Covering maxima

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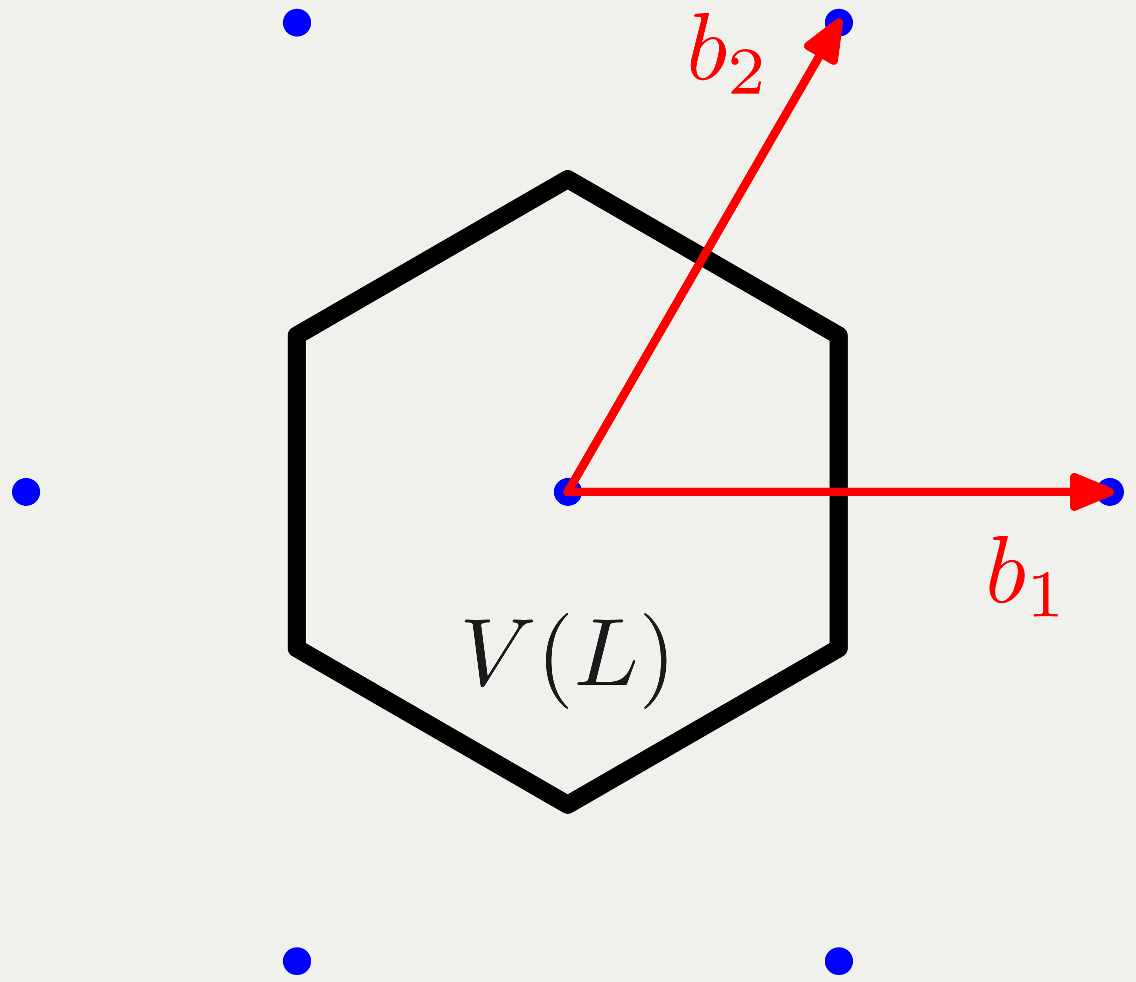
1999

Rudolf Scharlau

$b_1, \ldots, b_n$ basis of Euclidean space E

$L = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$ lattice

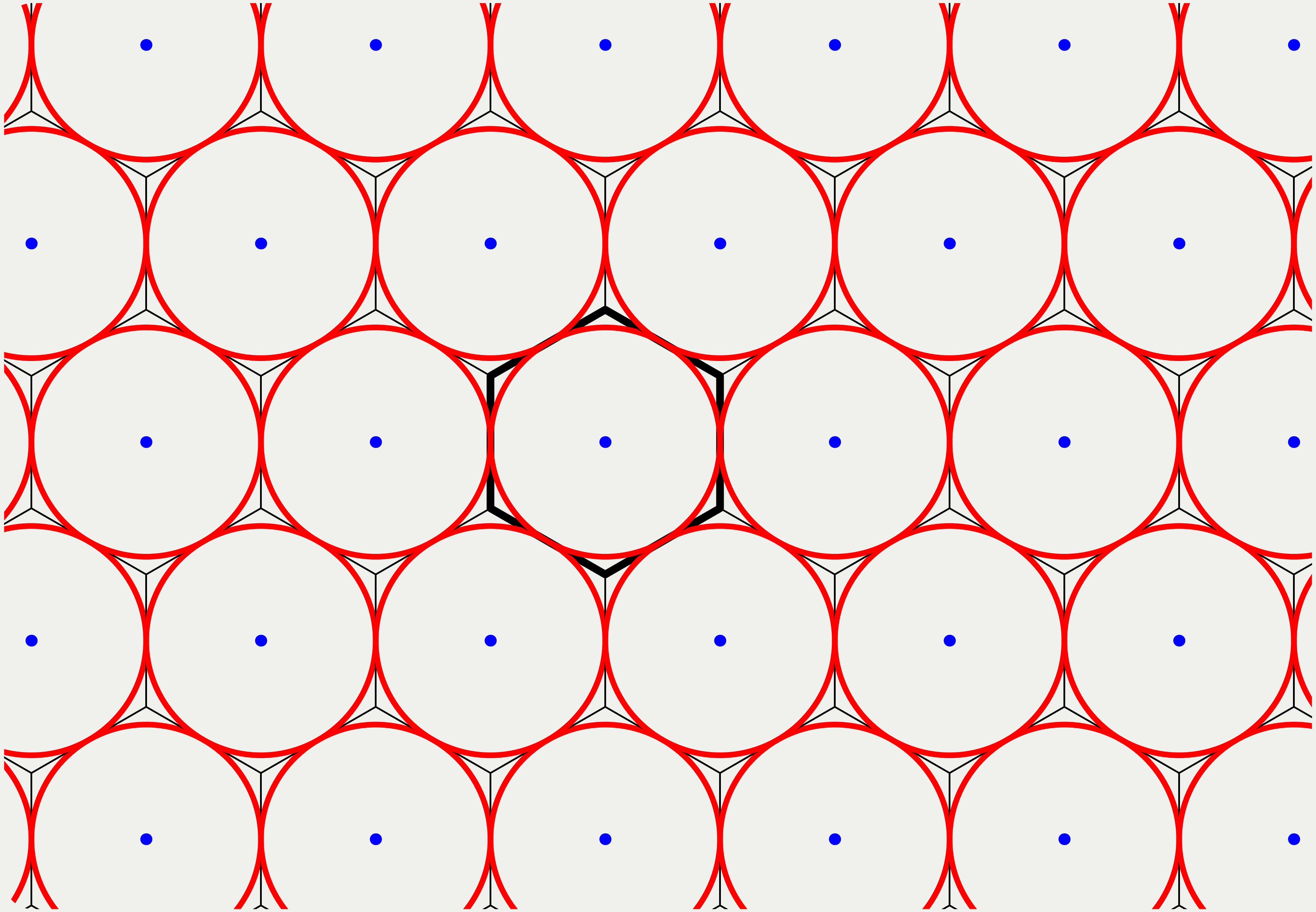


$V(L)$ = Voronoi cell

vol L = volume of $V(L)$

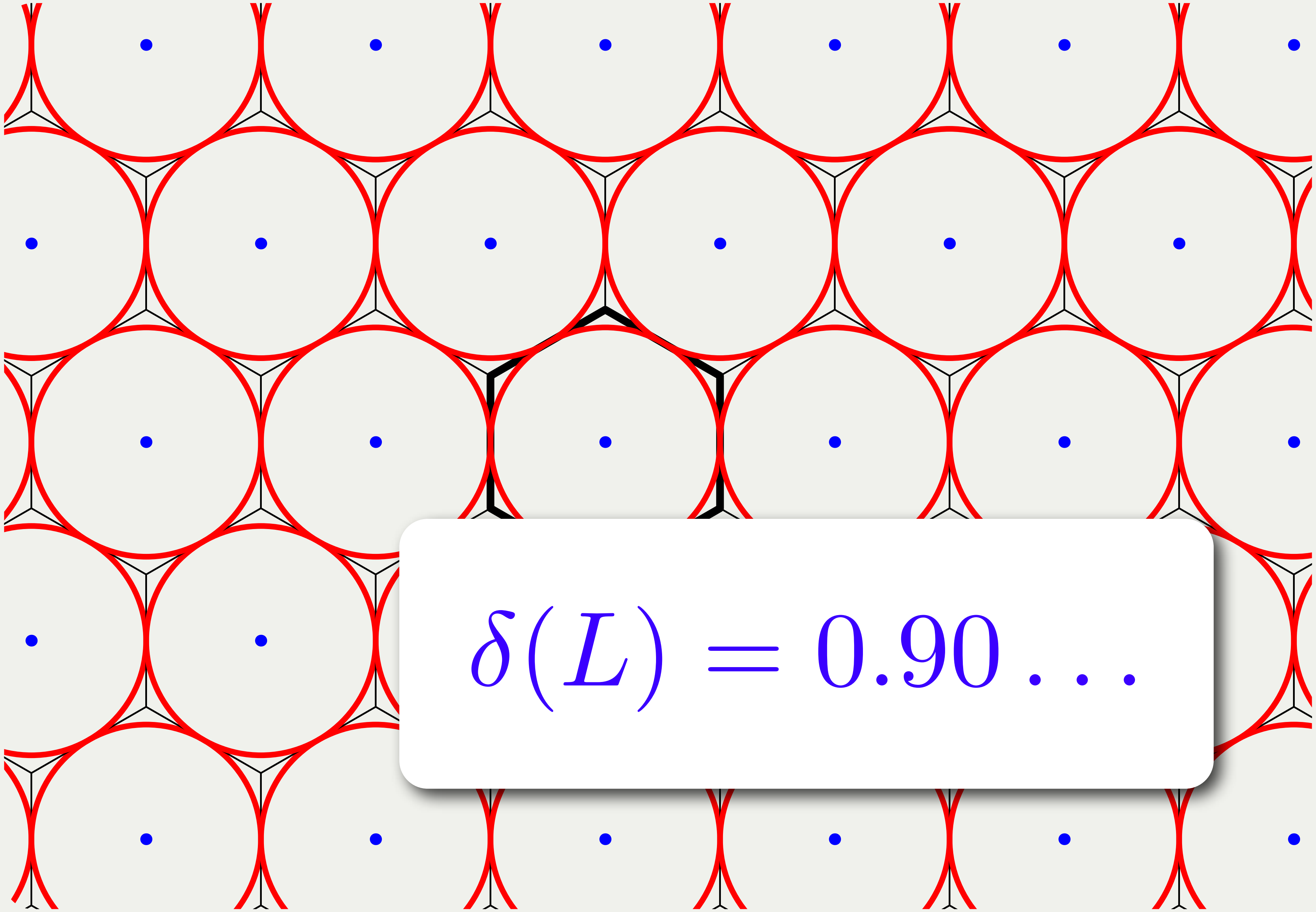
$\lambda(L)$ = inradius of $V(L)$

$\mu(L)$ = circumradius of $V(L)$



lattice packing problem $= \max_{L} \delta(L)$

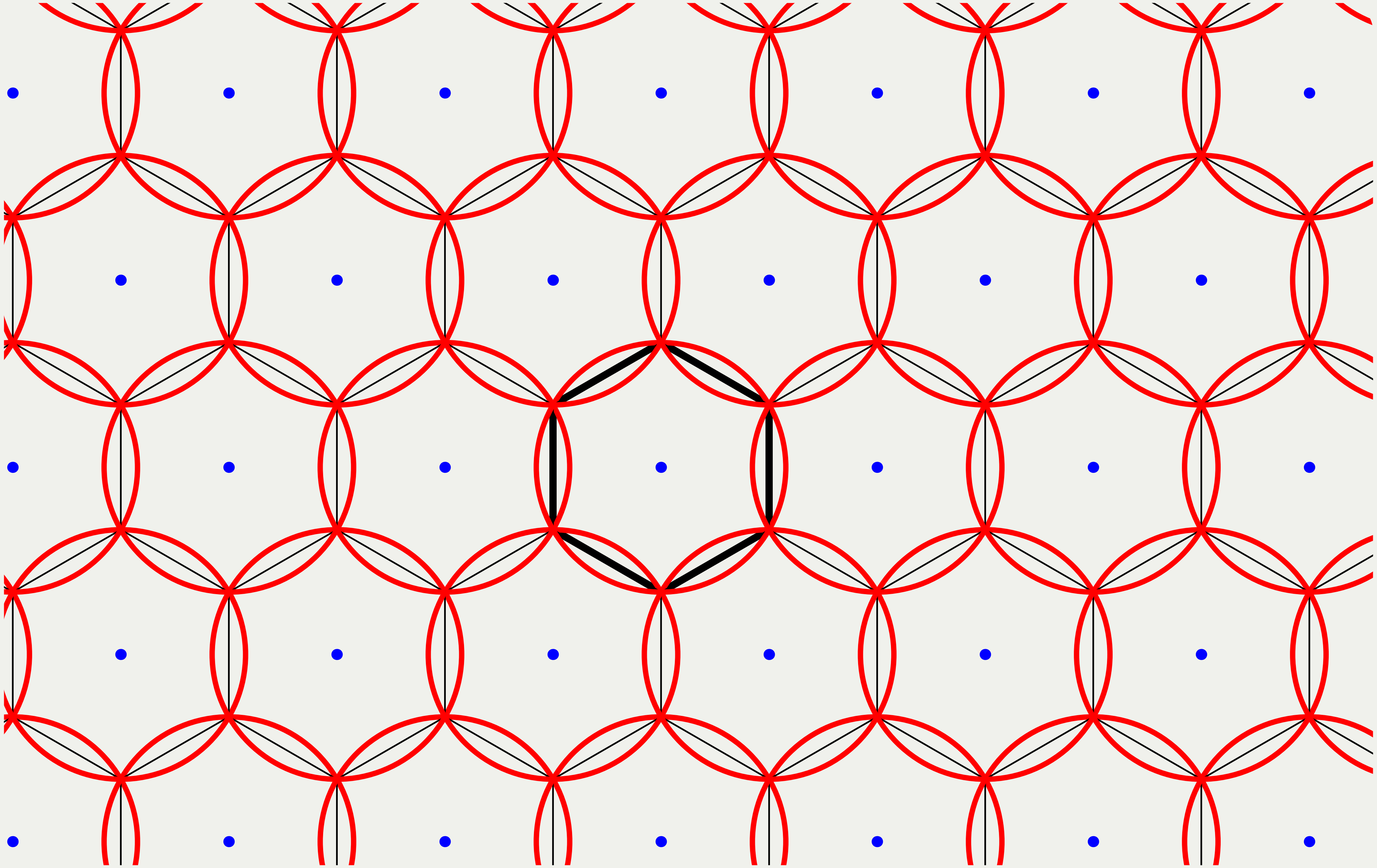
$$\delta(L) = \frac{\text{volume of ball w/ radius } \lambda(L)}{\text{vol } L} = \text{packing density}$$



$$\delta(L) = 0.90\ldots$$

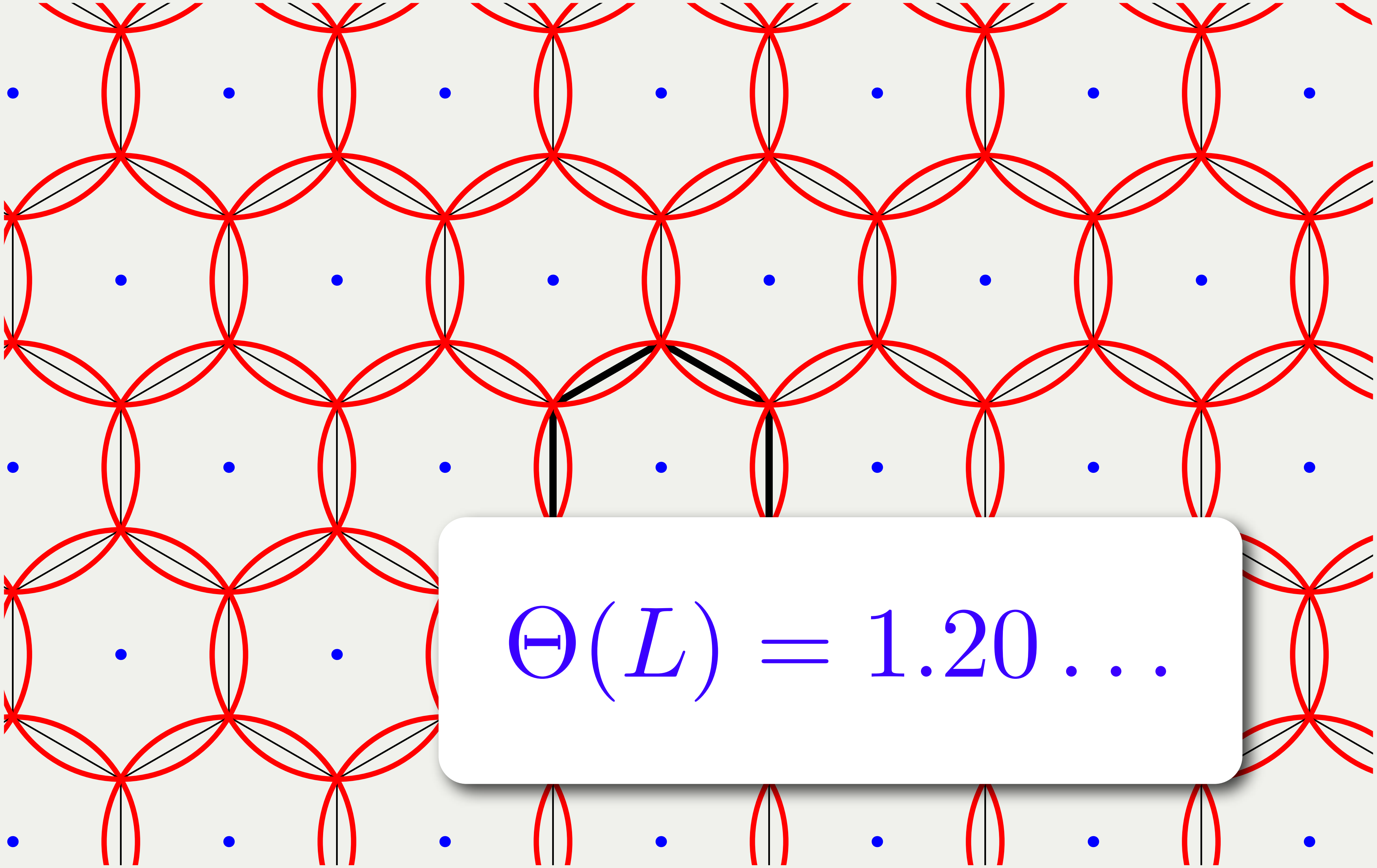
lattice packing problem $= \max_{L} \delta(L)$

$$\delta(L) = \frac{\text{volume of ball w/ radius } \lambda(L)}{\text{vol } L} = \text{packing density}$$



$$\text{lattice covering problem} = \min_{L} \Theta(L)$$

$$\Theta(L) = \frac{\text{volume of ball w/ radius } \mu(L)}{\text{vol } L} = \text{covering density}$$



$$\Theta(L) = 1.20\ldots$$

$$\text{lattice covering problem} = \min_{L} \Theta(L)$$

$$\Theta(L) = \frac{\text{volume of ball w/ radius } \mu(L)}{\text{vol } L} = \text{covering density}$$

Far easier than general sphere packing / covering

... but already difficult "enough"

- every dimension gives a new surprise
- many local optima
- computing packing density is NP-hard (Ajtai 1998)
- computing covering density conjectured to be NP-hard, probably not in NP

(DSV 2009: Counting vertices of a Voronoi cell is #P-hard.)

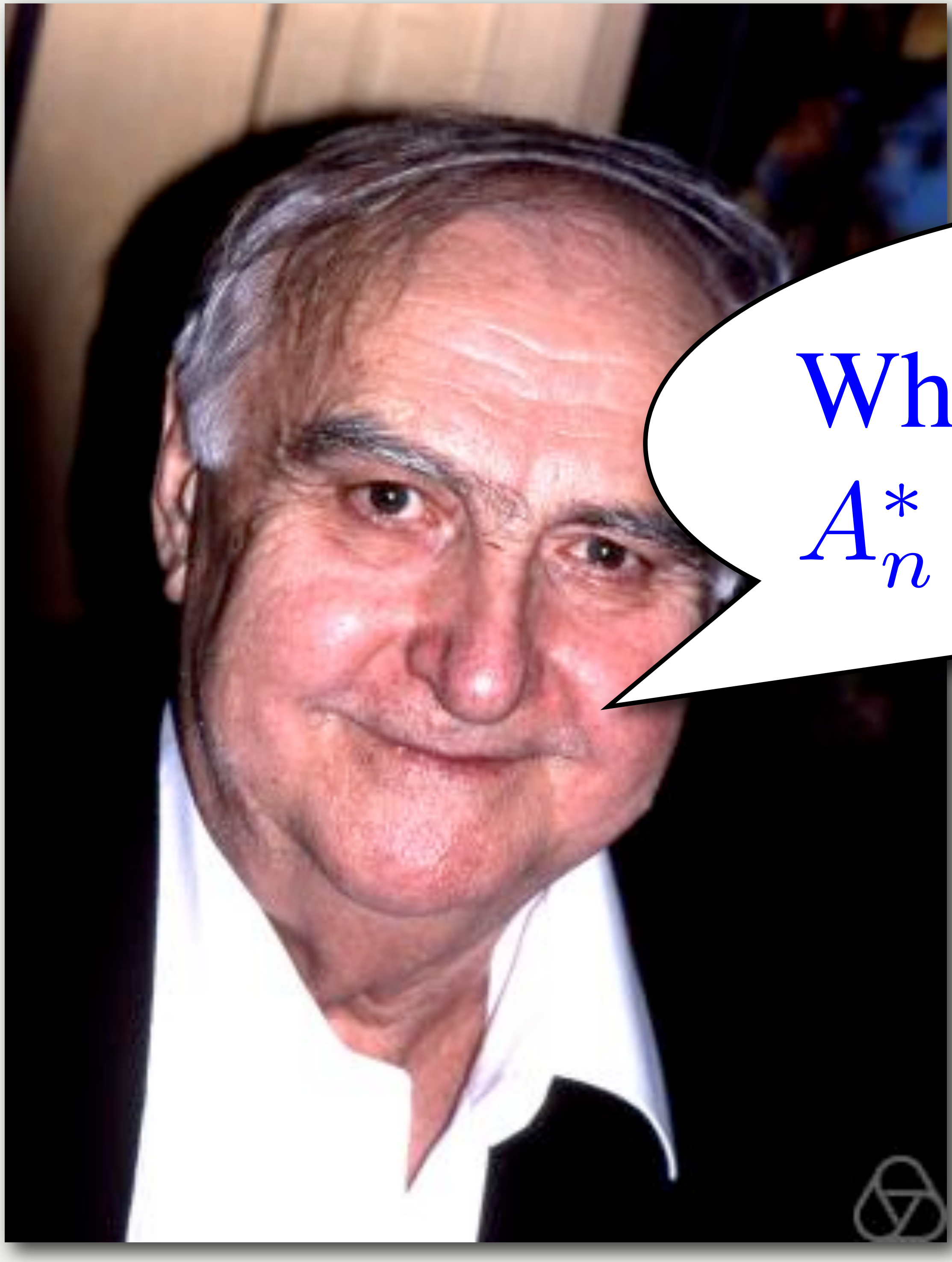
Known results

n	lattice	Θ
1	$\mathbb{Z}$	1
2	A_2^*	1.209199
3	A_3^*	1.463505
4	A_4^*	1.765529
5	A_5^*	2.124286

Solution in dimension 5 by Ryshkov, Baranovski (1975) in a 140-pages paper

$$\mathsf{A}_n = \left\{ x \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} x_i = 0 \right\} \qquad L^* = \{ y \in E : x \cdot y \in \mathbb{Z} \text{ for all } x \in L \}$$

1967



Sergey Ryshkov

What is the first n for which A_n^* is not optimal?

$n \leq 114$	Ryshkov 1967
$n \leq 24$	Conway, Parker, Sloane 1982
$n \leq 9$	Baranovski 1994
$n = 6$	Ph.D. thesis 2003

$$\begin{pmatrix} 2.0550 & -0.9424 & 1.1126 & 0.2747 & -0.9424 & -0.6153 \\ -0.9424 & 1.9227 & -0.5773 & -0.7681 & 0.3651 & -0.3651 \\ 1.1126 & -0.5773 & 2.0930 & -0.4934 & -0.5773 & -0.9804 \\ 0.2747 & -0.7681 & -0.4934 & 1.7550 & -0.7681 & 0.7681 \\ -0.9424 & 0.3651 & -0.5773 & -0.7681 & 1.9227 & -0.3651 \\ -0.6153 & -0.3651 & -0.9804 & 0.7681 & -0.3651 & 1.9227 \end{pmatrix}$$

the candidate for optimal lattice covering

Algorithm which finds all local minima

∗ *classify all combinatorial types of Voronoi cells*

subdivide space of all lattices in polyhedral cones

∗ *for every type solve a convex optimization problem*

local lattice covering problem is "SDP representable"

∗ *issues*

- far too many types in dimension 6, only 222 in dimension 5
- solution generally not nice (rational)

The space of lattices

$$\overbrace{\mathrm{GL}_n(\mathbb{Z}) \backslash \mathrlap{\underbrace{\mathrm{GL}_n(\mathbb{R}) \,/\, \mathrm{O}(\mathbb{R}^n)}_{\substack{\text{semidefinite matrices} \\ S^n_{\geq 0}}}}\phantom{\mathrm{GL}_n(\mathbb{R}) \,/}}^{\text{lattices}}\phantom{\,\mathrm{O}(\mathbb{R}^n)}$$

lattice basis $b_1, \ldots, b_n$ $\longleftrightarrow$ semidefinite matrix $(b_i \cdot b_j)$

$$\overbrace{\mathrm{GL}_n(\mathbb{Z}) \backslash \rlap{$\underbrace{\phantom{\mathrm{GL}_n(\mathbb{R}) \,/\, \mathrm{O}(\mathbb{R}^n)}}_{\text{semidefinite matrices } S^n_{\geq 0}}$}\mathrm{GL}_n(\mathbb{R})}^{\text{lattices}} \,/\, \mathrm{O}(\mathbb{R}^n)$$

fundamental domain in $S^n_{\geq 0}$ for $\mathrm{GL}_n(\mathbb{Z})$-action

Decompose $S^n_{\geq 0}$ into polyhedral cones.

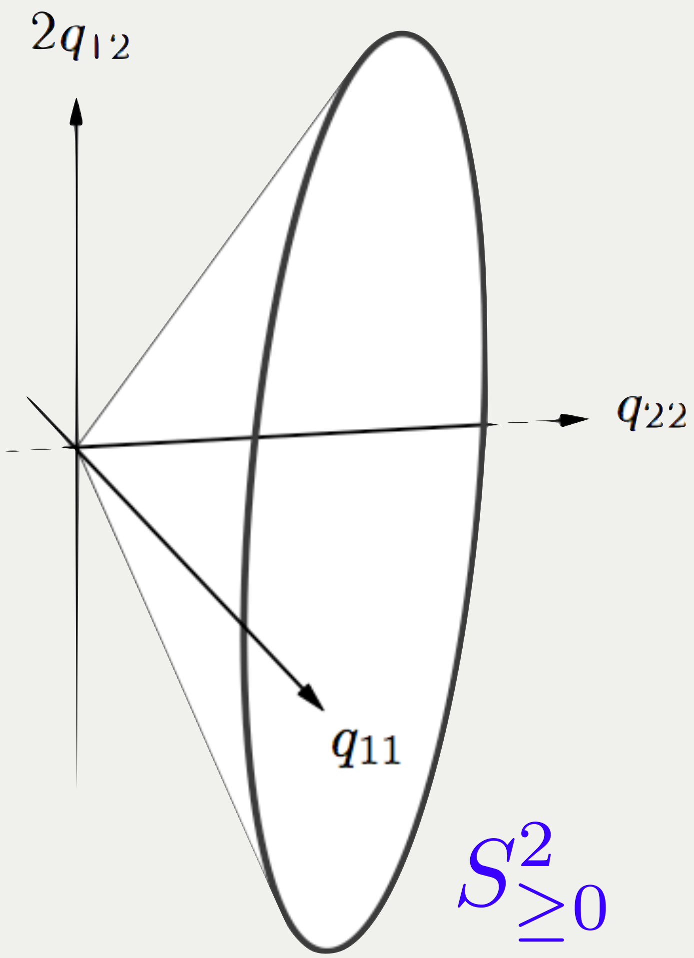
Voronoi I vs. Voronoi II

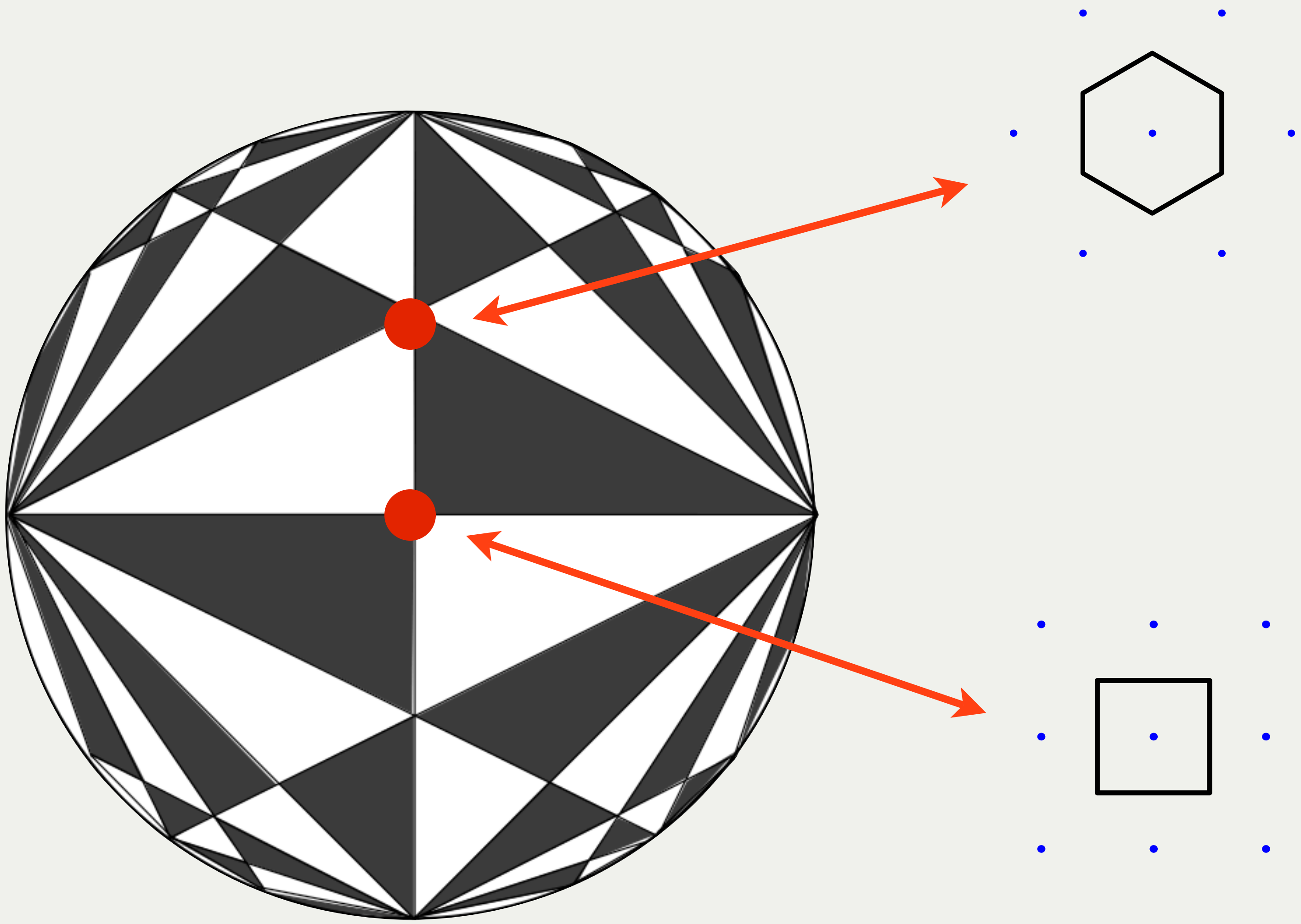
good for packing → Voronoi I

good for covering → Voronoi II

Voronoi II

polyhedral cone = domain where type of Voronoi cell is fixed





Running SDP gives

$$\begin{pmatrix} 2.0550 & -0.9424 & 1.1126 & 0.2747 & -0.9424 & -0.6153 \\ -0.9424 & 1.9227 & -0.5773 & -0.7681 & 0.3651 & -0.3651 \\ 1.1126 & -0.5773 & 2.0930 & -0.4934 & -0.5773 & -0.9804 \\ 0.2747 & -0.7681 & -0.4934 & 1.7550 & -0.7681 & 0.7681 \\ -0.9424 & 0.3651 & -0.5773 & -0.7681 & 1.9227 & -0.3651 \\ -0.6153 & -0.3651 & -0.9804 & 0.7681 & -0.3651 & 1.9227 \end{pmatrix}$$

with covering density 2.46...

covering density of A_6^* is 2.55...

1999

Rudolf Scharlau

Develop a (nice) theory for lattice coverings!

What happens to the nice lattices?

lattice	packing	covering
$\mathbb{Z}$	global opt.	global opt.
A_2	global opt.	global opt.
D_4	global opt.	not loc. opt.
E_6	global opt.	?
E_7	global opt.	?
E_8	global opt.	?
K_{12}	conj. global opt.	?
BW_{16}	conj. global opt.	?
Λ_{24}	global opt.	?

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E_8	global opt.	not loc. opt. (SV 2005)
K_{12}	conj. global opt.	?
BW_{16}	conj. global opt.	?
Λ_{24}	global opt.	loc. opt. (SV 2005)

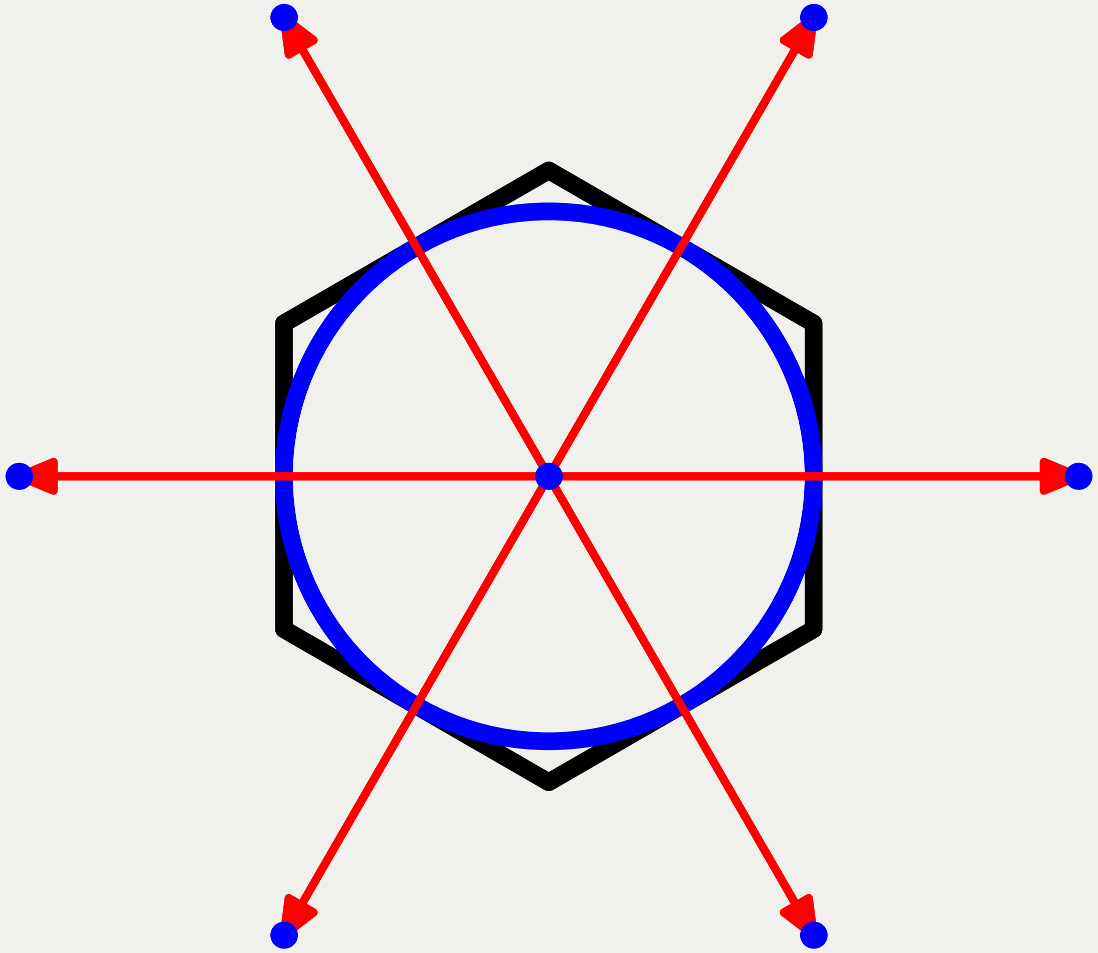
SECOND TRY

- Characterize local maxima for Θ
- Local minima for δ don't exist if n > 1
- New theory links Voronoi I and Voronoi II

Voronoi I

$Q \in S^n_{\geq 0}$

$Q[v] = v^\top Q v$



homogeneous minimum

$$\lambda(Q) = \min_{v \in \mathbb{Z}^n \setminus 0} Q[v]$$

minimal vectors

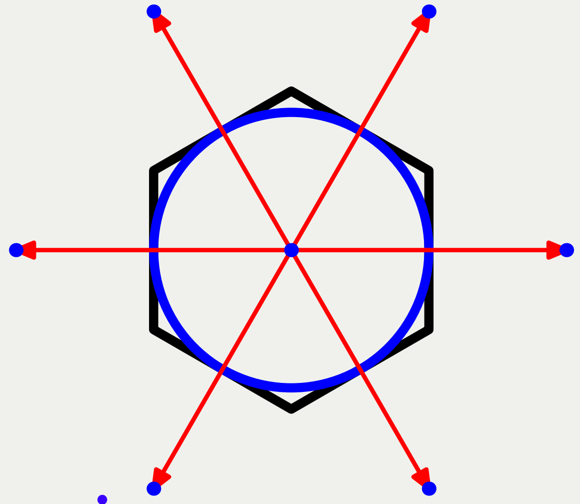
$$\mathbf{Min}\, Q = \{v \in \mathbb{Z}^n : Q[v] = \lambda(Q)\}$$

Hermite invariant

$$\gamma(Q) = \frac{\lambda(Q)}{(\det Q)^{1/n}}$$

maximizing Hermite = maximizing packing

Voronoi I



Theorem. *Voronoi 1907*

Q local maximum for $\gamma \iff Q$ perfect and eutactic.

perfectness:

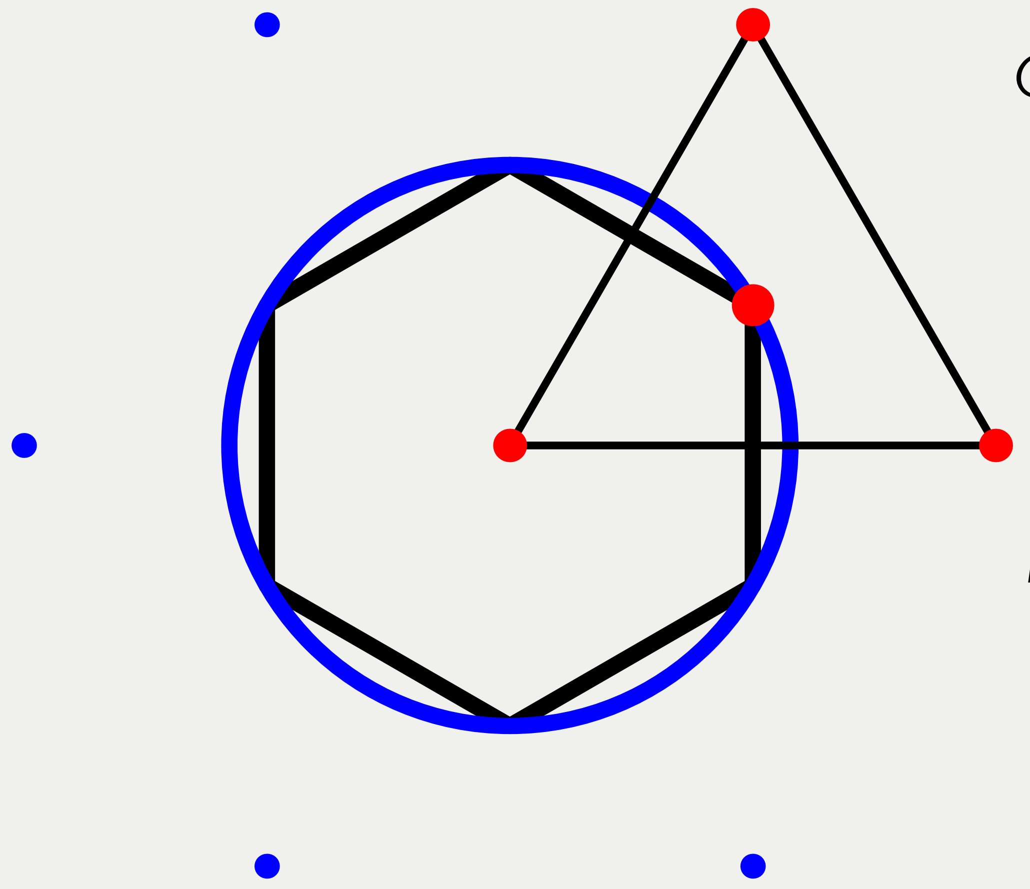
$$\mathrm{lin}\{vv^\top : v \in \mathrm{Min}\, Q\} \text{ has maximal dimension } \binom{n+1}{2}$$

eutaxy:

$$Q^{-1} = \sum_{v \in \mathrm{Min}\, Q} \alpha_v v v^\top \quad \text{with } \alpha_v > 0$$

Consequence: Local maxima are rational

Voronoi II



inhomogeneous minimum

$$\mu(Q) = \max_{x \in \mathbb{R}^n} \min_{v \in \mathbb{Z}^n} Q[x - v]$$

deep hole vectors

$$c \in \mathbb{R}^n \text{ s.t. } \mu(Q) = \min_{v \in \mathbb{Z}^n} Q[c - v]$$

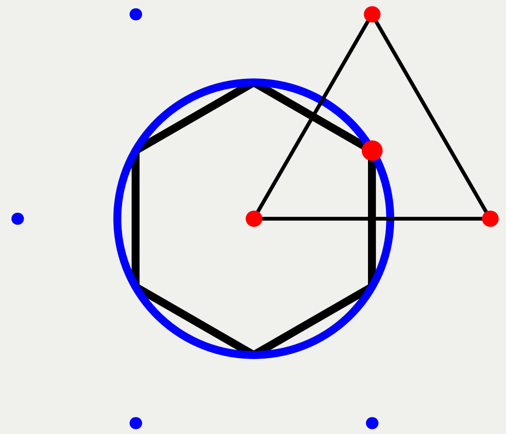
$$\mathbf{Min}_c\, Q = \{v \in \mathbb{Z}^n : Q[c - v] = \mu(Q)\}$$

inhomogeneous Hermite invariant

$$H(Q) = \frac{\mu(Q)}{(\det Q)^{1/n}}$$

maximizing i-Hermite = maximizing covering

Voronoi II



Theorem. *DSV 2009*

$$Q \text{ local maximum for } H \iff Q \text{ i-perfect and i-eutactic.}$$

inhomogeneous perfectness:

$$\operatorname{lin}\left\{ \begin{pmatrix} 1 \\ v \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix}^{\top} : v \in \operatorname{Min}_c Q \right\} \text{ has max. dim. } \binom{n+2}{2} - 1 \text{ for all deep holes } c$$

inhomogeneous eutaxy:

$$\begin{pmatrix} 1 & c^{\top} \\ c & cc^{\top} + \frac{\mu(Q)}{n} Q^{-1} \end{pmatrix} = \sum_{v \in \operatorname{Min}_c Q} \alpha_v \begin{pmatrix} 1 \\ v \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix}^{\top} \text{ with } \alpha_v > 0 \text{ for all deep holes } c$$

Consequence: Local maxima are rational

Venkov theory

Nice lattices are strongly perfect.

Q strongly perfect: Min Q is spherical 4-design

Theorem. *Venkov 2001*

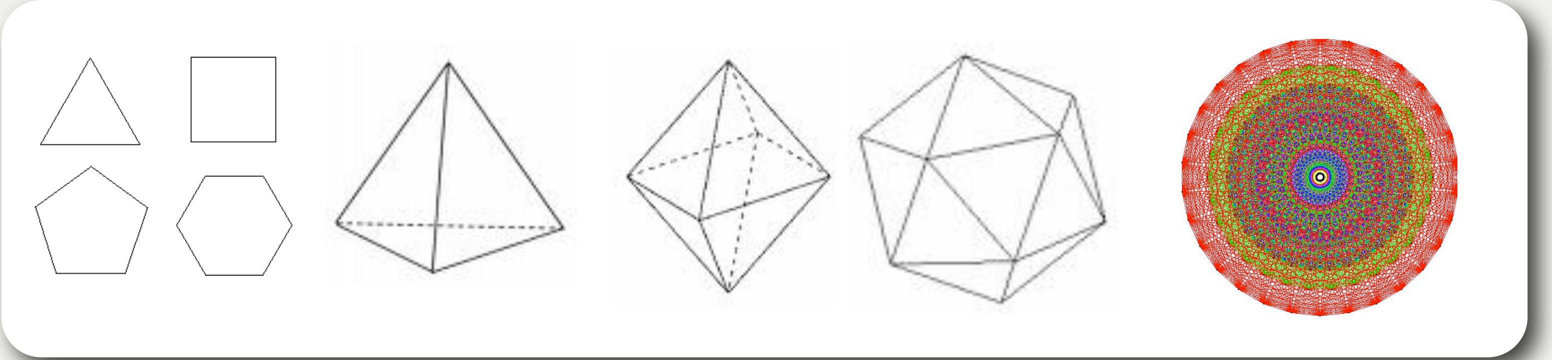
Q strongly perfect $\implies$ Q perfect and eutactic.

Spherical designs

$X \subseteq S^{n-1}$ spherical t-design

$$\int_{S^{n-1}} f(x) d\omega(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

for all polynomials of degree $\leq t$.



point conf.	strength
n-gon	n - 1
simplex	2
cross polytope	3
icosahedron	5
240	7
196560	11

Venkov i-theory

Q strongly i-perfect: $\mathrm{Min}_c\, Q$ is spherical 4-design for all deep holes c

Theorem. *DSV 2009*

Q strongly i-perfect $\quad\Longrightarrow\quad$ Q i-perfect and i-eutactic.

This happens to the nice lattices:

lattice	packing	covering
$\mathbb{Z}$	global opt.	global min.
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