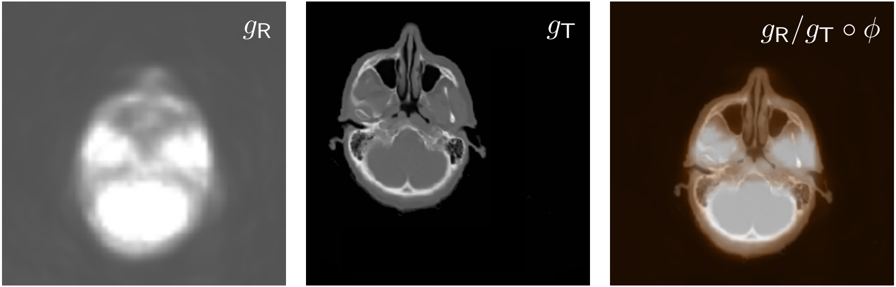

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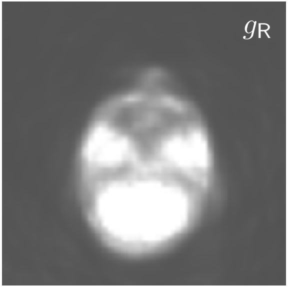

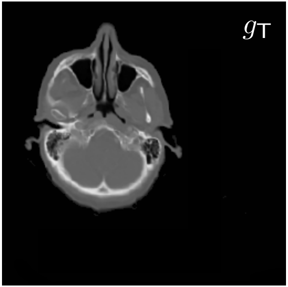

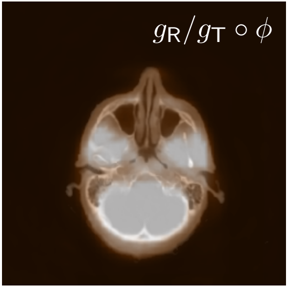

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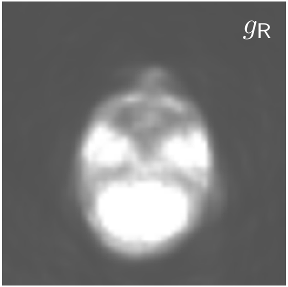

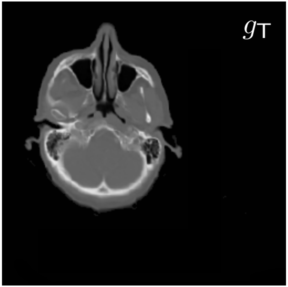

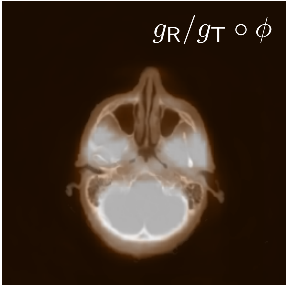

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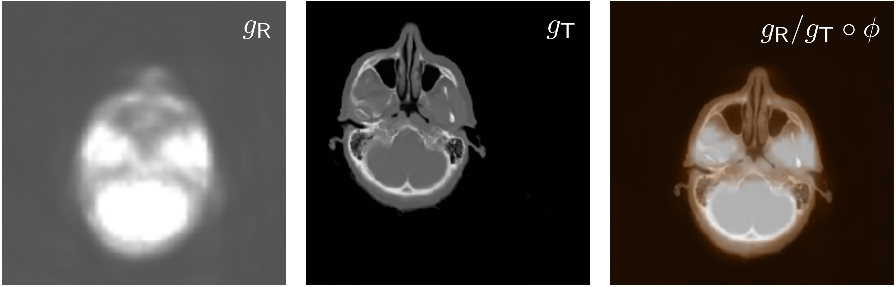

- What is a suitable notion of "$\sim$"? $\Rightarrow$ *Similarity measure*
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- How to compute the deformation ϕ numerically?

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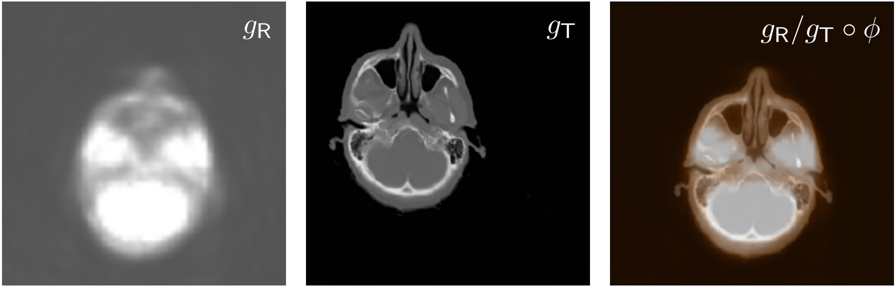

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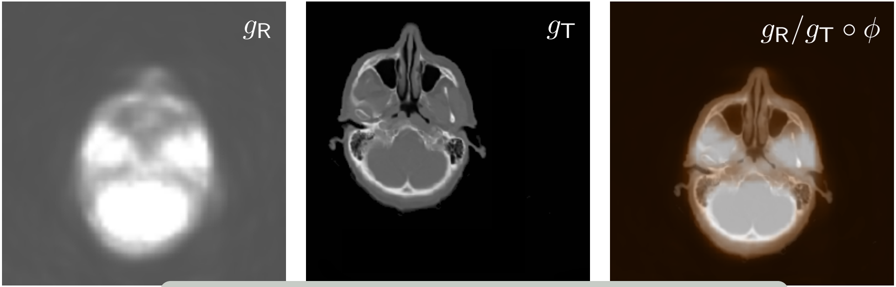

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Introductory text books on image registration:
[Modersitzki, '04][Modersitzki, '09]

Idea to approach a wide range of problems

Variational approaches - Image registration

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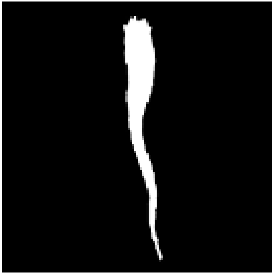
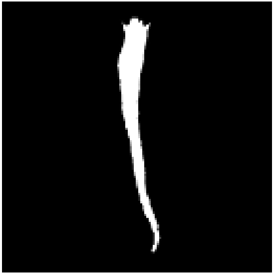
Example: Register two images $g_{\mathsf{T}}, g_{\mathsf{R}}$ with comparable gray values.



g_{R}



g_{T}

Variational approaches - Image registration

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Conditions on the non-parametric, non-rigid deformation ϕ:

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$$E[\phi] = \frac{1}{2} \int_\Omega |g_\mathsf{T}(\phi(x)) - g_\mathsf{R}(x)|^2 \,\mathrm{d}x$$

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$\Rightarrow g_\mathsf{T}$ and g_R are registered by minimizing E.

Variational approaches - Image registration

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$\Rightarrow$ g_T and g_R are registered by minimizing E.

Structure "$\mathcal{D}[\phi] + \lambda\mathcal{R}[\phi]$" typical for registration (and inverse problems).

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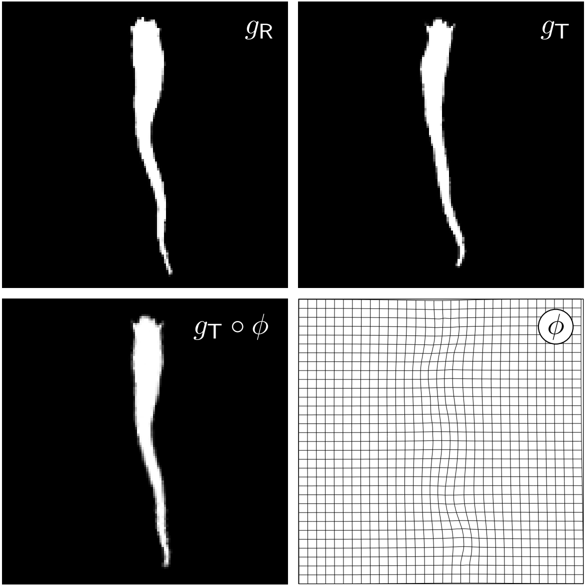
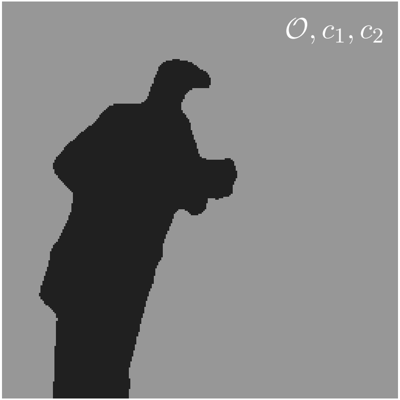
Note: Competing conditions are balanced by weights (like λ in E)

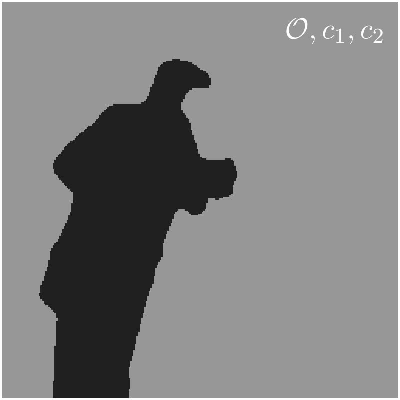
A simple registration result




g



Given an image g, we search for a piecewise constant segmentation, i.e. two gray values c_1, c_2 and a region $\mathcal{O}$.



Given an image g, we search for a piecewise constant segmentation, i.e. two gray values c_1, c_2 and a region $\mathcal{O}$. A valid segmentation minimizes

$$E[\mathcal{O}, c_1, c_2] = \int_{\mathcal{O}} (g(x) - c_1)^2 \, \mathrm{d}x + \int_{\Omega \setminus \mathcal{O}} (g(x) - c_2)^2 \, \mathrm{d}x + \nu \, \mathrm{Per}(\mathcal{O}).$$

(Piecewise constant binary Mumford-Shah model)

Denoising

Given a noise image f, i.e. $f = f_0 + n$, find f_0 by minimizing

$$J[u] = \underbrace{\int_\Omega (u - f)^2 \,\mathrm{d}x}_{\text{data term}} + \lambda \underbrace{\int_\Omega \|\nabla u(x)\| \,\mathrm{d}x}_{\text{regularizer}} \quad \text{(Rudin-Osher-Fatemi)}$$

Variational approaches - Denoising and deblurring

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Deconvolution/Deblurring

Given a blurry image/signal f, i.e. $f = Af_0$, find f_0 by minimizing

$$J[u] = \int_\Omega (Au - f)^2 \,\mathrm{d}x + \lambda \int_\Omega \|\nabla u(x)\| \,\mathrm{d}x\,.$$

General task

Given: Normed vector space $(X, \|\cdot\|), M \subset X, J : M \to \mathbb{R}$,

Find: $y^* \in M$ such that $J[y^*] \leq J[y]$ for all $y \in M$.

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Here,

- J is called *objective functional*,
- M is the *admissible set*.

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Central questions

- Existence of minimizers? $\dim(X) = \infty$ has large implications.
- Characterization of minimizers? (necessary/sufficient conditions)
- How can they be efficiently computed in practice?

Variational image registration

Non-parametric deformations

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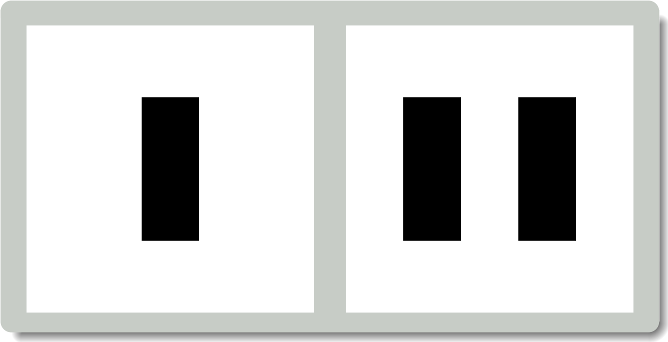
- solution not unique



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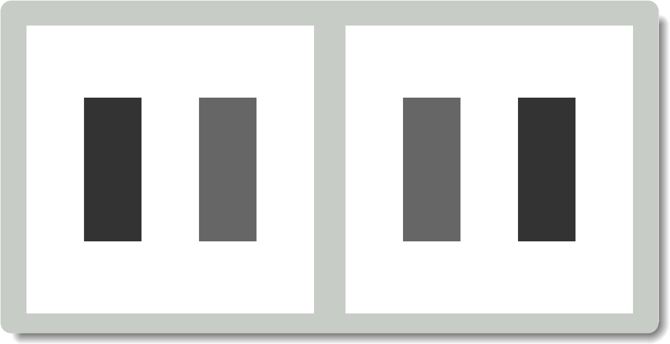

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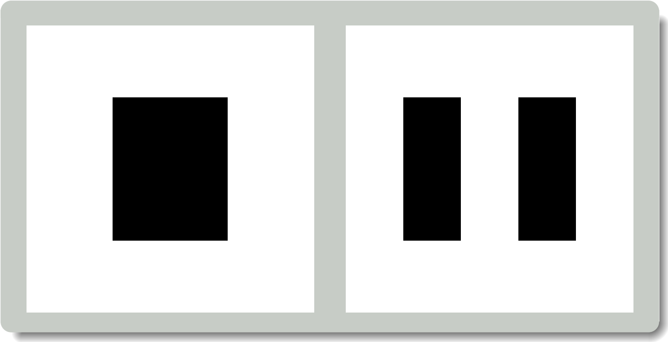

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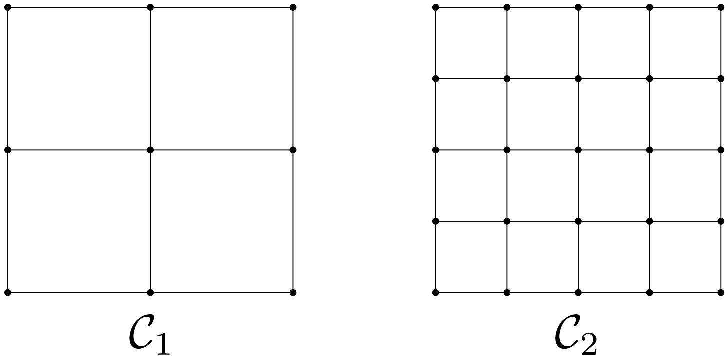
- regularized gradient descent (gradient flow)
- step size control (e. g. Armijo rule)

Multilevel minimization



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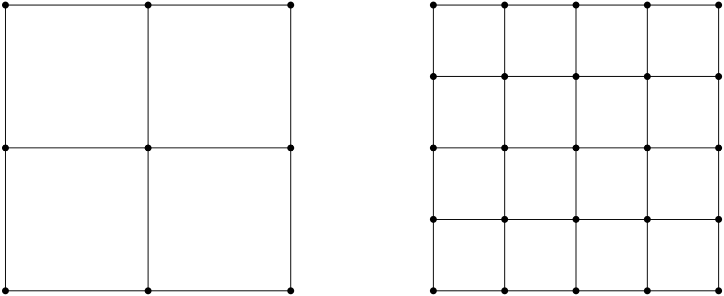
Multilevel minimization



$\mathcal{C}_1$ $\mathcal{C}_2$

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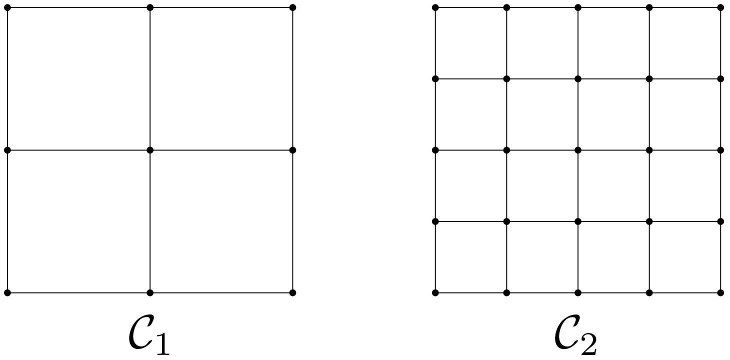

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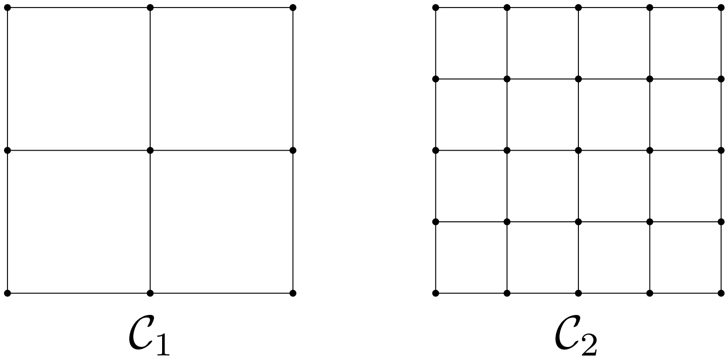

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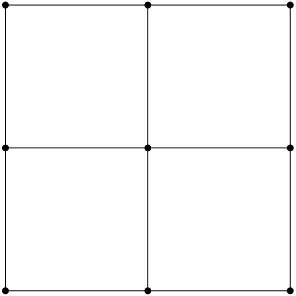

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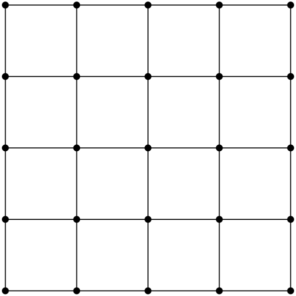

$\mathcal{C}_2$

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- solve the problem on a coarse starting level l
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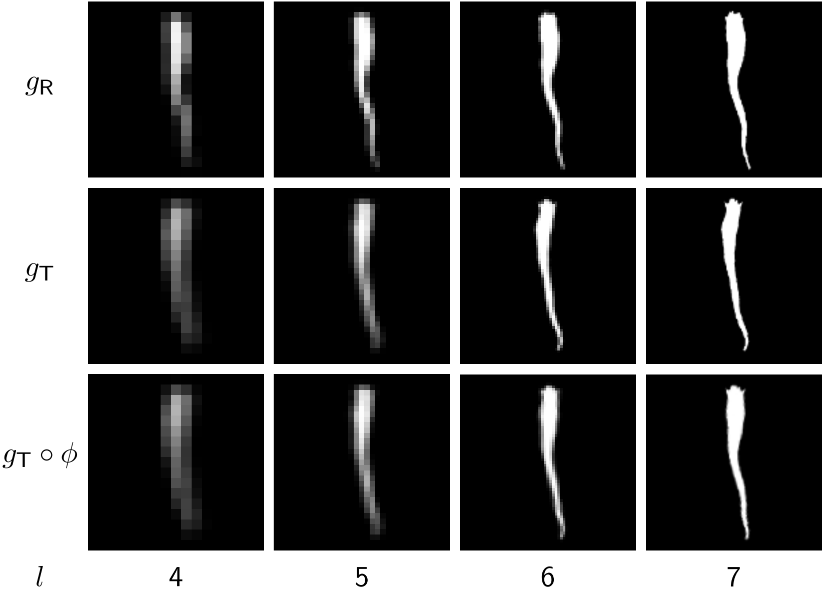
- solve on level $l+1$ with the prolongated result as initial guess
- iterate till reaching the finest level

Multilevel minimization



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$\Rightarrow$ Use an inner product favoring desired solution properties.

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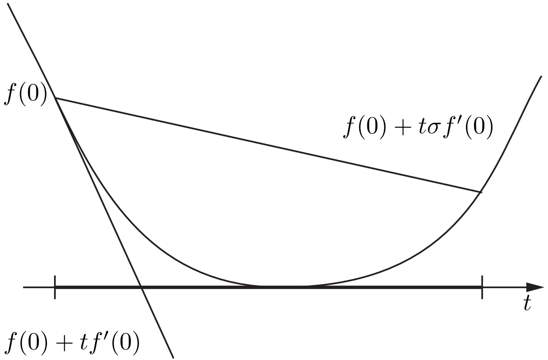
$\Rightarrow A^{-1}$ is regularizing.

Step size control with the Armijo rule

Finding a suitable step size is a 1D optimization problem:

$$f : \mathbb{R} \to \mathbb{R}, t \mapsto E[\phi + td]$$



Armijo condition

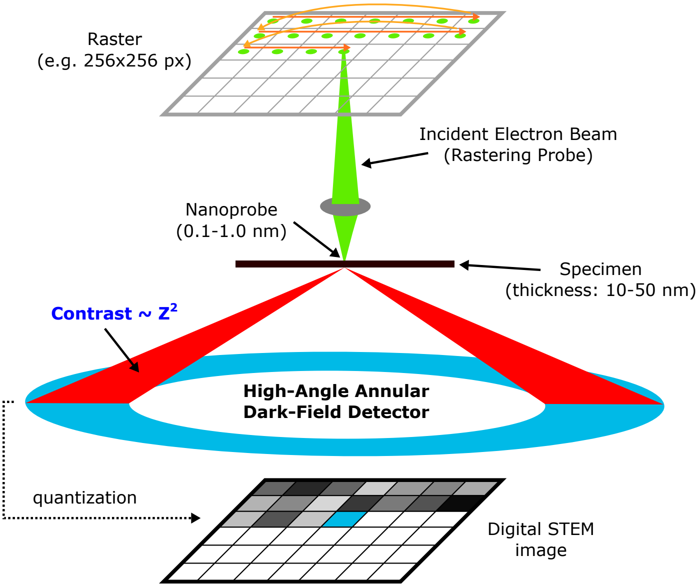
$$\frac{f(\tau) - f(0)}{\tau f'(0)} = \frac{\text{secant slope}}{\text{tangent slope}} \geq \sigma \in (0, 1)$$

High precision analysis of image series from electron microscopy

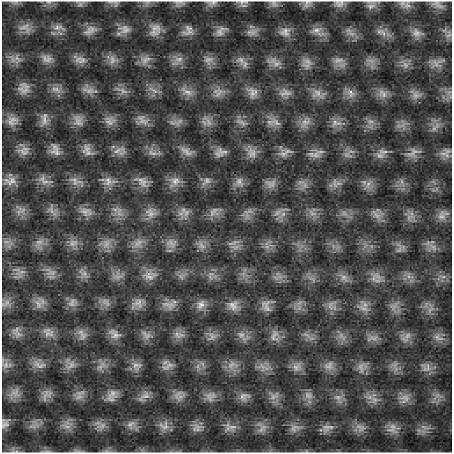
joint work with:

P. Binev, D. A. Blom, R. Sharpley, T. Vogt, W. Dahmen (University of SC), N. Mevenkamp (RWTH, now at Zeiss)
P. Voyles (University of Wisconsin), A. Yankovich (Chalmers)

Scanning transmission electron microscopy

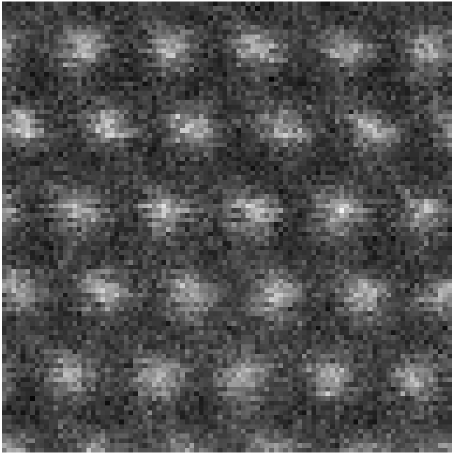


Scanning transmission electron microscopy (cont.)

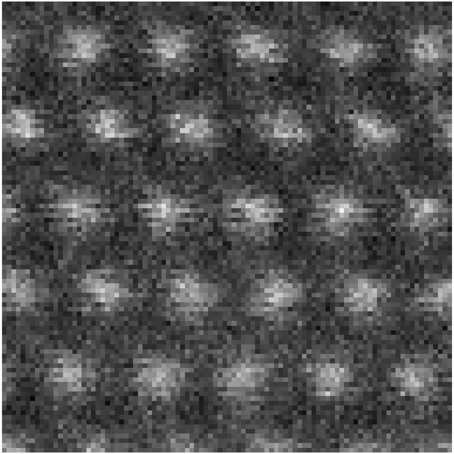


Gallium-Nitrogen lattice at 20.5Mx magnification

Scanning transmission electron microscopy (cont.)



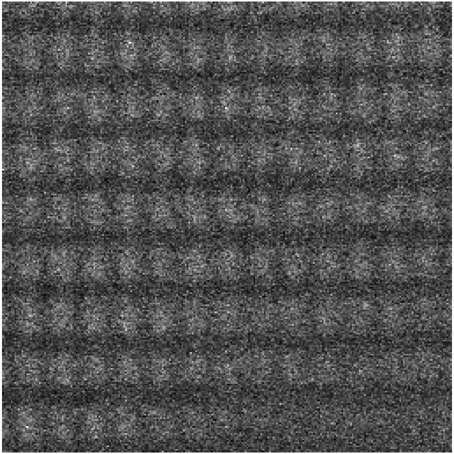
Close-up reveals horizontal distortions and noise



Close-up reveals horizontal distortions and noise

- Distortions arise from environmental and instrumental disturbances

STEM image series of silicon (112)



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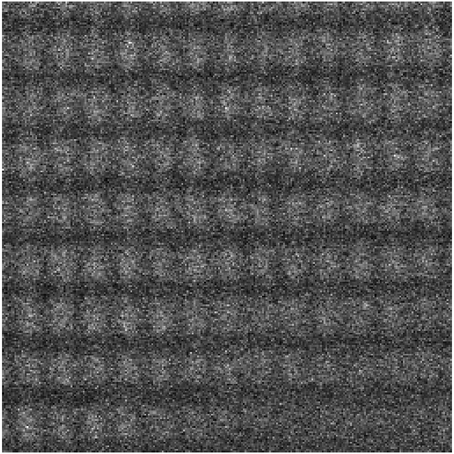

Challenges

- individual frames are very noisy
- large movement of the sample during the series acquisition

Normalized cross-correlation

Registration of noisy images requires a robust distance measure.

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Data functional $\quad E_{\mathsf{NCC}}[\phi] = -\mathsf{NCC}[g_{\mathsf{R}}, g_{\mathsf{T}} \circ \phi]$

Image reconstruction from an image series

Energy for joint registration and reconstruction

$$E[f,\phi_1,\dots,\phi_n] = \sum_{i=1}^{n}\left(-\mathsf{NCC}[f, f_i\circ\phi_i] + \frac{\lambda}{2}\int_\Omega \|D\phi_i - \mathbb{1}\|^2 \,\mathrm{d}x\right)$$

[B., Binev, Blom, Dahmen, Sharpley, Vogt Ultramicroscopy '14]

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Theorem Let Ω be a bounded domain and $f_1, \dots, f_n \in C(\mathbb{R}^d)$ bounded. Then, a minimizer of

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- initialize f with f_1, ϕ_1 with id and ϕ_i, $i > 2$, with $\phi_{i,1}$ (see next slide)
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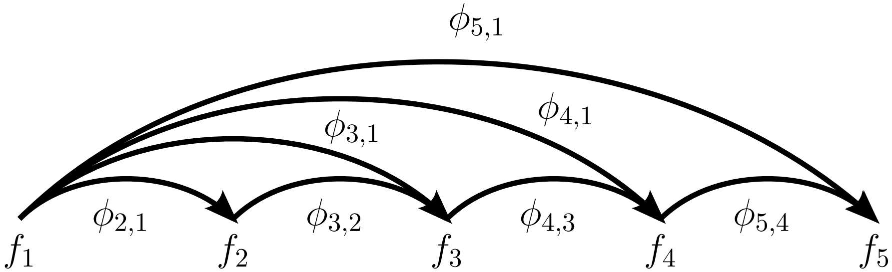
Note This strategy is biased to f_1. [B., Liebscher Ultramic. '19]

Registering long image series

Strategy



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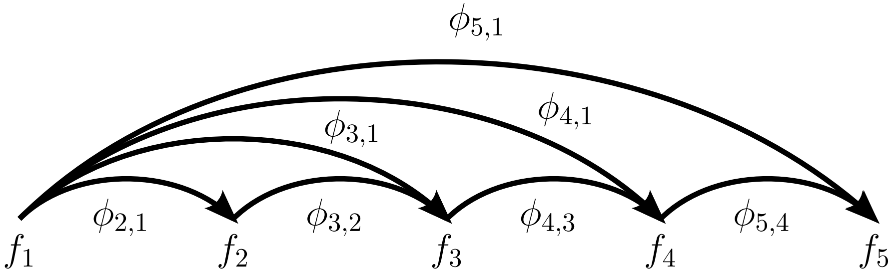
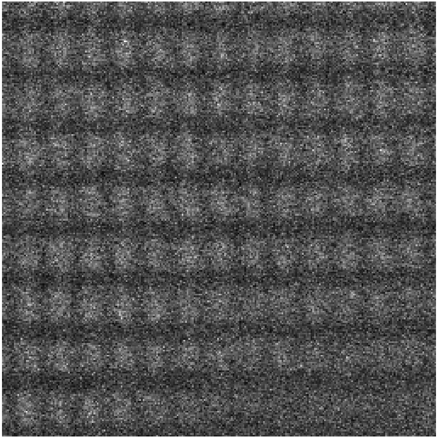
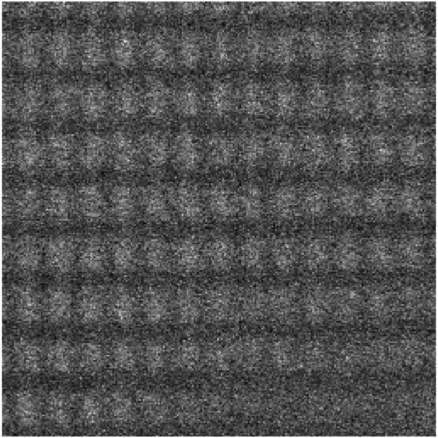


- compute $\phi_{i+1,i}$ for $i = 1, \ldots, n-1$ (initial guess id)
- compute $\phi_{3,1}$ (initial guess $\phi_{3,2} \circ \phi_{2,1}$)
- compute $\phi_{4,1}$ (initial guess $\phi_{4,3} \circ \phi_{3,1}$)
- $\ldots$
- compute $\phi_{n,1}$ (initial guess $\phi_{n,n-1} \circ \phi_{n-1,1}$)

Non-rigid registration – Aligning a series



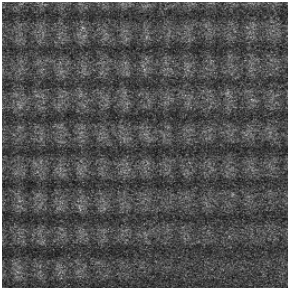
initial data



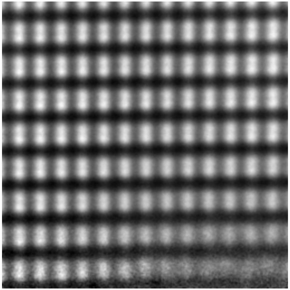
registered data

Silicon (112), magnification 82Mx, dwell time 3.2µs per pixel

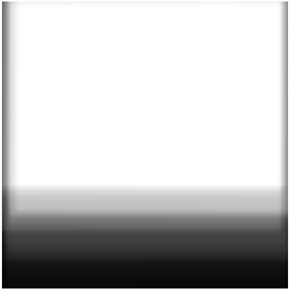
Non-rigid registration – Reconstruction from 512 frames



f_1



reconstruction



#samples

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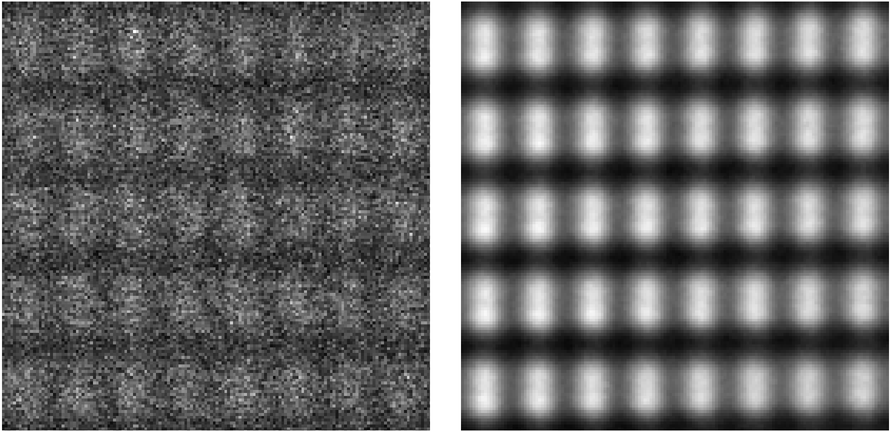

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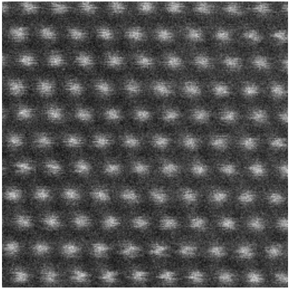

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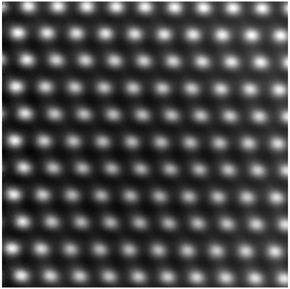
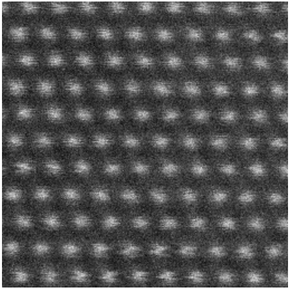


reconstruction

#samples

GaN [11-20], magnification 29Mx, dwell time 12μs



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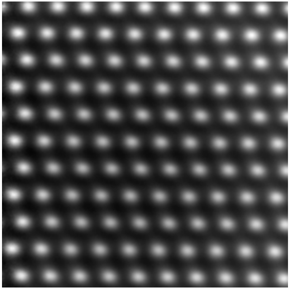

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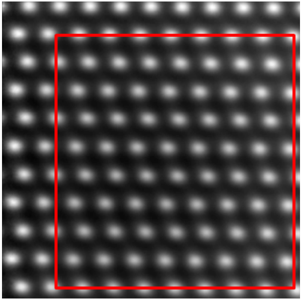
GaN [11-20], magnification 29Mx, dwell time 12μs

How to evaluate the quality of the reconstruction? Ground truth?

Results – Precision analysis on GaN [11-20]



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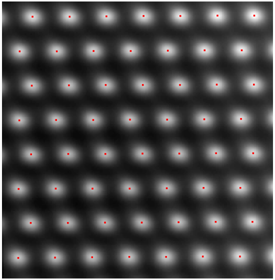

- detect atom centers

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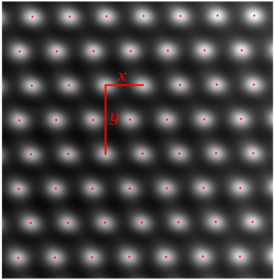

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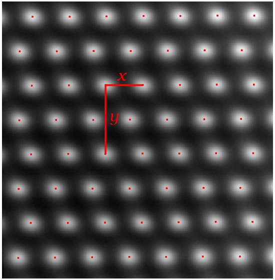

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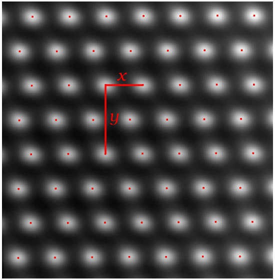

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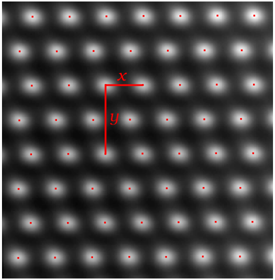

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[Yankovich, B., Dahmen, Binev, Sanchez, Bradley, Li, Szlufarska, Voyles Nature Comm. '14]

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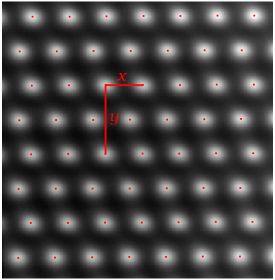

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Before best reported precision from STEM series was ~ 5pm.

[Kimoto et al. Ultramicr. '10]

Precision achievable in a single STEM image typically ~ 15pm.

[Schmid et al. Micron '12]

Non-rigid registration - Analysis of a Pt nanocatalyst

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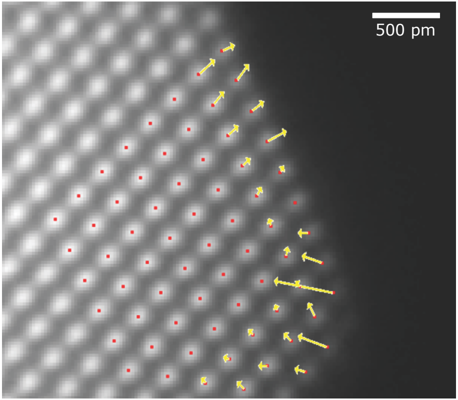

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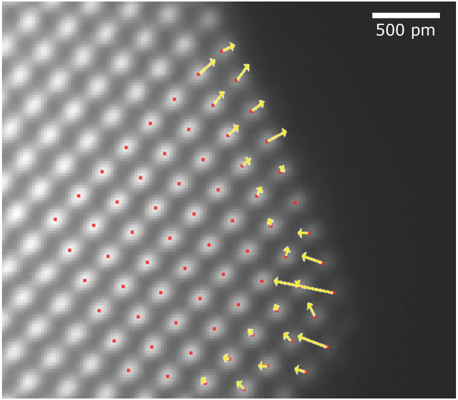

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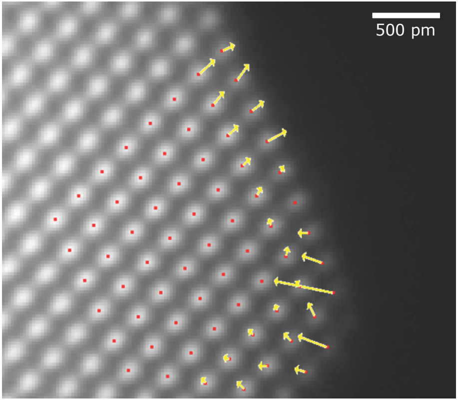

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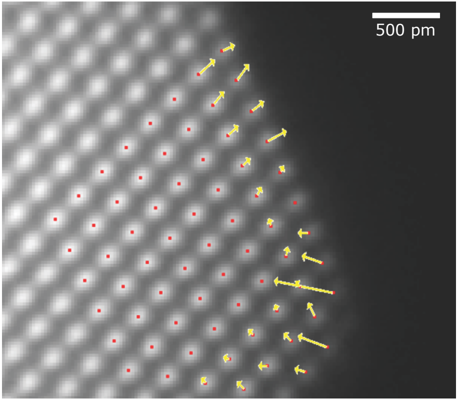

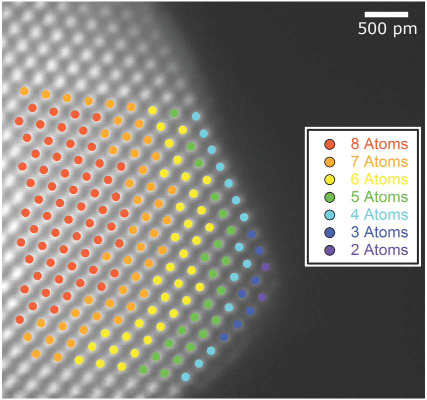

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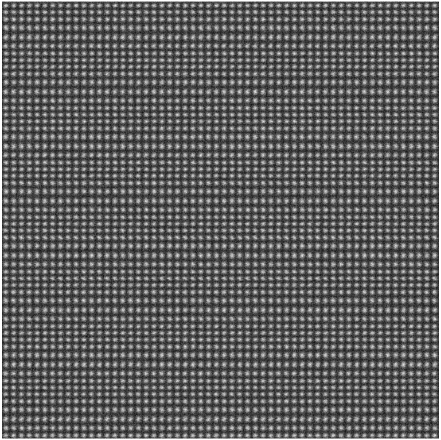
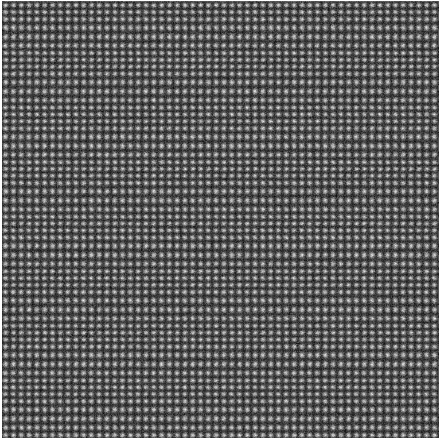
Limits of alternating minimization - Initialization bias



initial data



registered data

Artificial series with fast and slow scan noise

joint work with C. Liebscher (MPIE Düsseldorf)

Bias correction

Observation: If ψ is a translation, we have

$$E[f, \phi_1, \ldots, \phi_n] = E[f \circ \psi, \phi_1 \circ \psi, \ldots, \phi_n \circ \psi].$$

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After each alternate minimization step

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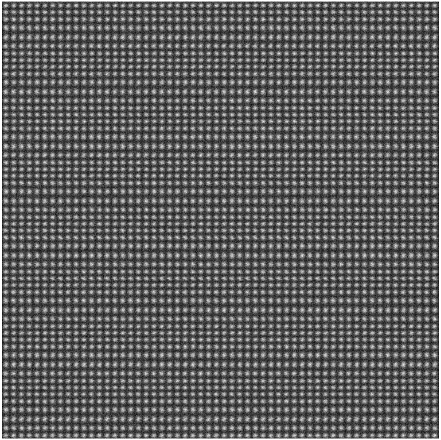
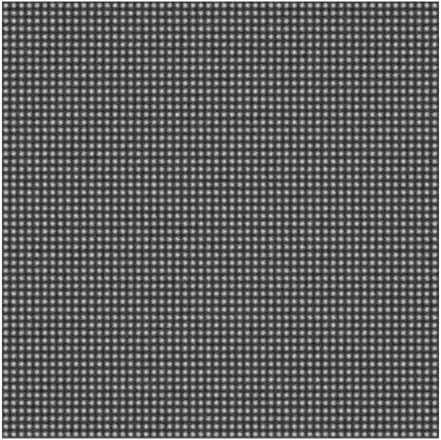
Bias correction - Artificial results



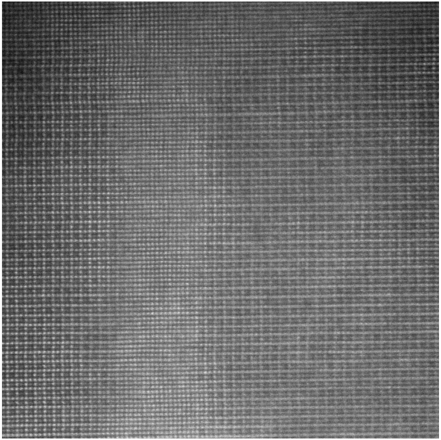
initial data



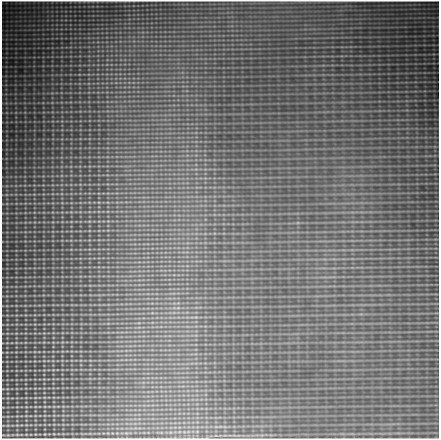
registered data with reduction

[B., Liebscher Ultramic. '19]

Bias correction - Reconstruction from 19 STEM frames



f_1



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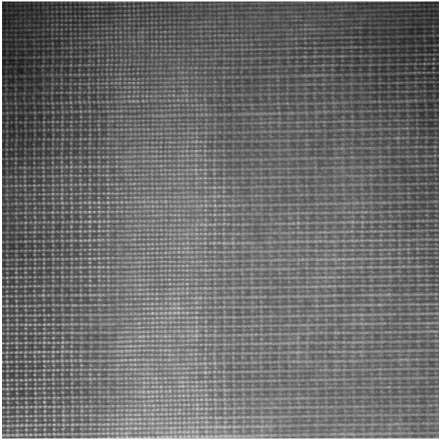

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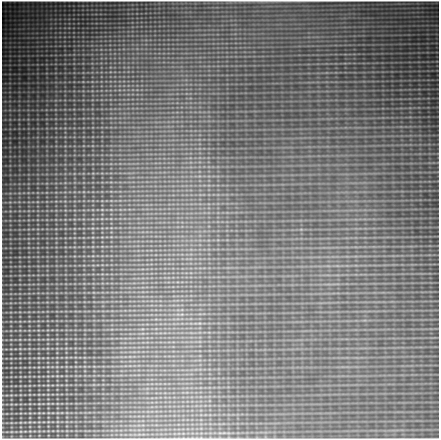

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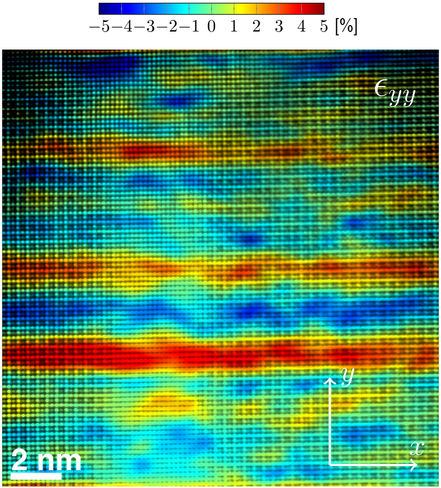
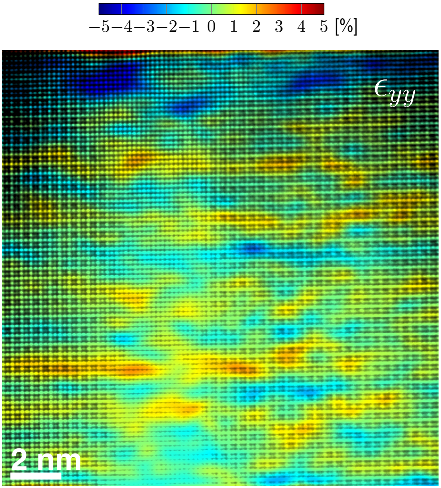
Bias correction - Effect on strain estimation



no bias-correction



with bias-correction

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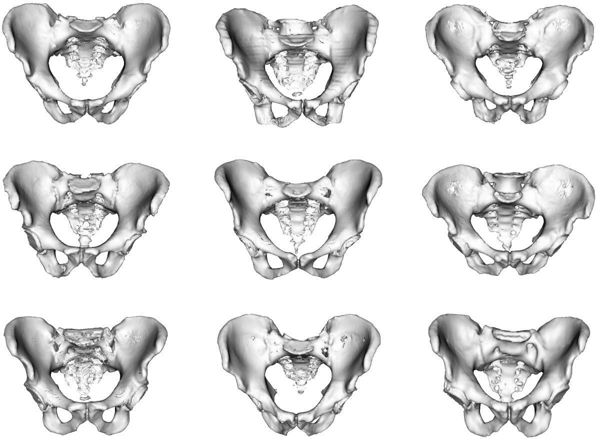
Usage via a command line tool configured via a parameter file.

Shape averaging for jawbone reconstruction



joint work with A. Modabber, F. Peters (University Hospital Aachen)

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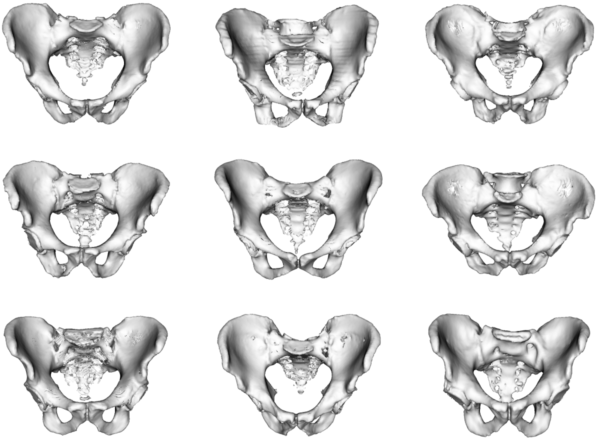

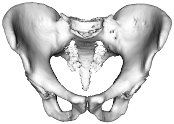

average pelvis

joint work with A. Modabber, F. Peters (University Hospital Aachen)

Existence of minimizers and some consequences

Existence of minimizers

J needs to be bounded from below on the admissible set, i.e.

$$\underline{J} := \inf_{x \in X} J[x] > -\infty.$$

(not sufficient, e.g. $J[x] = e^x$, this J is even strictly convex and analytic)
Then, there is a *minimizing sequence*, i.e. $(x_n)_n \in X^{\mathbb{N}}$ with $J[x_n] \to \underline{J}$ for $n \to \infty$.

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Direct method in the calculus of variations:

1. Selection of a minimizing sequence $(x_n)_n \in X^{\mathbb{N}}$
2. Getting a convergent subsequence $(x_{n_k})_k \in X^{\mathbb{N}}$ (with $x^* \in X$)
3. Proving *lower semi-continuity* of J, i.e.

$$J[y] \leq \liminf_{n \to \infty} J[y_n] \text{ for all } (y_n)_n \in X^{\mathbb{N}} \text{ with } y_n \to y \in X.$$

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Then, x^* is a minimizer, i.e. $J[x^*] = \underline{J}$, since

$$\underline{J} = \lim_{n \to \infty} J[x_n] = \lim_{k \to \infty} J[x_{n_k}] = \liminf_{k \to \infty} J[x_{n_k}] \geq J[x^*] \geq \underline{J}.$$

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The functional space setting gives us information about inherent properties of the solution, e.g. its regularity, that the numerical solution approximates.

A reason to check the existence of minimizers

With the direct method, one can show that minimizers of

$$J[y] = \frac{1}{2} \|y - g\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla y\|_{L^2}^2 \qquad \text{(simple denoising model)}$$

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This means that the simple denoising model cannot preserve edges!

A suitable space for denoised images

Let $y \in C^1[0,1]$ be increasing. Then,

$$|y|_{H^{1,1}} = \int_0^1 \left|y'(t)\right| \mathrm{d}t = \int_0^1 y'(t)\,\mathrm{d}t = y(1) - y(0).$$

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Still, we need to extend the $H^{1,1}$-norm to such functions, we need a more general concept than weak derivatives.

A suitable space for denoised images (cont.)

For $x \in \mathbb{R}^d$ with $x \neq 0$, we have

$$\|x\|_2 = x \cdot \frac{x}{\|x\|_2} \leq \sup_{\|p\|_2 \leq 1} -x \cdot p \leq \sup_{\|p\|_2 \leq 1} \|x\|_2 \|p\|_2 = \|x\|_2$$

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The latter only needs $y \in L^1(\Omega)$ to be defined.

Functions of bounded variation

For $y \in L^1(\Omega)$, the *total variation* is defined as

$$|y|_{BV(\Omega)} = \sup_{p \in C_c^\infty(\Omega,\mathbb{R}^d) \wedge \left\| \|p\|_2 \right\|_{L^\infty} \leq 1} \int_\Omega y \, \mathrm{div} p \, \mathrm{d}x \, .$$

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In particular, we have $H^{1,1}(\Omega) \subset BV(\Omega) \subset L^1(\Omega)$.

Total variation examples

Let $p \in C_c^\infty(\mathbb{R})$ with $\|p\|_{L^\infty} \leq 1$.

- For the *Heaviside Function*

$$H : \mathbb{R} \to \mathbb{R}, t \mapsto \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

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Since there is an admissible p with $p(a) = -1$ and $p(b) = 1$, the statement follows.

Total variation examples (cont.)

Consider $\chi_{B_r(0)}$ in $\mathbb{R}^2$. For $p \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\| \|p\|_2 \|_{L^\infty} \leq 1$, we get

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Thus, as regularizer, $|\cdot|_{BV}$ smoothens the boundary of the sublevel sets $\{x \in \Omega \,:\, y(x) < c\}$.

The ROF-functional (Rudin, Osher, Fatemi, 1992)

To denoise an image $g \in L^2(\Omega)$, we are looking for a minimizer of

$$J_{\mathsf{ROF}} : BV(\Omega) \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}, y \mapsto J[y] = \frac{1}{2} \|y - g\|_{L^2}^2 + \lambda |y|_{BV} .$$

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Note: $|\cdot|_{BV}$ is not differentiable, but convex.

Characterization of minimizers in classical optimization

Necessary condition Let $X = \mathbb{R}^d$, $M \subset X$ open and $J \in C^1(M)$. If $y^* \in M \subset X$ is a local extremum of J, then

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Proposition Let $M \subset \mathbb{R}^d$ be convex and open, and $J \in C^1(M)$. Then,

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Corollary Let M and J be as above and J additionally convex. Moreover, let $y^* \in M$ with $\nabla J[y^*] = 0$. Then,

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Proof Let $y \in M$ be arbitrary but fixed. Then,

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□

Sufficient conditions

Let X be a normed vector space and $J : X \to \mathbb{R}_\infty$ convex. $u \in X'$ is called *subgradient* of J, if

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Let X be a normed vector space and $J : X \to \mathbb{R}_\infty$ convex. $u \in X'$ is called *subgradient* of J, if

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Proposition Let $X = \mathbb{R}^d$ and $J \in C^1(X)$ convex. Then,

$$\partial J[y] = \{\nabla J[y]\} \text{ for all } y \in X.$$

Minimization using the proximal operator

Closed proper convex functionals

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- $\Gamma_0(X)$ denotes the set of closed proper convex functionals on X.

The proximal operator

Proposition Let X be a reflexive Banach space and $J \in \Gamma_0(X)$.

Then, the mapping

$$\mathrm{prox}_J : X \to X, y \mapsto \operatorname*{argmin}_{u \in X} \left(J[u] + \frac{1}{2} \|u - y\|^2 \right)$$

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$$\{\operatorname{prox}_{\tau J}[y]\} = (\mathrm{id} + \tau \partial J)^{-1}[y].$$

Here, for a set-valued mapping $A : X \to \mathcal{P}(Y)$, the inversion is defined by

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The proximal operator - Examples

Now, confine to $X = \mathbb{R}^n$ and $\|\cdot\| = \|\cdot\|_2$, i.e. *Discretize Then Optimize*.

- For $J \equiv c \in \mathbb{R}$, we have

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$$u^* := \operatorname{prox}_{\tau J}(y) = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left(\frac{\tau}{2} \|u - g\|_2^2 + \frac{1}{2} \|u - y\|_2^2 \right)$$

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- For $J(y) = \sum_{i=1}^{n} J_i(y_i)$ with $J_i \in \Gamma_0(\mathbb{R})$, we have

$$\operatorname{prox}_{\tau J}(y) = (\operatorname{prox}_{\tau J_1}(y_1), \ldots, \operatorname{prox}_{\tau J_n}(y_n)).$$

The proximal operator - Examples (cont.)

- For $J(y) := \|y\|_1$, $\operatorname{prox}_{\tau J}(y)$ is the *soft threshold* operator, i.e.

$$(\operatorname{prox}_{\tau J}(y))_i = \begin{cases} y_i - \tau & y_i \geq \tau \\ 0 & |y_i| < \tau \\ y_i + \tau & y_i \leq -\tau \end{cases} .$$

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- If $C \subset \mathbb{R}^n$ is a nonempty, closed, convex set, then

$$I_C : \mathbb{R}^n \to \mathbb{R}_\infty, y \mapsto \begin{cases} 0 & y \in C \\ \infty & y \notin C, \end{cases}$$

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  where Π_C is the Euclidean projection to C, i.e.

$$\Pi_C(y) = \operatorname*{argmin}_{z \in C} \|z - y\|_2 .$$

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$\Rightarrow$ Operator Splitting.

For $J = G + H$, we consider the optimization problem

$$\min_{y \in \mathbb{R}^n} (G(y) + H(y)),$$

where $G \in C^1(\mathbb{R}^n) \cap \Gamma_0(\mathbb{R}^n)$ and $H \in \Gamma_0(\mathbb{R}^n)$.

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This is also called *forward-backward splitting*, since it combines

- a forward Euler gradient descent step in G with
- a proximal point algorithm step in H (equivalent to a backward Euler gradient descent step in H).

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Assumptions as before and ∇G Lipschitz continuous with constant $L > 0$.

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$$J(y - \tau \mathcal{F}_\tau(y)) \leq J(z) + \mathcal{F}_\tau(y) \cdot (y - z) - \frac{\tau}{2} \left\| \mathcal{F}_\tau(y) \right\|_2^2 .$$

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Theorem Let $\tau_k \in [\tau_{\text{min}}, \frac{1}{L}]$, where $\tau_{\text{min}} \in (0, \frac{1}{L}]$, and let $\operatorname*{argmin}_{x \in X} J \neq \emptyset$. Then, the proximal gradient algorithm converges. More precisely,

$$0 \leq J(y^k) - J(y^*) \leq \frac{1}{2k\tau_{\text{min}}} \left\| y^0 - y^* \right\|_2^2 = O\left(\tfrac{1}{k}\right)$$

for $y^* \in \operatorname*{argmin}_{x \in X} J$. Moreover, $(y^k)_k$ converges to the set of minimizers, i.e.

$$\lim_{k \to \infty} \operatorname{dist}(y^k, \operatorname{argmin} J) = 0.$$

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  If ∇J is Lipschitz continuous, we get convergence for suitable τ_n.

The proximal gradient algorithm - Special cases

The above also proves convergence of other methods.

- $G = 0$, $H = J \Rightarrow$ proximal point algorithm
  Since $\nabla 0$ is Lipschitz continuous with constant 0, it follows the convergence for arbitrary, bounded step sizes.
- $G = J$, $H = 0 \Rightarrow$ fully explicit gradient descent
  If ∇J is Lipschitz continuous, we get convergence for suitable τ_n.
- $C \subset \mathbb{R}^n$ nonempty, convex and closed, $G = J$, $H = I_C$
  $\Rightarrow$ *projected gradient descent*, which minimizes $J(y)$ under the constraint $y \in C$. If ∇J is Lipschitz continuous, we get convergence for suitable τ_n.

Primal-dual approaches

Let $J : X \to \mathbb{R}_\infty$ be proper. Then, *Fenchel conjugate* of J denotes

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The necessary conditions for z and y are

$$0 \in \partial_z (Ay \cdot z - H^*(z)) = Ay - \partial H^*(z) \ \Rightarrow \ Ay \in \partial H^*(z),$$

$$0 \in \partial_y \left(A^T z \cdot y + G(y) \right) = A^T z + \partial G(y) \ \Rightarrow \ -A^T z \in \partial G(y).$$

Primal-dual approaches (cont.)

$Ay \in \partial H^*(z) \;\wedge\; -A^T z \in \partial G(y)$ is equivalent to

$$z = \mathrm{prox}_{\sigma H^*}[z + \sigma A y] \;\wedge\; y = \mathrm{prox}_{\tau G}[y - \tau A^T z]$$

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- avoids computing $\operatorname{prox}_{\sigma H(A\cdot)}$, just needs $\operatorname{prox}_{\sigma H^*}$, A, A^T
- typically $\operatorname{prox}_{\sigma H^*}$ can be computed pointwise
- proposed in [Chambolle, Pock '10], very popular ($4300+$ citations)
- also called *Primal-Dual Hybrid Gradient* (PDHG) method
- well suited for models that use the total variation as regularizer
- convergence guaranteed if $\theta = 1$, $\tau\sigma \|A\|^2 < 1$

PDHG extensions and variants

- $\theta = 0$: Arrow–Hurwicz algorithm, used for TV minimization before [Zhu, Chan '08]
- diagonal preconditioning [Pock, Chambolle '11]
- extension to Banach spaces with applications to inverse problems [Hohage, Homann '14]
- stochastic extension based on arbitrary sampling of the dual variables [Chambolle, Ehrhardt, Richtárik, Schönlieb, '18]
- stochastic PDHG to solve regularized stochastic minimization problems [Qiao, Lin, Qin, Lu Neurocomputing '18]
- learned primal-dual reconstruction [Adler, Öktem TMI '18]
- Riemannian Chambolle–Pock algorithm [Bergmann, Herzog, Louzeiro, Tenbrinck Vidal-Núñez '21]
- ...

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ROF model $J[y] = \int_\Omega (y - f)^2 \, \mathrm{d}x + \lambda \int_\Omega \|\nabla y(x)\| \, \mathrm{d}x$

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With $\mathbb{R}^n \simeq X$, $\mathbb{R}^m \simeq X \times X$ and $A \simeq \nabla^h$, $G(y) + H(Ay)$ is a discretization of the ROF-model fitting the framework.

A primal-dual approach for the ROF model (cont.)

One can show, that $H^* = I_P$, where

$$P = \left\{ p \in X \times X \;:\; \|p\|_\infty := \max_{i,j} |p_{i,j}| \leq 1 \right\}.$$

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Moreover, one can show $\left|\left|\left|\nabla^h\right|\right|\right|^2 \leq \frac{8}{h^2}$.

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`https://berkels.github.io/jupyterlite-demo`

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There are variants of the primal-dual method, which exploit the strict convexity of the data term G for an even faster convergence.

Mumford-Shah based image segmentation