A Geometric Interpretation of the Characteristic Polynomial of a Hyperplane Arrangement

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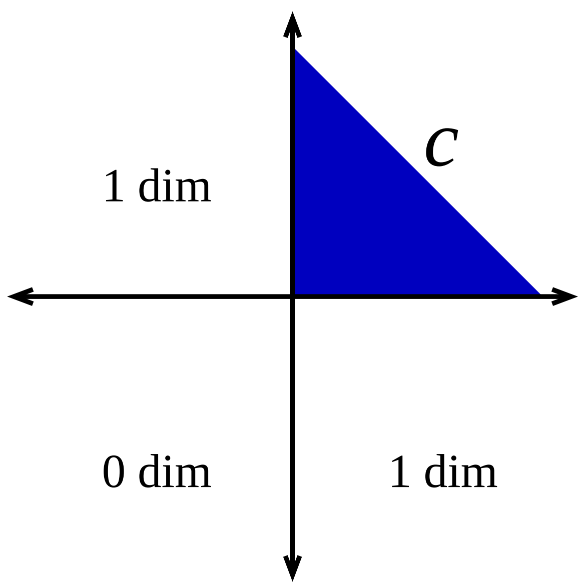
Projection Volumes

- $\mathcal{C}$ = Polyhedral cone in $\mathbb{R}^n$, $\pi_{\mathcal{C}}(x)$ = orthogonal projection onto $\mathcal{C}$
- $\pi_{\mathcal{C}}(x)$ is k-dim if it is in the relative interior of a k-dim face of $\mathcal{C}$
- ν_k = ratio of volume of $\mathbb{R}^n$ for which $\pi_{\mathcal{C}}(x)$ is k-dimensional

PROBLEM *Given a cone $\mathcal{C}$ compute the projection volumes ν_k*



(1/4, 1/2, 1/4)

STATISTICAL MOTIVATION: HYPOTHESIS TESTING

- $X_1 \sim N(\mu_1, \sigma_0^2)$ and $X_2 \sim N(\mu_2, \sigma_0^2)$
- Wish to test hypothesis about the unknown means $\mu_1, \mu_2 \in \mathbb{R}$:

$$H_0 : \ \mu_1 \geq \mu_2 \geq 0 \quad \text{versus} \quad H_1 : \ (\mu_1 < \mu_2 \text{ or } \mu_2 < 0)$$

EXAMPLE:

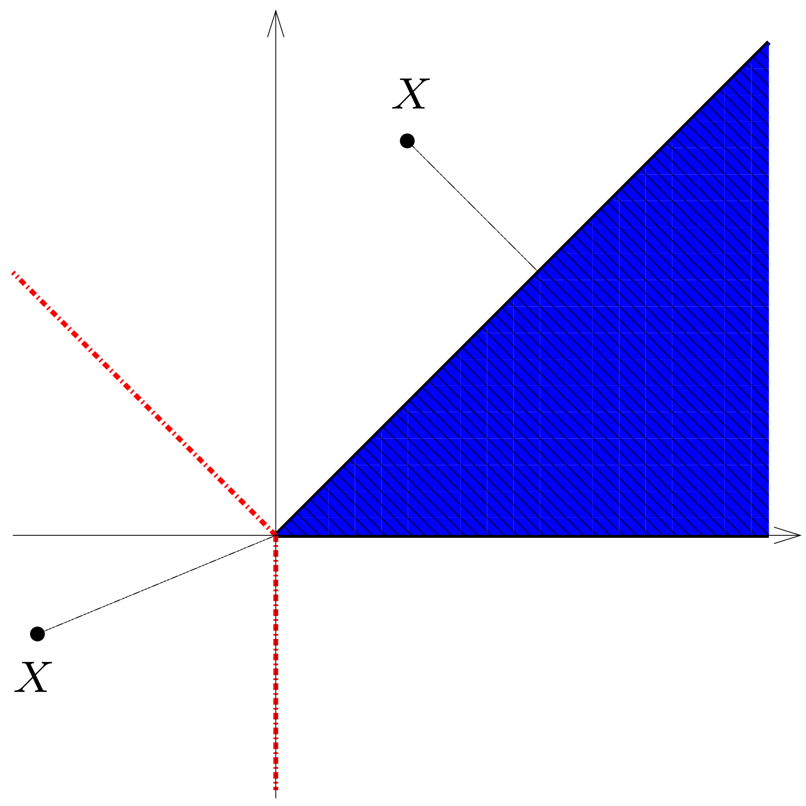
X_1 : Difference in blood pressure before and after taking ‘drug 1’

X_2 : Difference in blood pressure before and after taking ‘drug 2’

(X_1 and X_2 could be average differences for n patients)

Likelihood ratio test

Reject H_0 if (squared) Euclidean distance between $X = (X_1, X_2)$ and the cone $\mathcal{C} = \{\mu \in \mathbb{R}^2 \mid \mu_1 \geq \mu_2 \geq 0\}$ is 'too large'.



WHAT IS 'TOO LARGE'?

- Project data point $X = (2, 4)$ onto cone: $\pi_{\mathcal{C}}(X) = (3, 3)$
- Squared distance: $||X - \mathcal{C}||^2 = 1 + 1 = 2$
- Compute 'p-value':

$$p = \sup_{\mu \in \mathcal{C}} P_\mu(||X - \mathcal{C}||^2 \geq 2) = P_{\mu=0}(||X - \mathcal{C}||^2 \geq 2)$$

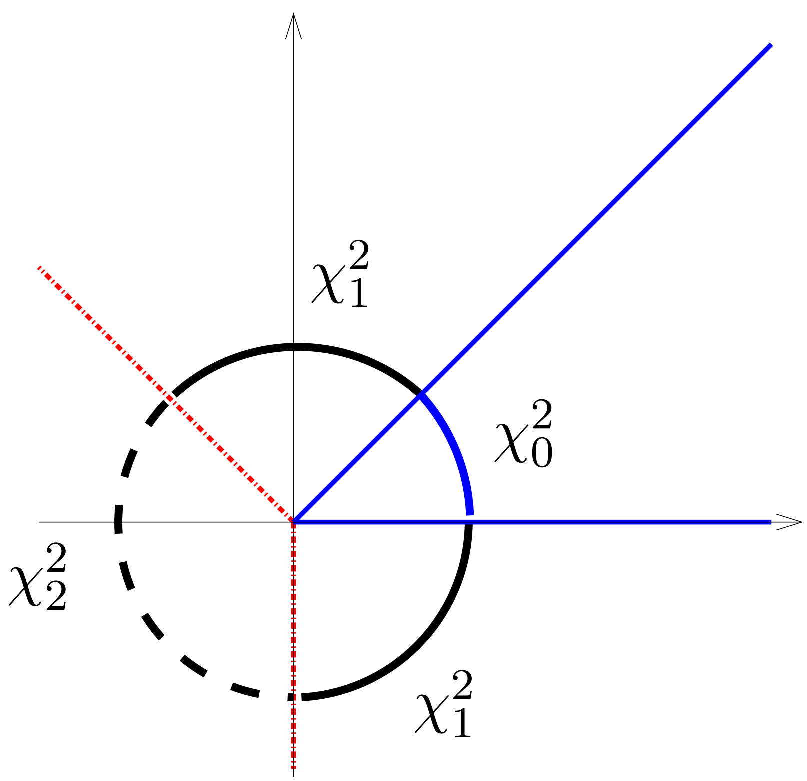
- If p is small, say smaller than 0.05, then reject H_0.

How does one compute $P_{\mu=0}(||X - \mathcal{C}||^2 \geq 2)$?

Mixture of chi-square distributions

$$P_{\mu=0}(||X - \mathcal{C}||^2 \geq 2) =$$

$$\frac{1}{8} P(\chi_0^2 \geq 2) + \frac{1}{2} P(\chi_1^2 \geq 2) + \frac{3}{8} P(\chi_2^2 \geq 2) \approx 0.22$$



AVERAGES OVER ARRANGEMENTS

- Consider cones $\mathcal{C}$ given by the closure of a region of a linear hyperplane arrangement.
- Consider the average projection volumes over all regions.

EXAMPLE: Any two lines in $\mathbb{R}^2$

Average volumes will always be $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$

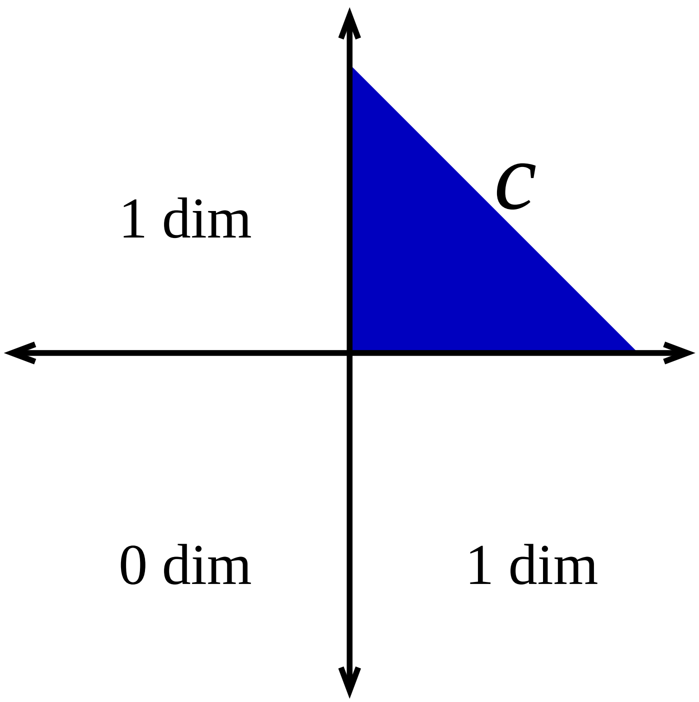
EXAMPLE: Any n lines in $\mathbb{R}^2$

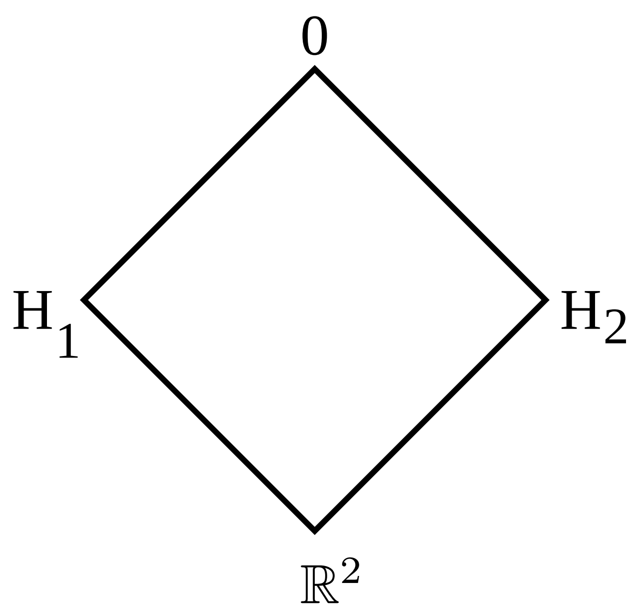
Average volumes will always be $\left(\frac{1}{2n}, \frac{n}{2n}, \frac{n-1}{2n}\right)$

Hyperplane Arrangements

- $\mathcal{L}(\mathcal{H})$ = Set of all intersections of collections of hyperplanes of $\mathcal{H}$ (include $\mathbb{R}^n$ for the empty intersection)

  $\mathcal{L}(\mathcal{H})$ forms a lattice under reverse inclusion of intersections.





Characteristic Polynomial

- The Characteristic polynomial:

$$\chi_{\mathcal{H}}(t) = \sum_{x \in L(\mathcal{H})} \mu(x) t^{\dim(x)}$$

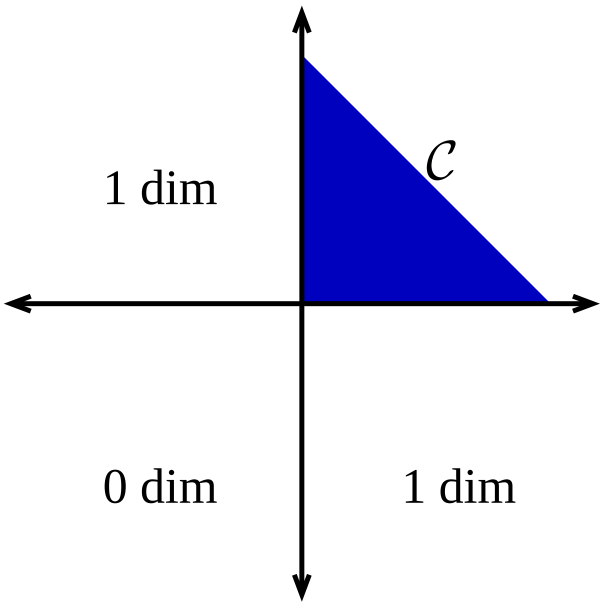
Möbius function $\mu : L(\mathcal{H}) \to \mathbb{Z}$

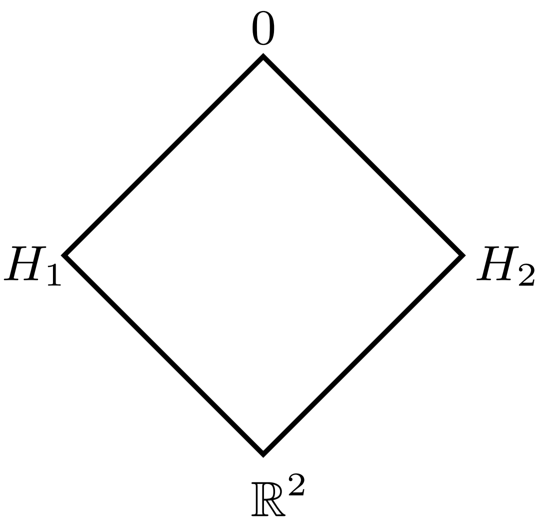
$$\mu(\mathbb{R}^n) = 1 \text{ and } \sum_{z \leq y} \mu(z) = 0$$

- The Poincaré polynomial $\pi(\mathcal{H}, t)$ is related by:

$$\chi_{\mathcal{H}}(t) = t^{\mathsf{Rk}(\mathcal{H})} \pi(\mathcal{H}, -t^{-1})$$

Example





$$\chi(t) = t^2 - 2t + 1$$

COEFFICIENTS

THEOREM *The average projection volumes are given by the absolute values of the coefficients of* $\chi(t)$*:*

$$\frac{\sum_{\mathcal{C}} \nu_k}{\#\mathcal{C}} = \frac{|\alpha_k|}{|\alpha_n| + \cdots + |\alpha_0|} = \frac{|\alpha_k|}{\#\mathcal{C}}$$

COROLLARY *If all regions of* $\mathcal{H}$ *are isometric, then the projection volumes of any region* $\mathcal{C}$ *are given by the coefficents of* $\chi(t)$:

$$\nu_k = \frac{|\alpha_k|}{\#\mathcal{C}}$$

Angle Sums of Polytopes

- F = k-dimensional face of a polytope P
- z = some point in the relative interior of F
- $\alpha_k(P, F)$ = the ratio of volume of an epsilon ball centered at z that lies in the interior of P
- The kth angle sum of P:

$$\phi_k(P) = \sum_{\text{dim } F = k} \alpha_k(P, F)$$

$\phi(P) = (\phi_0(P), \ldots, \phi_{d-1}(P))$ satisfy relations similar to f-vectors

EQUIPROJECTIVE POLYTOPES

DEFINITION A polytope P is called equiprojective if for all hyperplanes H in sufficiently generic position to P, the orthogonal projections of P onto all H have the same face numbers.

Example: 3-Cube $\longrightarrow$ Hexagon

THEOREM (Perles & Shephard) Let P be an equiprojective polytope, and P_0 the orthogonal projection of P onto a hyperplane in generic position.

$$2\phi(P) = (f(P) - f(P_0)).$$

ZONOTOPES

THEOREM (Shephard) Zonotopes are equiprojective.

- Z a zonotope
- L the intersection lattice of the associated arrangement

THEOREM (Zaslavsky)

$$f_i(Z) = \sum_{x \leq y, \mathrm{rk}(x)=i} (-1)^{\mathrm{rk}(y)-\mathrm{rk}(x)} \mu_L(x, y)$$

Note: The face structure of the arrangement is dual to that of the associated zonotope. In particular, any $x \in L(\mathcal{H})$ is associated to a collection of (isometric) faces of the zonotope Z.

INTERSECTION VOLUMES

LEMMA Let $x \in L(\mathcal{H})$ be an intersection of dim k. Let F_x be one of the faces associated to x of the associated zonotope. Then,

$$\mu_L(0, x) = \phi_0(F_x) = \nu_k(x)$$

Namely, for a fixed intersection x in the hyperplane arrangement, the vertex angle sums of the associated face of the zonotope gives both:

- The Möbius function at x

  Equiprojection corresponds to matroid truncation

- The total projection volume onto the intersection.

  Project onto the subspace orthogonal to x

Reflection groups

- $\mathcal{W} \subset GL(\mathbb{R}^n)$: Finite real reflection group
- Reflection in $\mathbb{R}^n$ is an isometry fixing the points of a hyperplane (mirror of reflection)
- Reflection arrangement or Coxeter arrangement $\mathcal{H}$ is the collection of all mirrors of a finite reflection group.
- A Fundamental chamber is the closure $\mathcal{C}$ of a region of $\mathbb{R}^n \backslash \mathcal{H}$ (All chambers are isometric)

Coefficients

Theorem *Let $\mathcal{W}$ be a finite reflection group, and $\chi_{\mathcal{W}}(t)$ the associated characteristic polynomial. The projection volumes ν_k are given by the coefficients of $\chi(t)$:*

$$\nu_k = \frac{|\alpha_k|}{|\alpha_n| + \cdots + |\alpha_0|} = \frac{|\alpha_k|}{\#\mathcal{W}}$$

- Connection to the group:

  Let x be a generic point in the fundamental chamber $\mathcal{C}$. Then the coefficient $|\alpha_k|$ is equal to the number of group elements $g \in \mathcal{W}$ for which the projection $\pi_{\mathcal{C}}(gx)$ is k-dimensional.

Coxeter Arrangements

- $|\alpha_k|$ is also known to be the number of group elements in $\mathcal{W}$ that leave fixed all points of some linear space of dimension $n-k$
- Top Coefficient: Action of $\mathcal{W}$ is simply transitive

$$\nu_n = 1/\#\mathcal{W}$$

- If $e_1, e_2, \ldots, e_n$ are the exponents of the group $\mathcal{W}$

$$\chi_{\mathcal{W}}(t) = (t-e_1)(t-e_2)\ldots(t-e_n)$$

- Bottom Coefficient: De Concini, Procesi, Stembridge, Denham

$$\nu_0 = \frac{|e_1 \cdots e_n|}{\#\mathcal{W}}$$

Infinite Families

- A_n, B_n, D_n
- Decompose $\mathcal{W}_n$ into $e_n + 1$ pieces.

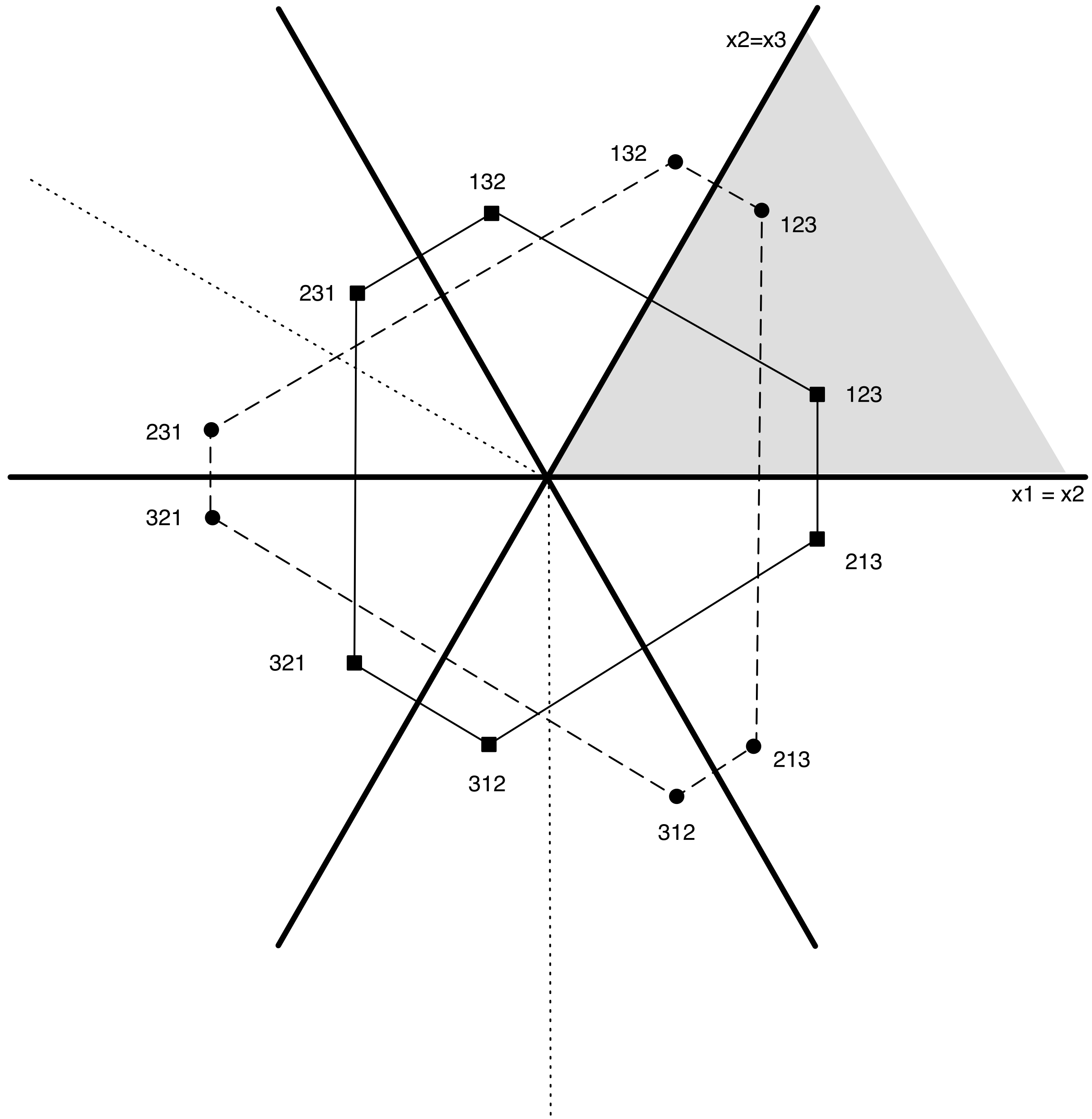
  One isomorphic to a reflection group of lower rank $\mathcal{W}_{n-1}$

  e_n in one-to-one correspondance with $\mathcal{W}_{n-1}$ s.t.

$$t\,\chi_{\mathcal{W}_{n-1}}(t) - e_n\,\chi_{\mathcal{W}_{n-1}}(t) = \chi_{\mathcal{W}_n}(t)$$

  Subgroup: Dimension of projection goes up by one

  Copies: Each maps a point in $\mathcal{C}$ to the “wrong side” of a bounding hyperplane, utilize a weighted projection


x2=x3
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x1 = x2
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