JARNIK'S CONVEX LATTICE POLYGON FOR NON-SYMMETRIC NORMS

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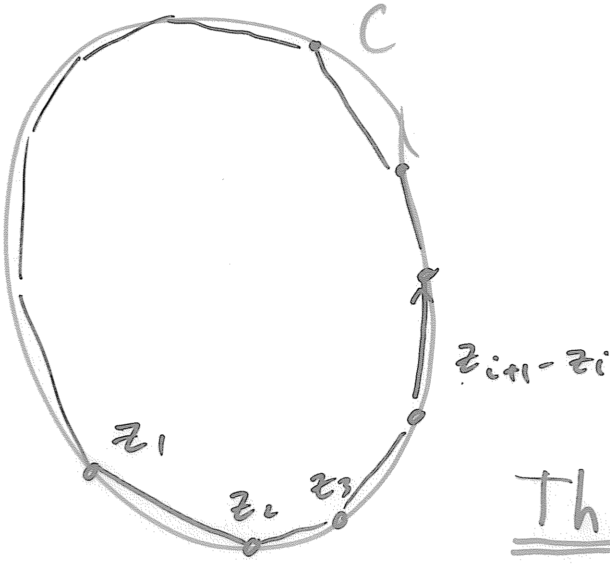
Jarník (1926) C is a strictly convex (closed) curve in R^2, of length ℓ. Then

$$|C \cap \mathbb{Z}^2| \leq \frac{3}{\sqrt[3]{2\pi}} \ell^{\frac{2}{3}} (1 + o(1)).$$

Moreover, this is best possible.

Here $\mathrm{conv}(C \cap \mathbb{Z}^2)$ is a convex lattice n-gon with $n = |C \cap \mathbb{Z}^2|$

$\mathcal{P}_n = \{$convex lattice n-gons$\}$



Def

$$L_n = \min\{\operatorname{per} P : P \in \mathcal{P}_n\}$$

Thm (Jarnik)

$$L_n = \frac{\sqrt{6\pi}}{9}\, n^{\frac{3}{2}}\left(1 + o(1)\right).$$

Proof $z_{i+1} - z_i \in \mathbb{P}$ (primitive vectors) $\forall i$ for the minimizer $P \in \mathcal{P}_n$.

$E(P) = \{ z_{i+1} - z_i : i = 1..n \}$ distinct vectors $\in \mathbb{P}$

$L_n = \sum_1^n |z_{i+1} - z_i| \geqq$ total length of the n shortest distinct primitive vectors

can be computed

density principle

Take the smallest $\lambda > 0$ with

$$|\lambda B \cap \mathbb{P}| \geqq n$$

↑ unit ball (Eucl.)

$\Downarrow$

$$\frac{6}{\pi^2} \lambda^2 \pi \approx n$$

$$\sum_{p \in \lambda B \cap \mathbb{P}} |p| \approx \frac{6}{\pi^2} \int_{\lambda B} |x| \, dx = \frac{6}{\pi^2} \cdot \frac{2}{3} \cdot \lambda^3 \pi$$

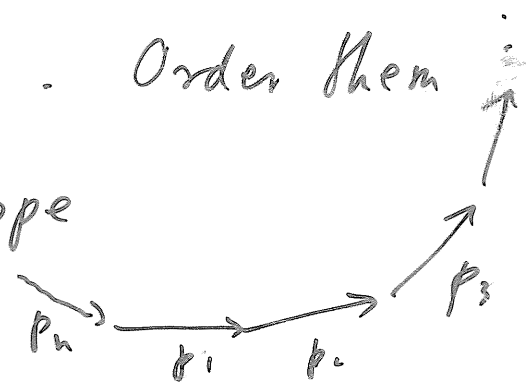
$$\Rightarrow \quad L_n \geqq \frac{\sqrt{6\pi}}{9} n^{3/2} (1 + o(1)).$$

Construction for $L_n \leqq$ same:

n even, take pairs $p, -p \in \lambda B \cap \mathbb{P}$

then $\sum p = 0$. Order them

by increasing slope



n odd

$L_n < L_{n+1}$

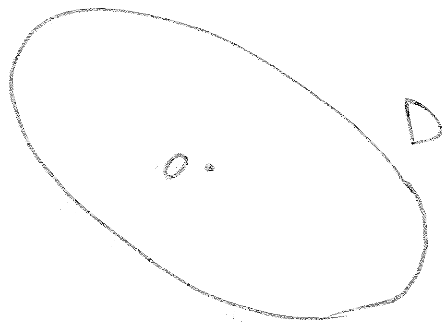
$\boxed{\text{increasing slope construction}}$

FACT Assume $P_n \in \mathcal{P}_n$ is a minimizer.

then $n^{-\frac{3}{2}} P_n$ tends to a circle

P_n has a limit shape (after scaling)

Assume D is the unit ball of another norm in $\mathbb{R}^2$ (symmetric)



for $P \in \mathcal{P}_n$

$$\mathrm{per}_D\, P = \sum_1^n \| z_{i+1} - z_i \|_D$$

$$L_n(D) = L_n = \min \{ \mathrm{per}_D\, P : P \in \mathcal{P}_n \}$$

Thm (M. PRODROMOU 2006)

$\lim n^{-\frac{3}{2}} L_n(D)$ exists and equals

$$\frac{\pi \int_D |x|\, dx}{(6 \text{ Area } D)^{\frac{1}{3}}}$$

Thm (same)

the minimizers have a limit shape (after scaling)

Note!

(*)

$K \subset \mathbb{R}^2$ convex, compact. $0 \in \operatorname{int} K$

$d = \max\{|x| : x \in K\}$

$w =$ width of K . ($w \geqq 3$ say)

$$\big| |K \cap \mathbb{Z}^2| - \text{Area } K \big| \ll \frac{1}{w} \text{ Area } K$$

$$\Big| |K \cap \mathbb{P}| - \frac{6}{\pi^2} \text{Area } K \Big| \ll \frac{\log d}{w} \text{ Area } K$$

$f : \mathbb{R}^2 \to \mathbb{R}$ 1-homogeneous $\left(\begin{array}{c} f(\lambda x) = \lambda f(x) \\ \forall \lambda > 0 \ \forall x \end{array} \right)$

then

$$\Big| \sum_{z \in K \cap \mathbb{P}} f(z) - \frac{6}{\pi^2} \int_K f(x)\,dx \Big| \ll \frac{\log d}{w} \cdot \int_K |f(x)|\,dx$$

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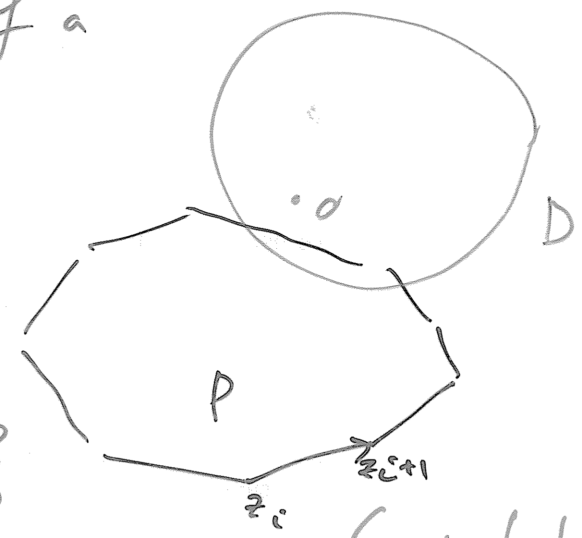
standard

Let D be the unit ball of a non-symmetric norm in $\mathbb{R}^2$



$$\mathrm{per}_D P = \sum_1^n \|z_{i+1} - z_i\|_D$$

(orientation!)

$$L_n(D) = \min \{ \mathrm{per}\, P : P \in \mathcal{P}_n \}$$

Results (I. B., N. ENRIQUEZ, 2009)

$$L_n(D) = c(D)\, n^{3/2} (1 + o(1))$$

there is a limit shape:

$$n^{-\frac{3}{2}} P_n \longrightarrow C$$ a fixed convex body in $\mathbb{R}^2$

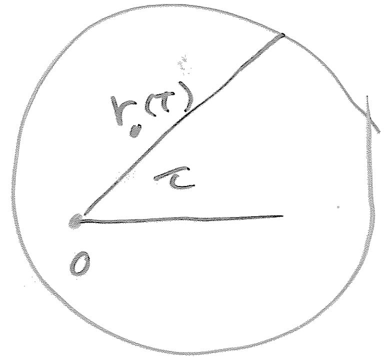
Assume D is strictly convex (convenience)

$\Rightarrow$ $L_n < L_{n+1}$ and

$\Rightarrow$ every edge of a minimizer $\in \mathbb{P}$

VARIATIONAL PROBLEM $VP(r_0)$

$r_0 : [0, 2\pi] \to \mathbb{R}^+$ is the radial function of D



find $r : [0, 2\pi] \to \mathbb{R}^+$, cont. $r(0) = r(2\pi)$ with

$$\int_0^{2\pi} \frac{r^3(\tau)}{r_0(\tau)}\, d\tau \longrightarrow \min$$

subject to

$$\left.\begin{aligned} \int_0^{2\pi} r^3(\tau) \cos\tau \, d\tau &= 0 \\ \int_0^{2\pi} r^3(\tau) \sin\tau \, d\tau &= 0 \end{aligned}\right\} \text{centre of gravity of } K \text{ in } 0$$

$$\left.\frac{1}{2}\int_0^{2\pi} r^2(\tau)\, d\tau = 1 \right\} \text{Area } K = 1$$

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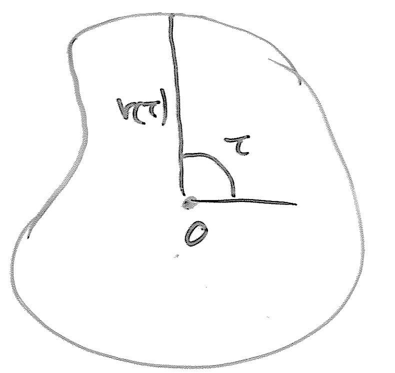
r is the radial function of a convex (starshaped) body K



V2

KNOWN all optimal sol'ns to $VP(r_0)$ satisfy

$$\frac{1}{r(\tau)} = \frac{a}{r_0(\tau)} + b\cos\tau + c\sin\tau$$

NOT KNOWN existence of opt. sol'ns to $VP(r_0)$

(later)

FACT If $r(\tau)$ is a feasible sol'n to $VP(r_0)$, then there is a sequence $Q_n \in \mathcal{P}_n$ with

$$\lim n^{-\frac{3}{2}} \operatorname{per} Q_n = \frac{2}{3\sqrt{6\pi}} \int_0^{2\pi} \frac{r^3(\tau)}{r_0(\tau)} \, d\tau$$

Proof $r(\tau)$ is the radial function of K

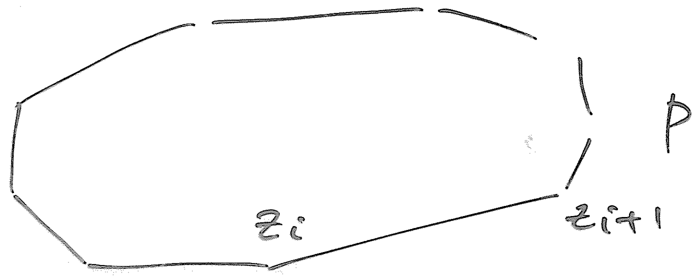
Area $K = 1$, center of gravity of $K = 0$

choose $\lambda > 0$ minimal with $|\lambda K \cap \mathbb{P}| = n$

. . . . .

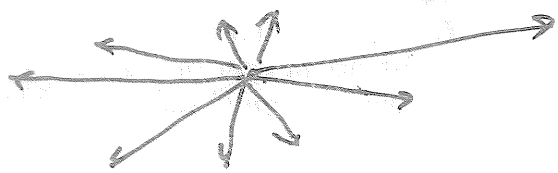
$\square$

$P \in \mathcal{P}_n$



$E(P) = \{ z_{i+1} - z_i : i = 1, \dots, n \}$

$C(P) = \text{conv}\, E(P)$



Suppose $P \in \mathcal{P}_n$ is a minimizer.

<u>FACT</u> $E(P) \subset \mathbb{P}$

<u>FACT</u> $L_n < L_{n+1}$

<u>HOPE</u> $E(P)$ contains all (almost all) primitive vectors from $C(P)$

and $C(P)$ is nice

<u>OF COURSE</u> $c_1 |x| \le \|x\|_D \le c_2 |x|$

OR $|x| \ll \|x\|_D \ll |x|$

OR $c_1 B \subset D \subset c_2 B$

Claim 1. $L_n \gg n^{\frac{3}{2}}$

Proof (simple) The n shortest (in D-norm) distinct vectors in $\mathbb{P}$ are $p_1, \ldots, p_n$.

Then for $P \in \mathcal{P}_n$

$$\text{per } P \geqq \sum_1^n \|p_i\| \approx \frac{6}{\pi^2} \int_{\lambda D} \|x\| \, dx$$

$$= \frac{6}{\pi^2} \lambda^3 \int_D \|x\| \, dx$$

where $\lambda > 0$ is minimal with $\{p_1, \ldots, p_n\} \subset \lambda D$.

$$\Updownarrow$$

$$n \approx \frac{6}{\pi^2} \lambda^2 \text{ Area } D$$

$$\Rightarrow L_n \gg n^{3/2}. \quad \square$$

Claim 2. $L_n \ll n^{3/2}$.

Proof $p_0 = -\sum_1^n p_n$. $\|p_0\| \ll n^{3/2}$.

increasing slope construction on $p_0, p_1, \ldots, p_n$

gives a convex lattice n or $(n+1)$-gon

with perimeter $\ll n^{3/2}$ $\square$

<u>Corollary</u> $\liminf n^{-\frac{3}{2}} L_n(D) = \alpha > 0$

Now $P_n \in \mathcal{P}_n$ is a minimizer.

$E_n = E(P_n)$, $C_n = C(P_n)$

$w = w(E_n) =$ width of E_n

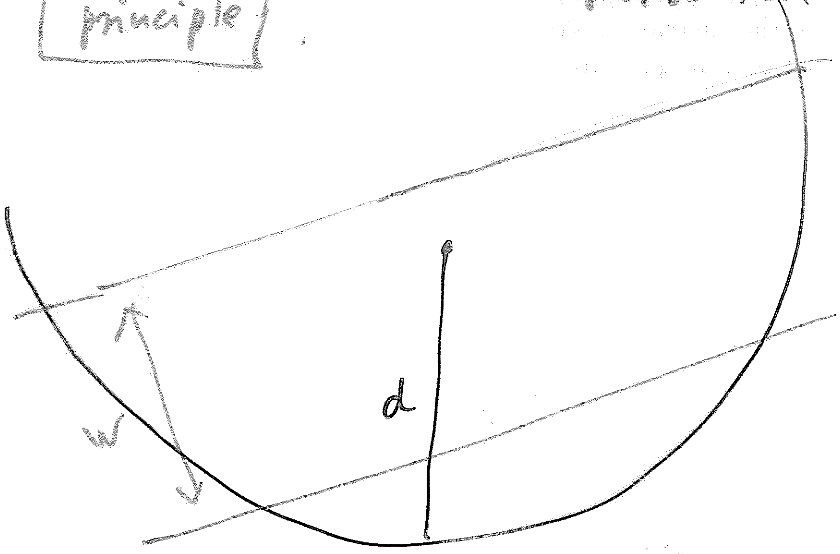
<u>Claim 3.</u> $w \gg n^{\frac{1}{2}}$

<u>Proof</u> (sketch) Set $w = \gamma n^{\frac{1}{2}}$. $\boxed{\text{NEED } \gamma \gg 1.}$

$$L_n = \sum_{v \in E_n} \|v\| \gg \sum_{v \in E_n} |v| \geq M_n(w)$$

where $M_n(w) =$ min sum of the n shortest (Euclidean) distinct vectors contained in a strip of width w

$\boxed{\text{density principle}}$.



... turns out to be $\gg \frac{1}{\gamma} n^{\frac{3}{2}}$. □

Claim 4 $R > 0$ minimal with $E_n \subset RB$

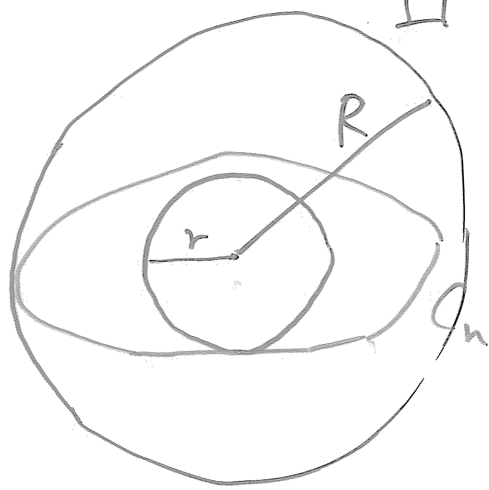
$$\Rightarrow \quad R \ll n^{\frac{1}{2}}$$

$\square$

Claim 5 $r > 0$ maximal with $rB \subset C_n$

$$\Rightarrow \quad r \gg n^{\frac{1}{2}}$$

$\square$



Cor $\exists\ 0 < r < R$ such that

$$rB \subset \frac{1}{\sqrt{n}} C_n \subset RB$$

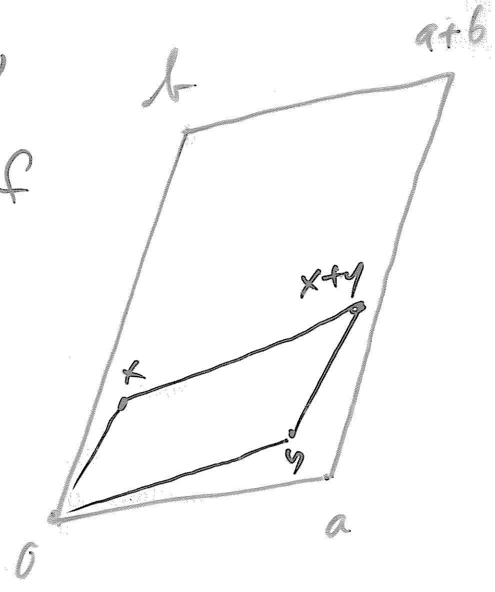
LEMMA $\forall \varepsilon > 0\ \exists\ n_0 = n_0(\varepsilon, D)$ such that for all $n > n_0$

$$\mathbb{P} \wedge (1-\varepsilon) C_n \subset E_n$$

Proof is based on a special approximation lemma (omitted here)

and on

Claim 6. $a, b \in E_n$, $a \neq \pm b$, $T = \operatorname{conv}\{0, a, b, a+b\}$. If $x, y \in (\mathbb{P} \cap T) \setminus E_n$ and $x \neq y$ then $x+y \notin T$.



Cor $E_n \approx \mathbb{P} \cap C_n$

Now one can choose a subsequence $n_1 < n_2 < \ldots$ with

$$\lim n_k^{-\frac{3}{2}} L_{n_k} = \alpha$$

and

$$\lim \frac{1}{\sqrt{\text{Area } C_{n_k}}} C_{n_k} = C$$

(using the Blaschke selection thm)

where C is a convex body with

$$rB \subset C \subset RB.$$

Claim 7. The radial function of C, $r(\tau)$, is a feasible sol'n to $VP(r_0)$

Proof Area $C = 1$ and the center of gravity is 0. $\square$

Thm $r(\tau)$ is an optimal sol'n to $VP(r_0)$

Proof $r(\tau)$ is feasible.

If r^* is also feasible and

$$\int_0^{2\pi} \frac{r^{*3}(\tau)}{r_0(\tau)}\, d\tau < \int_0^{2\pi} \frac{r^3(\tau)}{r_0(\tau)}\, d\tau,$$

then (from FACT) one gets a sequence $Q_n \in \mathcal{P}_n$ with

$$\lim n^{-\frac{3}{2}} \operatorname{per} Q_n < \alpha$$

$\square$

Claim 8 If the centre of gravity of D is 0, then the opt. sol'n to $VP(r_0)$ is unique.

Proof If r is a feasible sol'n, then by Hölder

$$\underbrace{\int_0^{2\pi} r^2(\tau)\,d\tau}_{=2} \le \left(\int_0^{2\pi} \frac{r^3(\tau)}{r_0(\tau)}\,d\tau\right)^{\frac{2}{3}} \underbrace{\left(\int_0^{2\pi} r_0^2(\tau)\,d\tau\right)^{\frac{1}{3}}}_{=2\,\text{Area}\,D}$$

with equality iff $r = \gamma r_0$ $(\gamma > 0)$. $\square$

Thm the opt. sol'n to $VP(r_0)$ is unique

Proof Assume $r : [0, 2\pi] \to \mathbb{R}^+$ is an opt sol'n to $VP(r_0)$. Then

$$\frac{1}{r(\tau)} = \frac{a}{r_0(\tau)} + b\cos\tau + c\sin\tau$$

with suitable $a, b, c \in \mathbb{R}$.

Claim 9. Here $a \int_0^{2\pi} \frac{r^3}{r_0} = 2.$

$\Rightarrow$ the coefficient a is the same for all optimal solutions.

Proof $$2 = \int_0^{2\pi} r^2 = \int_0^{2\pi} r^3 \frac{1}{r} - b \int_0^{2\pi} r^3 \cos\tau - c \int_0^{2\pi} r^3 \sin\tau$$

$$= \int_0^{2\pi} r^3 \Big[\underbrace{\frac{1}{r} - b\cos\tau - c\sin\tau}_{\frac{a}{r_0}} \Big] =$$

$$= a \int_0^{2\pi} \frac{r^3}{r_0} \, d\tau \quad \square$$

Suppose now $r_i : [0, 2\pi] \to R^+$ $(i = 1, 2)$ are opt sol'ns ~~to~~ $VP(r_0)$. Then

$$\frac{1}{r_i} = \frac{a}{r_0} + b_i \cos\tau + c_i \sin\tau .$$

$$\int_0^{2\pi} \frac{r_i^3}{r_1} = \int_0^{2\pi} r_i^3 \left(\frac{a}{r_0} + b_1 \cos\tau + c_1 \sin\tau \right) = a \int_0^{2\pi} \frac{r_i^3}{r_0} .$$

r_1 is the radial function of a starshaped body whose centre of gravity is O.

Claim 8 $\Rightarrow$ $r_1 = r_2$. $\square$