A Hierarchy of Theta Bodies for Polynomial Systems

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The Theta Body of a Graph

$G = ([n], E)$ graph

$\chi^S \in \{0,1\}^n$ incidence vector of stable set S in G

- STAB(G) := conv$\{\chi^S : S$ stable set in $G\}$

- theta body of G (Lovász):

TH(G) := $\{\mathbf{x} \in \mathbb{R}^n : \begin{bmatrix} 1 & \mathbf{x}^t \\ \mathbf{x} & U \end{bmatrix} \succeq 0, \text{diag}(U) = \mathbf{x}, U_{ij} = 0 \, \forall \{i,j\} \in E\}$

- QSTAB(G) := $\{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \, (i \in [n]), \; \sum_{i \in C} x_i \leq 1 \, (C \text{ clique})\}$

STAB$(G) \subseteq$ TH$(G) \subseteq$ QSTAB(G)

STAB(G) = TH(G) = QSTAB(G) $\Leftrightarrow$ G is perfect

Connection to Sums of Squares of Polynomials

$I_G := \langle x_i - x_i^2 \, (i \in [n]), \; x_i x_j \, (\{i,j\} \in E) \rangle \subseteq \mathbb{R}[x_1, \ldots, x_n]$

$\mathcal{V}_{\mathbb{R}}(I_G) = \{\chi^S : S$ stable in $G\}$ (real zeros of I_G)

Recall: If I ideal in $\mathbb{R}[\mathbf{x}]$ then:

- $f \equiv g \bmod I \Leftrightarrow f - g \in I \Leftrightarrow \forall \, \mathbf{p} \in \mathcal{V}_{\mathbb{R}}(I), \; f(\mathbf{p}) = g(\mathbf{p})$
- $f \equiv \sum h_j^2 \bmod I \Rightarrow f(\mathbf{x}) \geq 0$ valid on $\mathcal{V}_{\mathbb{R}}(I)$.

  If f is linear then $f(\mathbf{x}) \geq 0$ is valid on $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$.

(Lovász): $\mathrm{TH}(G) = \{\mathbf{x} \in \mathbb{R}^n : \; f(\mathbf{x}) \geq 0 \; \forall$ linear f such that

$f \equiv \sum h_j^2$ mod I_G with h_j linear $\}$

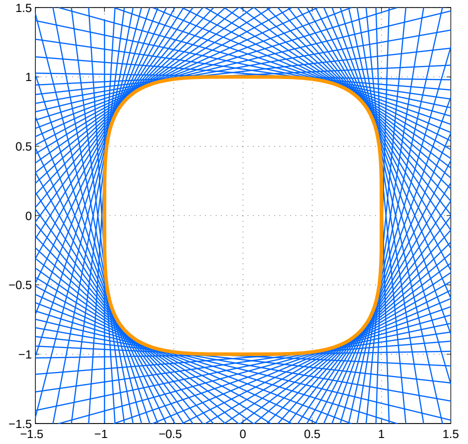
Approximating $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ via Sums of Squares

⋆ $\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$ cut out by all linear $f \in \mathbb{R}[\mathbf{x}]$ s.t.

$$f(\mathbf{p}) \geq 0 \;\forall\, \mathbf{p} \in \mathcal{V}_{\mathbb{R}}(I)$$

⋆ A Certificate of $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I)$:

$$f \equiv \sum h_j^2 \bmod I$$



Definitions: $f \in \mathbb{R}[\mathbf{x}]$, $I \subseteq \mathbb{R}[\mathbf{x}]$ ideal:

- If $f \equiv \sum h_j^2 \bmod I$, $\deg(h_j) \leq k$, say f is k-sos mod I
- I is $(1,k)$-sos if $f \geq 0$ on $\mathcal{V}_{\mathbb{R}}(I)$ & f linear $\Rightarrow$ f k-sos mod I

Lovász: Which ideals are $(1,1)$-sos or “perfect”?

Motivation: I_G is $(1,1)$-sos $\Leftrightarrow$ G is perfect

Theta bodies of polynomial ideals (Gouveia-Parrilo-T)

$$\text{TH}_k(I) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq 0 \; \forall \; f \text{ linear \& k-sos mod } I\}$$

$$\text{TH}_1(I) \supseteq \text{TH}_2(I) \supseteq \cdots \supseteq \overline{\text{conv}(\mathcal{V}_{\mathbb{R}}(I))} \text{ —}(*)$$

$$\boxed{\text{TH}(G) = \text{TH}_1(I_G)}$$

Theorem (GPT):

(i) $\text{TH}_k(I)$ is the projection of a spectrahedron (is a Lasserre relaxation)

(ii) Compute using combinatorial moment matrices (Laurent)

Definition: I is TH_k-exact if $\text{TH}_k(I) = \overline{\text{conv}(\mathcal{V}_{\mathbb{R}}(I))}$

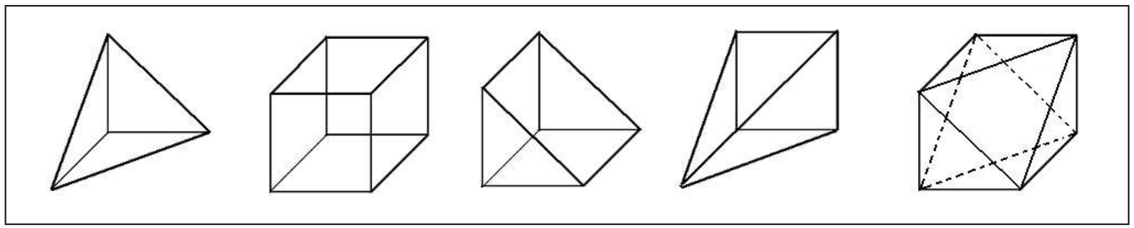
$$\boxed{I \text{ is } (1,k)\text{-sos} \Rightarrow I \text{ is } \text{TH}_k\text{-exact}}$$

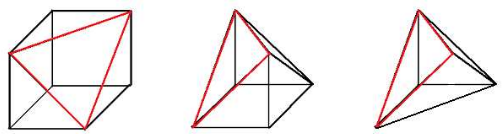
Finite Point Sets

I is the vanishing ideal of a finite set $S \subset \mathbb{R}^n$ (eg. I_G)

Theorem (GPT):

- I is $(1,k)$-sos $\Leftrightarrow$ I is TH$_k$-exact
- (Answer to Lovász Qn) The following are equivalent:
  - I is TH$_1$-exact (i.e., $\text{conv}(S) = \text{TH}_1(I)$)
  - $\text{conv}(S)$ has a (finite) linear inequality description in which $\forall$ $f(\mathbf{x}) \geq 0$, $S \subseteq \{f(\mathbf{x}) = 0\} \cup \{f(\mathbf{x}) = 1\}$ (S is 2-level)





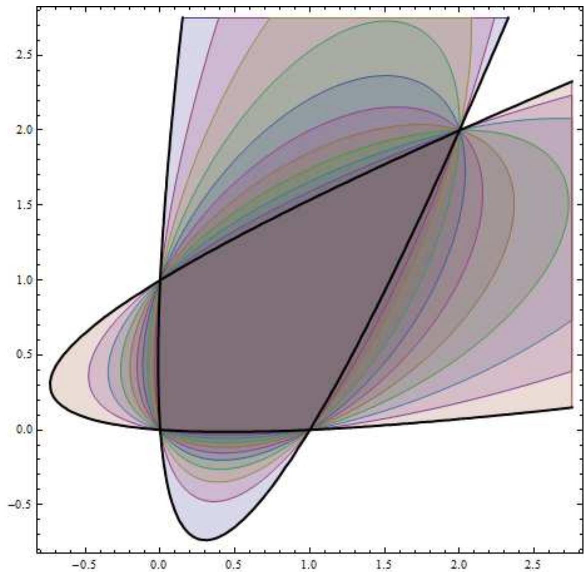
Corollaries: If I is TH_1-exact then:

- conv(S) affinely equivalent to a $0/1$-polytope.
- conv(S) has at most 2^n vertices and facets. Both bounds are sharp — cube and cross-polytope.
- A down-closed $0/1$-polytope is 2-level if and only if it is STAB(G) for a perfect graph.
- G is perfect $\Leftrightarrow$ STAB(G) is 2-level

Geometry of theta bodies

Theorem (GPT): $\text{TH}_1(I) = \bigcap\{\text{conv}(\mathcal{V}_{\mathbb{R}}(F)) \,:\, F \text{ convex quadric in } I\}$

Ex. $I =$ Vanishing ideal of $\{(0,0),(1,0),(0,1),(2,2)\}$



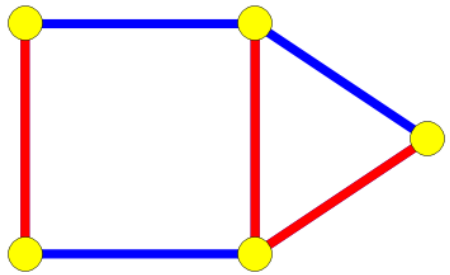
Question: What is the geometry of higher theta bodies?

Cuts in $G = ([n], E)$ (GPT+Laurent)

$D \subseteq E$ cut in G

$\chi^D : (\chi^D)_e = \begin{cases} -1 & \text{if } e \in D \\ 1 & \text{if } e \notin D \end{cases}$

CUT(G) := conv$\{\chi^D : D$ cut in $G\}$



$IG = \langle x_e^2 - 1 \ (e \in E),\ 1 - \mathbf{x}^C \ (C$ chordless circuit in $G)\rangle$

Theorem (GLPT): G is "cut-perfect" (i.e.,CUT$(G) =$ TH$_1(IG)$) $\Leftrightarrow$ G has no K_5-minor and no induced cycles of length ≥ 5. (a Lovász qn)

More generally: $\mathcal{M} = (E, \mathcal{D})$ binary matroid without $F_7^*, R_{10}, \mathcal{M}_{K_5}^*$ minors $\Rightarrow$ $I\mathcal{M}$ is TH$_1$-exact $\Leftrightarrow$ $\mathcal{M}$ does not have chordless cocircuits of length ≥ 5. (uses Barahona-Grötschel)

Theta Bodies for Cuts

- For $T \subseteq [n]$, $F \subseteq E$ is a T-join if $T = \{v \in [n] : \deg_F(v) \text{ odd}\}$.
- $\mathcal{F}_k$ – all T-joins of size $\leq k$ no two of which share the same T.

  Eg. $\mathcal{F}_1 = \{\emptyset\} \cup E$

$\text{TH}_1(IG) = \{\mathbf{y} \in \mathbb{R}^E : \exists\, X \succeq 0,$ indexed by $\{\emptyset\} \cup E$ with

- $X_{\emptyset,e} = y_e$, • $\text{diag}(X) = \mathbf{1}$, • $X_{e,f} = y_g$ if e, f, g triangle,
- $X_{e,f} = X_{g,h}$ if e, f, g, h circuit $\}$

Remark: If G^* is the suspension of G from a new vertex, then

$\text{proj}_E(\text{TH}_1(IG^*)) \subseteq$ Goemans-Williamson relaxation.

Higher order exactness

(Not much known)

Stable sets:

- G odd cycle $\Rightarrow I_G$ is TH_2-exact
- For any G, the odd-cycle, odd-wheel, clique inequalities are valid for $\text{TH}_2(I_G)$

Cuts:

- G has diameter ≤ 2 and no K_5 minor $\Rightarrow IG$ is TH_2-exact
- For any G, all cycle inequalities

$$\mathbf{x}(F) - \mathbf{x}(C \setminus F) \geq 2 - |C|, \ F \subseteq C, \ |F| \text{ odd}$$

from cycles C of length at most $4k$ are valid for $\text{TH}_k(IG)$.