Face numbers of spheres, manifolds, and pseudomanifolds

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Plan

I. Simplicial complexes + invariants

II. Stacked polytopes and the Lower Bound Theorem

III. New results

IV. Ideas of proofs ; more new results

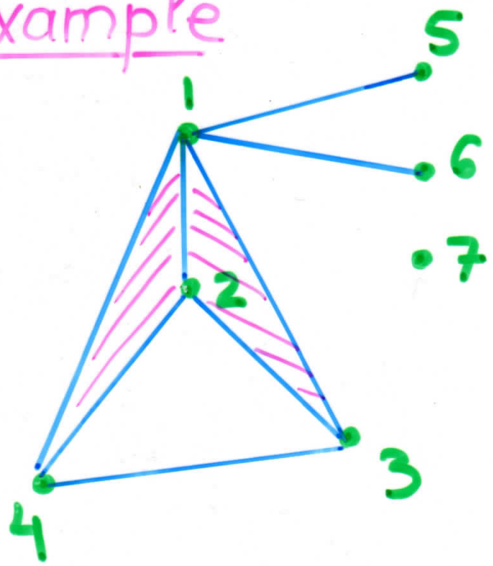
V. Open Problems

Simplicial complexes

= (finite) families of sets that are closed under inclusion

faces

Example



$$\Delta = \left\{ \begin{array}{c} 123,\ 124,\ 34, \\ 15,\ 16,\ 7 \\ \text{+ all their} \\ \text{subsets} \end{array} \right\}$$

Main examples for this talk:

The bdry of simplic. polytopes $\subset$ simplic. spheres $\subset$ simplic. manifolds $\subset$ simplic. pseudo-manif.

Invariants of simplic. complexes

- face numbers (or f-numbers)

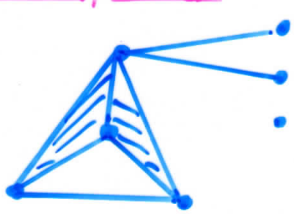
$$f_i(\Delta) := \text{number of } \underbrace{i\text{-diml faces}}_{\text{(sets of size } i+1)}$$

$$f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{d-1}) \qquad d-1 = \dim \Delta$$

- Betti numbers

$$\beta_i(\Delta) = \dim \tilde{H}_i(\Delta; k) = \begin{cases} \#\, i\text{-diml holes}, & i > 0 \\ \#\, \text{conn. comp.} - 1, & i = 0 \end{cases}$$

Example 1



dim = 2

$f(\Delta) = (1, 7, 8, 2)$

$\beta_0 = 1, \quad \beta_1 = 1, \quad \beta_2 = 0$

Example 2

$$\beta_i(S^{d-1}) = \begin{cases} 0, & i < d-1 \\ 1, & i = d-1 \end{cases}$$

$\beta(S^1 \times S^1) = (0, 2, 1)$

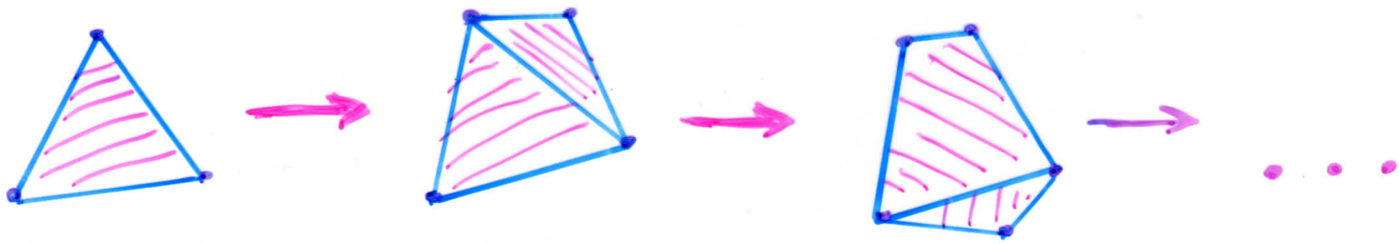
Central problem to give substantial (new) necess. conds. on the f/β – numbers of various classes of simplic. complexes.

Our starting point – The Lower Bound Thm

(Barnette, 1973 Kalai, 1987)

Among all connected simplic. manifolds (without bdry) of dim $d-1$ and with n vertices, the brdy of a stacked polytope $P(n,d)$ simultan. minimizes ALL the f-numbers.

Stacked Polytopes



$$ST(n, d-1) := \partial P(n,d)$$

$$= \underbrace{\partial(d\text{-simplex}) \# \ldots \# \partial(d\text{-simplex})}_{(n-d)\text{ copies}}$$

$$f_j\big(ST(n,d-1)\big) = \begin{cases} \binom{d+1}{j+1} + (n-d-1)\binom{d}{j}, & j < d-1 \\ (d+1) + (n-d-1)\cdot(d-1), & j = d-1. \end{cases}$$

More convenient invariants - <u>h-numbers</u>

Δ - simplicial, $(d-1)$-dim $\rightsquigarrow h(\Delta)=(h_0, h_1, \dots, h_d)$

$$\sum_{i=0}^{d} f_{i-1}(\Delta)\cdot(x-1)^{d-1} = \sum_{i=0}^{d} h_i(\Delta)\cdot x^{d-i}$$

Properties

* Δ is a sphere $\Rightarrow$ $h_i(\Delta) \geq 0 \quad \forall i$;
  $h_i(\Delta) = h_{d-i}(\Delta) \quad \forall i$

* Δ is a manifold $\Rightarrow \begin{cases} h_i(\Delta)=h_{d-i}(\Delta), & d\text{-even} \\ h_{d-i}(\Delta)=h_i(\Delta)+(-1)^i\binom{d}{i}\cdot(\tilde{\chi}-1) & d\text{-odd} \end{cases}$

* For $d \geq 3$, the $\angle$BT is equiv. to $h_2 \geq h_1$.

We'll discuss generaliz./strengthenings

of the LBT to

balanced spheres

manifolds

manifolds with bdry

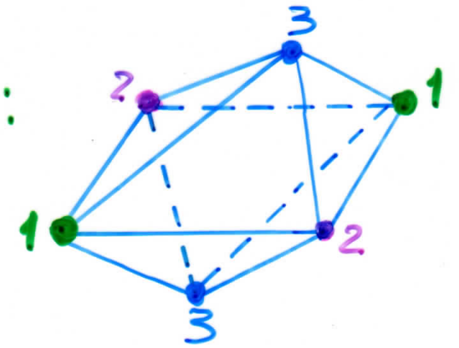
pseudomanif. with isolated singularities

Balanced spheres —

a $(d-1)$-dim spheres whose graph is d-colorable

Ex d-dim cross-polytope

C_d^* :

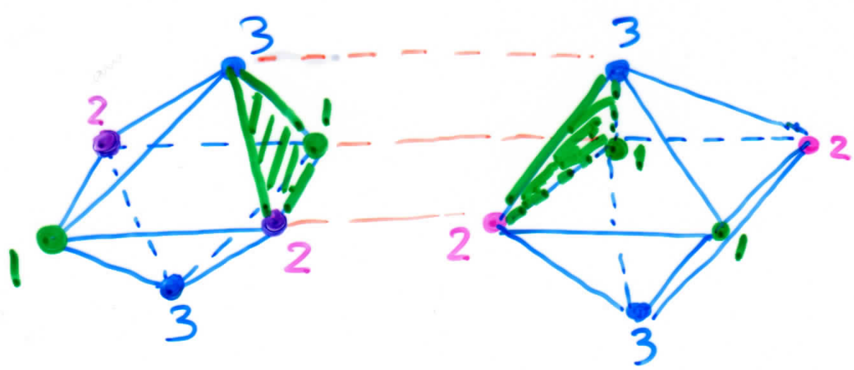


$h_i = \binom{d}{i}$

$\Rightarrow (d-1)\cdot h_1 = 2h_2$

Ex "Stacked cross-polytope" $\rightleftharpoons ST^*(n, d-1)$

$:=$ connected sum of (bdries of) C_d^*



$$\psi_j(n,d-1) := (j+1)\cdot f_j(ST^*(n,d-1)) = \begin{cases} (2^{j+1}-1)\cdot\binom{d-1}{j}\cdot(n-d) + d\cdot\binom{d-1}{j}, & j<d-1 \\ (2^d-2)\cdot(n-d)+2d, & j=d-1 \end{cases}$$

The LBT for balanced spheres

Thm (Goff, S. Klee, N)

If Δ is a $(d-1)$-dim balanced sphere, $d \geq 3$, then

$$2h_2(\Delta) \geq (d-1) \cdot h_1(\Delta),$$

and so $\quad (j+1) \cdot f_j(\Delta) \geq \varphi_j(n, d-1).$

Thus, among all balanced spheres, $ST^{\times}(n, d-1)$ has the smallest f-numbers.

Q: Does this result continue to hold for balanced manifolds?

YES! (very recent work of J. Browder and S. Klee)

Recall:

LBT (Barnette, 1973 ; Kalai, 1987)

Δ - connected manifold without bdry, $\dim \Delta \geq 2$

$$\Downarrow$$

$$h_2(\Delta) \geq h_1(\Delta).$$

Thm (Kalai, 1987)

Δ - connected manifold with bdry, $\dim \Delta \geq 2$

$$\Downarrow$$

$$h_2(\Delta) \geq f_0^\circ(\Delta) = \#\text{interior vertices}$$

Q: Sharper bounds that involve topology of Δ ?

New results

Δ - $(d-1)$-dim, connected, $d \geq 4$.

Conjecture (Kalai, 1987)

If Δ is a connected manifold without bdry, then

$$h_2 - h_1 \geq \binom{d+1}{2} \cdot \beta_1(\Delta; k)$$

(k is arbitrary, if Δ is orient.; char $k=2$ otherwise).

Thm (N-Swartz)

Kalai's conjecture holds for ALL manifolds.

Thm (N-Swartz)

If Δ has non-empty orientable bdry, then

$$h_2(\Delta) \geq f_0^{\circ}(\Delta) + \begin{cases} \binom{d}{2} \cdot \beta_1(\partial\Delta) + d \cdot \beta_0(\partial\Delta), & d \geq 5 \\ 3 \cdot \beta_1(\partial\Delta) + 4 \cdot \beta_0(\partial\Delta), & d = 4 \end{cases}$$

and these bounds are sharp.

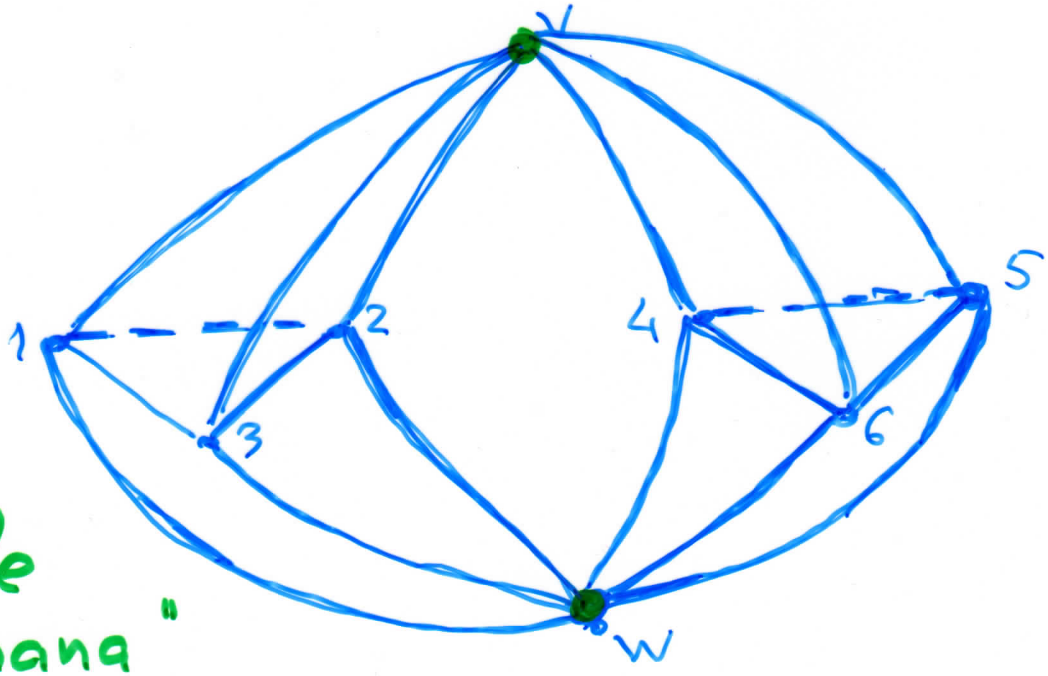
Manifolds → Pseudomanifolds

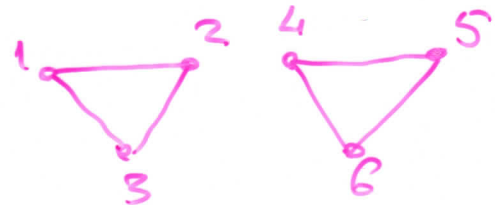
* A simplic. complex Δ is a pseudomanifold if it is pure and every ridge is contained in exactly two facets.

* Links of faces $\mathrm{lk}_\Delta\, G = \{F - G : G \subseteq F \in \Delta\}$

Ex

"double Banana"



$\mathrm{lk}\, v = \mathrm{lk}\, w =$ 

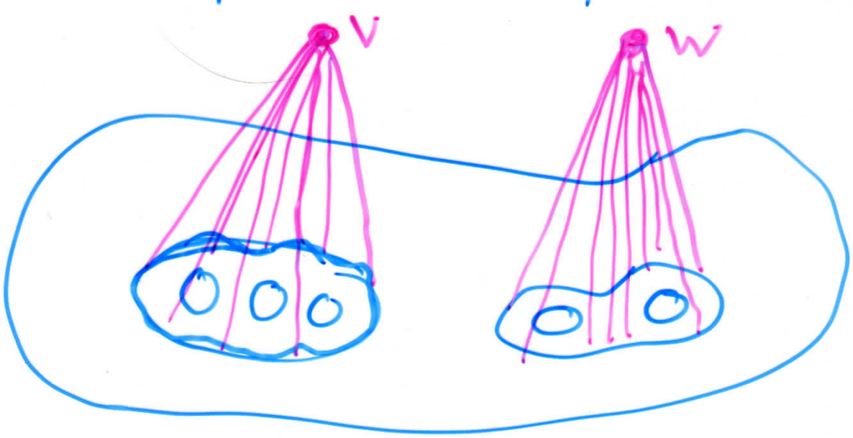
Isolated singularities

A pseudomanifold Δ has isolated singularities if the link of each vertex is a manifold (without bdry).

Examples

* double banana

* M - manifold with bdry



$\Delta = M$ with each bdry comp. coned off

A vertex v is singular if $\operatorname{lk} v \neq S^{d-2}$.

Homologically isolated singularities

- Δ – simplic. complex, v – vertex of Δ

$$i^*: H^j(\Delta, \Delta - v; k) \longrightarrow H^j(\Delta; k)$$

- If Δ is a pseudomanif. with isolated singularities at $v_1, \dots, v_p$, $\dim \Delta = d-1$, we say that Δ has homol. isol. sing. if

$$i^*\left(H^j(\Delta, \Delta - v_s)\right), \quad s = 1, \dots, p$$

are k-independent subspaces of $H^j(\Delta)$, $\forall j < d-1$

Examples

- only one isol. sing.
- a handlebody with homol. indep. handles pinched off.

More new results (work in progress)

Thm (N-Swartz)

If Δ is a $(d-1)$-dim pseudomanifold with homol. isol. sing., and $d \geq 5$, then

$$h_2 - h_1 \geq \binom{d+1}{2} \cdot \left(\beta_{d-2}(\Delta) - \beta_{d-1}(\Delta) + 1 \right)$$

$$+ d \cdot \sum_v \dim \operatorname{Ker} \left[H^{d-2}(\Delta, \Delta - v) \xrightarrow{i^*} H^{d-2}(\Delta) \right]$$

$$- d \cdot \sum_v \left(\beta_{d-3}(\operatorname{lk} v) - \beta_{d-2}(\operatorname{lk} v) + 1 \right)$$

Note: if Δ is a connected, k-orientable manif., this reduces to

$$h_2 - h_1 \geq \binom{d+1}{2} \cdot \beta_1(\Delta)$$

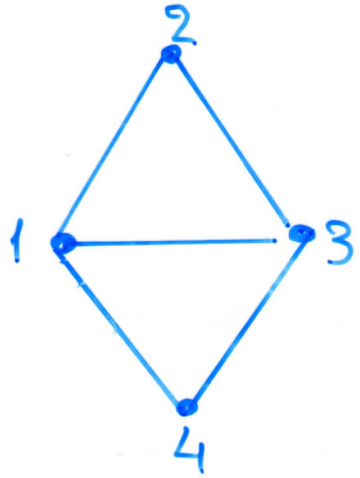
Ideas of proofs: Stanley-Reisner rings

h-numbers have an algebraic meaning!

Simplicial compl. Δ on $\{1,2,\ldots,n\}$ $\rightsquigarrow$ Stanley-Reisner ring $k[\Delta] = k[x_1,\ldots,x_n]/I_\Delta$

$$I_\Delta = \left(x_{i_1} x_{i_2} \cdots x_{i_s} : \{i_1 < i_2 < \ldots < i_s\} \notin \Delta \right)$$

Ex



$$\rightsquigarrow k[x_1,x_2,x_3,x_4] \Big/ \left(x_1x_2x_3,\ x_1 \cdot x_3 \cdot x_4,\ x_2x_4 \right)$$

d=2

A linear system of parameters

Δ — $(d-1)$-dim simplic. complex, $\theta_1, \ldots, \theta_d \in k[\Delta]_1$.

Say that $\theta_1, \ldots, \theta_d$ is a linear system of param. if $k[\Delta]/(\theta_1, \ldots, \theta_d)$ is a finite-dim k-space.

- **Comments** (assume k is infinite)

* If $\theta_1, \ldots, \theta_d$ are chosen "generically", then they form an l.s.o.p.

* $k[\Delta]$ - graded, $\theta_1, \ldots, \theta_d \in k[\Delta]_1 \implies$

$$k(\Delta) := k[\Delta]/(\theta_1, \ldots, \theta_d) \text{ is graded}$$

$$\| $$

$$\bigoplus_{i=0}^{d} k(\Delta)_i$$

Dimensions of $k(\Delta)_j = k(\Delta)/(\theta_1, \dots, \theta_d)_j$

Thm (Stanley, 1975) Δ - $(d-1)$-dim sphere

$$\Rightarrow \dim_k k(\Delta)_j = h_j \quad \forall\, 0 \le j \le d$$

Thm (Schenzel, 1981) Δ - $(d-1)$-dim manif

$$\Rightarrow \dim_k k(\Delta)_j = h_j + \binom{d}{j} \cdot \left(\beta_{j-2}(\Delta) - \beta_{j-3}(\Delta) + \beta_{j-4}(\Delta) - \dots\right)$$

Thm (N-Swartz) Δ - $(d-1)$-dim pseudo-manif. with homol. isol. singul. $\Rightarrow$

$$\begin{aligned}\dim_k k(\Delta)_j = h_j &+ \binom{d}{j}\left(\beta_{j-2}(\Delta) - \beta_{j-3}(\Delta) + \beta_{j-4}(\Delta) - \dots\right) \\ &- \binom{d-1}{j} \sum_v \left(\beta_{j-3}(\operatorname{lk} v) - \beta_{j-4}(\operatorname{lk} v) + \dots\right) \\ &+ \binom{d-1}{j} \sum_v \dim \operatorname{Ker}\left(H^{j-1}(\Delta, \Delta - v) \xrightarrow{i^*} H^{j-1}(\Delta)\right)\end{aligned}$$

What goes into the proofs ?

- Local cohomology modules, H^i_m, of $k[\Delta]$ and its quotients

- Gräbe's description of $H^i_m(k[\Delta])$ in terms of simplicial cohomology of links of Δ and maps between them.

- Connections between $k(\mathrm{lk}\, v)$ and $k(\Delta)$

  – due to Swartz

- rigidity ineq. $h_2 \geq h_1$ for manifolds

...

Open Problems – Pseudomanif.

- LBT for homol. isol. sing when $d=4$?
- What about isolated, but NOT homol. isol. sing?
  - In this case dims of $k[\Delta]/(\theta_1, \dots, \theta_d)_j$ **depend** on the choice of θ's.

* What are these dims for generic θ's ?

* Does topology of Δ + f-numbers of Δ determine these dims ?

That is, if $|\Delta_1|$ homeo to $|\Delta_2|$ and $f(\Delta_1) = f(\Delta_2)$, is

$$\dim\left(k[\Delta_1]/\Theta\right)_j = \dim\left(k[\Delta_2]/\Theta\right)_j \ ?$$

Cases of equality in the LBT

Kalai, 1987: Δ – connected, $(d-1)$-dim. manifold without bdry, $d \geq 4$. Then

$$h_2(\Delta) = h_1(\Delta) \Longleftrightarrow \Delta \text{ is a stacked sphere.}$$

N-Swartz: Δ – connected, $\mathbb{k}$-orient., $(d-1)$-dim manifold without bdry, $d \geq 5$. Then

$$h_2(\Delta) - h_1(\Delta) = \binom{d+1}{2} \cdot \beta_1(\Delta)$$

$$\Updownarrow$$

all vertex links of Δ are stacked spheres.

Problem does the same hold for $d = 4$?

Manifolds with Boundary

- For a connected manif. with non-empty orient. Bdry, we have

$$h_2 \geq f_0^{\circ} + \begin{cases} \binom{d}{2} \cdot \beta_1(\partial\Delta) + d \cdot \beta_0(\partial\Delta), & d \geq 5 \\ 3 \cdot \beta_1(\partial\Delta) + 4 \cdot \beta_0(\partial\Delta), & d = 4 \end{cases}$$

When does equality hold?

Conj Equality

$\Updownarrow$

all vertex links of Δ are stacked balls or stacked spheres

Balanced spheres

Know that $2h_2 \geq (d-1)h_1 \iff \binom{d}{1} h_2 \geq \binom{d}{2} h_1$

- Does every balanced sphere satisfy

$$\binom{d}{i-1} \cdot h_i \geq \binom{d}{i} \cdot h_{i-1} \quad \forall i \leq \lfloor \tfrac{d}{2} \rfloor ?$$

* Stacked cross-polytopes are defined only for n divis. by d.

Can we have $2h_2 = (d-1) \cdot h_1$ for a sphere with $d \nmid n$?