Acute triangulations of polytopes

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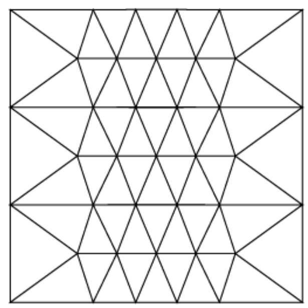


The Problem:

A simplex $\Delta \subset \mathbb{R}^d$ is *acute* if all its dihedral angles are $< \pi/2$.

An *acute triangulation* is a finite subdivision into acute simplices.

1. For a convex polytope $P \subset \mathbb{R}^d$, find an acute triangulation.

2. Find an acute partition of $\mathbb{R}^d$.

FIGURE 1. An acute triangulation and an acute dissection of a square.

Main Corollary [Kopczyński–P.–Przytycki (2009)]

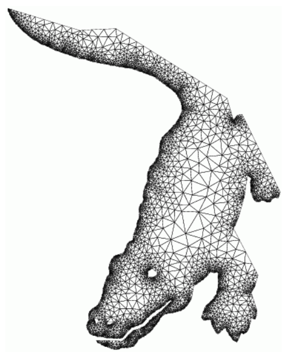
A d-cube has an acute triangulation only for $d \leq 3$.

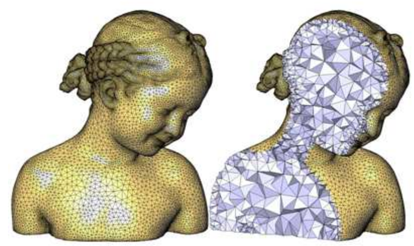
This resolves an old folklore open problem:
Martin Garner (1960), Burago-Zalgaller (1960),
Eppstein–Sullivan–Üngör (2004), Křížek (2006), etc.

Note: There is an easy triangulation of a d-cube into $d!$ non-obtuse simplices.

Why acute triangulations?

- Classical geometric problem.
- Finite element method.
- Crystallography.
- Large recreational literature.



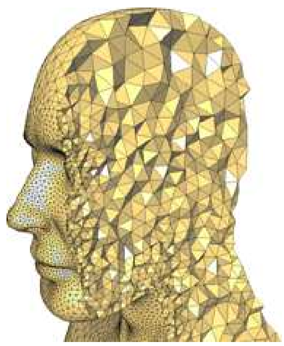


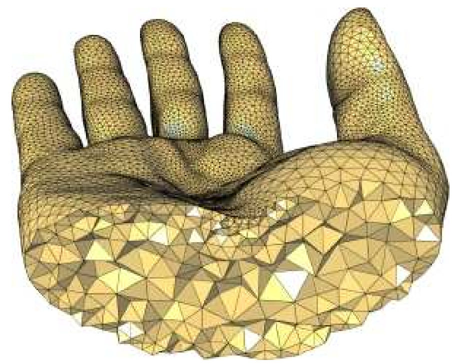
Finite element applications:

Input: triangulated surface $S \subset \mathbb{R}^3$.

Goal: find a *good** triangulation of the interior of S.

* *good* means all tetrahedra are as close to regular as possible.





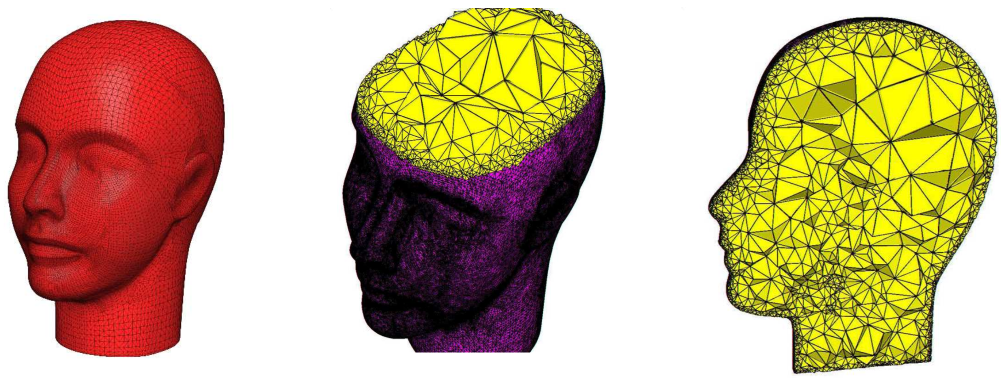


Figure 2. TetGen group from Berlin: triangulation of the head.

Statistics:

Input points: 20,796. Input facets: 41,588.

Mesh points: 350,980. Mesh tetrahedra: 1,366,269.

Acute triangulations in the plane

- ▷ Proposed by Martin Garner (*Scientific American*, 1960)
- ▷ Resolved (independently) and extended by Burago–Zalgaller (1960) (Existence only, no complexity bounds follow from the proof).
- ▷ Easy to do in practice (Delaunay triangulations).
- ▷ Beginning 1980's heavily studied in the DCG community.

Theorem [Bern–Mitchell–Ruppert (1995) + Maehara (2002)]
Every n-gon in the plane has an acute triangulation with $O(n)$ triangles, which can be computed in linear time.

Burago–Zalgaller's proof

Theorem: Every polygon in the plane has an acute triangulation.

1) triangulate the polygon
2) make an acute triangulation of every triangle
3) assuming everything is rational: subtriangulate all triangles by the common denominator
4) otherwise, use an approximation argument

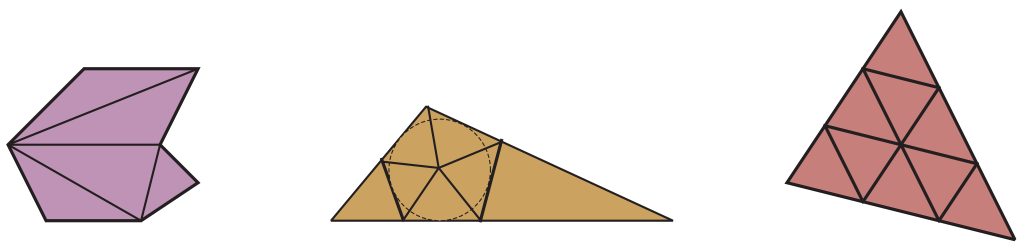


FIGURE 3. Steps of the BZ proof.

Flips on triangulations

Idea: use 2-flips to improve your triangulation (in any order).

Theorem: This always gives the Delaunay triangulation.

Observation: This always maximizes the min angle in a triangulation.

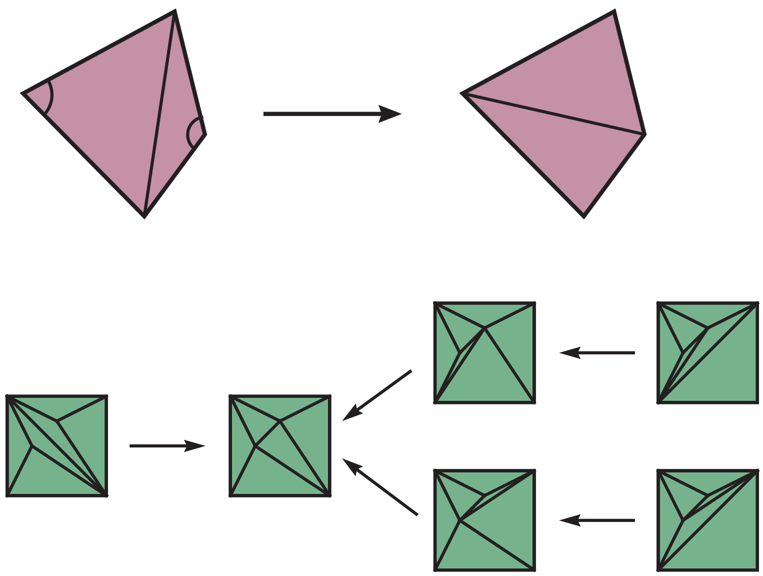


Figure 4. 2-flips in a triangulation.

Dimensionality curse

Philosophy: the higher the dimension, the harder it is to make acute triangulations (both theoretically and practically).

$d = 2$ – relatively easy

$d = 3$ – possible sometimes; perhaps, always

$d = 4$ – impossible sometimes; perhaps, very rarely

$d \geq 5$ – always impossible

Observation: Faces of an acute d-simplex are also acute simplices. Thus, acute triangulation of a d-cube contains acute triangulations of all n-cubes, for $n < d$.

$d = 3$ case: the beginning of a beautiful friendship

♡ Studied for 30+ years. Until recently, very little progress.

♡ In practice: tile with Sommerville's tetrahedra:

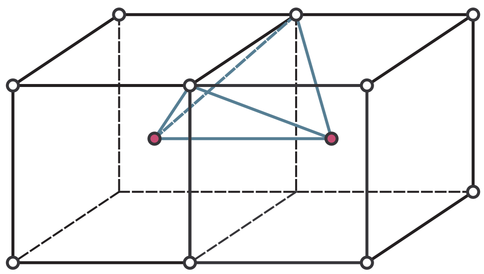


Figure 5. Sommerville's tetrahedron.

Diversion: acute partition of the space

- ♡ Aristotle: regular tetrahedron tiles the space (*On the Heaven*, 350 BC)
- ♡ Sommerville: four space-filling tetrahedra (1923).
- ♡ M. Goldberg: two new space-filling families (1974).

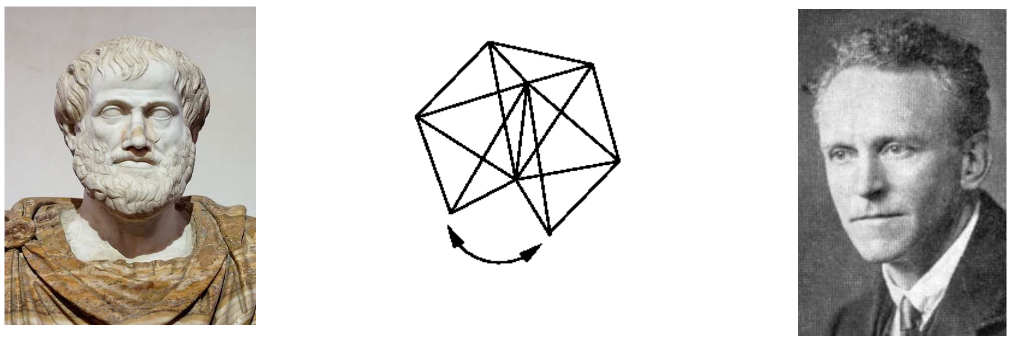


FIGURE 6. Aristotle, his error, Sommerville.

More on acute partitions of the space

♡ Edmonds: if reflections are not allowed, then Sommerville's 1923 classification is complete (2007)

♡ Eppstein–Sullivan–Üngör: a periodic acute partition of $\mathbb{R}^3$ (2004)

♡ Delgado Friedrichs and Huson: no *transitive* acute partition of $\mathbb{R}^3$ (1999)

Open: Find an acute partition of $\mathbb{R}^3$ into congruent tetrahedra.

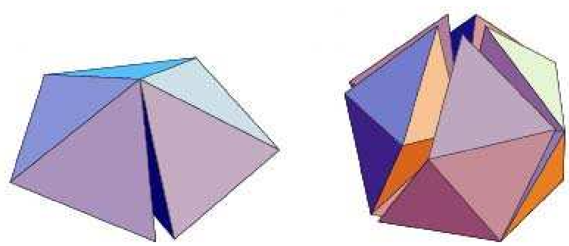


FIGURE 7. Trying to tile with regular tetrahedra.

Back to polytopes

VanderZee–Hirani–Zharnitsky–Guoy, Kopczyński–P.–Przytycki (2009) :

- ◇ acute triangulation of a cube (VHZG: 1370, KPP: 2715 tetrahedra)
- ◇ VHZG proof uses advanced simulation (mesh-improving technique)
- ◇ KPP proof uses the 600–cell (a regular polytope in $\mathbb{R}^4$)

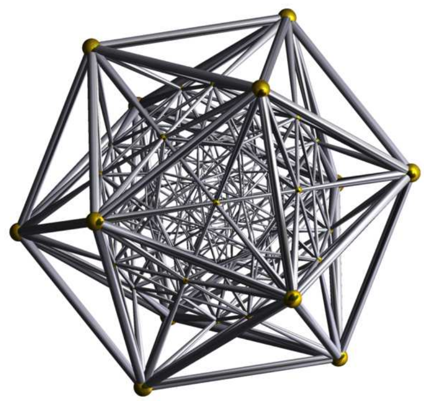


FIGURE 8. Graph drawn in the perspective projection of the 600-cell.

KPP construction step by step:

1) Make a nontrivial acute triangulation of the regular tetrahedron:
   - ✓ Take the 600–cell, remove facets adjacent to a fixed tetrahedron to obtain a 543–cell 3-dim surface.
   - ✓ Make a stereographic projection; check the dihedral angles.
   - ✓ Where angles are large, move vertices a bit; push exterior points to the boundary of a regular tetrahedron.
2) Do the same for the standard tetrahedron.
3) Assemble a cube from four standard and one regular tetrahedra.

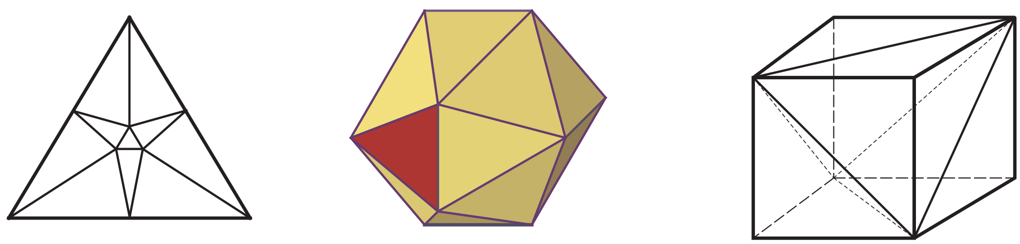


Figure 9. Steps of the KPP construction. (see more pictures...)

Theorem [KPP, 2009]
There is a non-trivial acute triangulation of all Platonic solids.

Conjecture: Every convex polytope in $\mathbb{R}^3$ has an acute triangulation.

Still open for non-obtuse triangulations; known in a number of special cases (see Bern-Chew-Eppstein-Ruppert, Brandts–Korotov–Křížek–Šolc).

$d \geq 5$ case: completely impossible

Theorem* [KPP, 2009]: A point in $\mathbb{R}^5$ cannot be surrounded with acute simplices.

Proof steps:

1) A triangulation of a d-manifold M is *rich* if every codim 2 face is surrounded with at least 5 simplices.
2) Use the generalized Dehn–Sommerville equations to show that for every rich 4-manifold M, we have:

$$\#\text{ of points in}\, M \;\leq\; \chi(M)\,.$$

3) For $d = 5$, take simplices containing a given point.
They give a rich triangulation of a 4-sphere, a contradiction.
4) The $d = 5$ case implies all $d > 5$ (Křížek).

* Křížek (2006) gave an erroneous proof of the theorem.

Kalai (1990) showed that every polytope in $\mathbb{R}^5$ has a 2-face with 3 or 4 vertices.

Dehn–Sommerville eq. for simplicial manifolds:

Theorem [Klee (1964), Macdonald (1971)]

Let M be a compact m–dimensional triangulated manifold with boundary. For $k = 0, \ldots, m$ we have:

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M).$$

Corollary: For $d = 4$ and $\partial M = \varnothing$, we have:

$$2f_1 = 3f_2 - 6f_3 + 10f_4, \qquad 2f_3 = 5f_4.$$

Lemma: If M is rich and closed, then $f_0 \leq \chi$.

Proof: Let N be the number of $(\Delta_2 \subset \Delta_4)$ flags. Then:

$$N = 10f_4, \quad N \geq 5f_2, \quad \text{which implies } f_2 \leq 2f_4.$$

Recall that $\chi = f_0 - f_1 + f_2 - f_3 + f_4$. We conclude:

$$2(\chi - f_0) = -2f_1 + 2(f_2 - f_3 + f_4) = -(3f_2 - 6f_3 + 10f_4) + 2(f_2 - f_3 + f_4)$$
$$= -f_2 + 4f_3 - 8f_4 = -f_2 + 10f_4 - 8f_4 = 2f_4 - f_2 \geq 0. \qquad \square$$

$d = 4$ case: clouds are gathering

♠ The 4-cube does not have an acute triangulation. (KPP)

♠ There is no periodic acute partition of $\mathbb{R}^4$. (KPP)

Main Theorem [KPP] :
For every $\varepsilon > 0$, there is no partition of $\mathbb{R}^4$ into simplices with all dihedral angles $< \pi/2 - \varepsilon$.

Proof of corollaries:

1) Acute triangulation of a 4–cube can be repeatedly reflected to make a periodic acute partition of the whole space $\mathbb{R}^4$.

2) A periodic acute partition of $\mathbb{R}^4$ gives a *rich triangulation* of a 4-torus, a contradiction.

$d = 4$ **case continued:**

♠ The 600–cell has an acute triangulation. (easy)

♠ What about other regular polytopes: the 16–cell (cross-polytope), the 24–cell, the 120–cell? (open*)

Conjecture****:** Space $\mathbb{R}^4$ has an acute partition.

* I am willing to bet $100 that the answer is NO for the regular cross-polytope.
Note: space $\mathbb{R}^4$ can be tiled with regular cross-polytopes.

** Brandts–Korotov–Křížek–Šolc (2009) make the opposite conjecture.

Outline of the proof of Main Theorem:

Lemma 1. If all dihedral angles are $< \pi/2 - \varepsilon$, then the simplices have *bounded geometry* (the ratio of the edge lengths in every tetrahedron is bounded).

Lemma 2. Let M be a compact 4–manifold which is a subcomplex of a rich partition of $\mathbb{R}^4$. Then $f_0 \leq 1 + f_2^\partial + f_1^\partial/2$.

Lemma 3. Let G be the graph (1-skeleton) of a rich partition of $\mathbb{R}^4$. Then: $|X| \leq C|\partial X|$ for all $X \subset G, |X| < \infty$, and some $C = C(\varepsilon)$.

Proofs: L1 is easy. L2 uses D–S equations (boundary version) + homology calculations. L3 follows directly from L1 and L2.

Definition [$p-$parabolicity of graphs]

$G = (V, W)$ is a locally finite infinite graph, $\Gamma(v)$ be the set of all semi-infinite self-avoiding paths γ in G starting from $v \in V$.

$L^p(V)$ is the L^p space of functions $f : V \to \mathbb{R}_+$ on vertices.

The *length* of a path γ in G is defined by $\mathrm{Length}_f(\gamma) = \sum_{w \in \gamma} f(w)$.

If graph G is *p–parabolic*, then

$$\mathrm{EL}(G) = \sup_{f \in L^p(V)} \inf_{\gamma \in \Gamma(v)} \frac{\mathrm{Length}_f(\gamma)^p}{\left(\|f\|_p\right)^p} = \infty.$$

Note: This definition does not depend on the choice of $v \in V$.

Theorem. [KPP, variation on Bonk–Kleiner (2002), etc.]
Graph of a triangulation of $\mathbb{R}^d$ *with bounded geometry is d-parabolic.*

Lemma. [Benjamini–Curien (2009)]
Let $G = (V, E)$ be a d–parabolic infinite locally finite connected graph, and $\mu : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\mu(|X|) \leq |\partial\Omega|$ holds for every finite $X \subset V$. Then for $p > d$, we have:

$$\sum_{k=1}^{\infty} \frac{1}{\mu(k)^{\frac{p}{p-1}}} = \infty.$$

Note: this is an extension of the Benjamini–Schramm inequality for impossibility of certain kissing sphere configurations in higher dimensions.