Acute triangulations of polytopes

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Los Angeles, October 20, 2009
The Problem:

A simplex $\Delta \subset \mathbb{R}^d$ is acute if all its dihedral angles are $< \pi/2$.
An acute triangulation is a finite subdivision into acute simplices.

1. For a convex polytope $P \subset \mathbb{R}^d$, find an acute triangulation.
2. Find an acute partition of $\mathbb{R}^d$.

Figure 1. An acute triangulation and an acute dissection of a square.
Main Corollary [Kopczyński–P.–Przytycki (2009)]

A $d$-cube has an acute triangulation only for $d \leq 3$.

This resolves an old folklore open problem:
Martin Garner (1960), Burago-Zalgaller (1960),

Note: There is an easy triangulation of a $d$-cube into $d!$ non-obtuse simplices.
Why acute triangulations?

- Classical geometric problem.
- Finite element method.
- Crystallography.
- Large recreational literature.
Finite element applications:

**Input:** triangulated surface $S \subset \mathbb{R}^3$.

**Goal:** find a good* triangulation of the interior of $S$.

*good means all tetrahedra are as close to regular as possible.
Figure 2. TetGen group from Berlin: triangulation of the head.

Statistics:

Input points:  20,796.  
Input facets:  41,588.  
Mesh points:  350,980.  
Mesh tetrahedra:  1,366,269.
Acute triangulations in the plane

▷ Proposed by Martin Garner (Scientific American, 1960)

▷ Resolved (independently) and extended by Burago–Zalgaller (1960)
  (Existence only, no complexity bounds follow from the proof).

▷ Easy to do in practice (Delaunay triangulations).

▷ Beginning 1980’s heavily studied in the DCG community.

Every $n$-gon in the plane has an acute triangulation with $O(n)$ triangles, which can be computed in linear time.
**Theorem:** Every polygon in the plane has an acute triangulation.

1) triangulate the polygon
2) make an acute triangulation of every triangle
3) assuming everything is rational: subtriangulate all triangles by the common denominator
4) otherwise, use an approximation argument
**Flips on triangulations**

**Idea:** use 2-flips to improve your triangulation (in any order).

**Theorem:** This always gives the Delaunay triangulation.

**Observation:** This always maximizes the min angle in a triangulation.

![Figure 4. 2-flips in a triangulation.](image-url)
Dimensionality curse

Philosophy: the higher the dimension, the harder it is to make acute triangulations (both theoretically and practically).

\( d = 2 \) – relatively easy
\( d = 3 \) – possible sometimes; perhaps, always
\( d = 4 \) – impossible sometimes; perhaps, very rarely
\( d \geq 5 \) – always impossible

Observation: Faces of an acute \( d \)-simplex are also acute simplices. Thus, acute triangulation of a \( d \)-cube contains acute triangulations of all \( n \)-cubes, for \( n < d \).
\[ d = 3 \text{ case: the beginning of a beautiful friendship} \]

♡ Studied for 30+ years. Until recently, very little progress.

♡ In practice: tile with Sommerville’s tetrahedra:

\[ 	ext{Figure 5. Sommerville’s tetrahedron.} \]
Diversion: acute partition of the space

- Aristotle: regular tetrahedron tiles the space (*On the Heaven*, 350 BC)
- Sommerville: four space-filling tetrahedra (1923).

![Figure 6. Aristotle, his error, Sommerville.](image)
More on acute partitions of the space

♥ Edmonds: if reflections are not allowed, then Sommerville’s 1923 classification is complete (2007)
♥ Delgado Friedrichs and Huson: no transitive acute partition of $\mathbb{R}^3$ (1999)

Open: Find an acute partition of $\mathbb{R}^3$ into congruent tetrahedra.

Figure 7. Trying to tile with regular tetrahedra.
Back to polytopes


- acute triangulation of a cube (VHZG: 1370, KPP: 2715 tetrahedra)
- VHZG proof uses advanced simulation (mesh-improving technique)
- KPP proof uses the 600–cell (a regular polytope in $\mathbb{R}^4$)

Figure 8. Graph drawn in the perspective projection of the 600-cell.
KPP construction step by step:

1) Make a nontrivial acute triangulation of the regular tetrahedron:
   ✓ Take the 600-cell, remove facets adjacent to a fixed tetrahedron to obtain a 543-cell 3-dim surface.
   ✓ Make a stereographic projection; check the dihedral angles.
   ✓ Where angles are large, move vertices a bit; push exterior points to the boundary of a regular tetrahedron.

2) Do the same for the standard tetrahedron.

3) Assemble a cube from four standard and one regular tetrahedra.

![Figure 9. Steps of the KPP construction. (see more pictures...)](image)
Theorem [KPP, 2009]
There is a non-trivial acute triangulation of all Platonic solids.

Conjecture: Every convex polytope in $\mathbb{R}^3$ has an acute triangulation.

Still open for non-obtuse triangulations; known in a number of special cases (see Bern-Chew-Eppstein-Ruppert, Brandts–Korotov–Křížek–Šolc).
\( d \geq 5 \) case: completely impossible

**Theorem** [KPP, 2009]: A point in \( \mathbb{R}^5 \) cannot be surrounded with acute simplices.

**Proof steps:**

1) A triangulation of a \( d \)-manifold \( M \) is *rich* if every codim 2 face is surrounded with at least 5 simplices.

2) Use the generalized Dehn–Sommerville equations to show that for every rich 4-manifold \( M \), we have:

\[
\#\text{ of points in } M \leq \chi(M).
\]

3) For \( d = 5 \), take simplices containing a given point.
   They give a rich triangulation of a 4-sphere, a contradiction.

4) The \( d = 5 \) case implies all \( d > 5 \) (Křížek).

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Kalai (1990) showed that every polytope in \( \mathbb{R}^5 \) has a 2-face with 3 or 4 vertices.
Dehn–Sommerville eq. for simplicial manifolds:

**Theorem** [Klee (1964), Macdonald (1971)]

Let $M$ be a compact $m$–dimensional triangulated manifold with boundary. For $k = 0, \ldots, m$ we have:

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M).$$

**Corollary:** For $d = 4$ and $\partial M = \emptyset$, we have:

$$2f_1 = 3f_2 - 6f_3 + 10f_4, \quad 2f_3 = 5f_4.$$
Lemma: If $M$ is rich and closed, then $f_0 \leq \chi$.

Proof: Let $N$ be the number of $(\Delta_2 \subset \Delta_4)$ flags. Then:

$$N = 10f_4, \quad N \geq 5f_2,$$

which implies $f_2 \leq 2f_4$.

Recall that $\chi = f_0 - f_1 + f_2 - f_3 + f_4$. We conclude:

$$2(\chi - f_0) = -2f_1 + 2(f_2 - f_3 + f_4) = -(3f_2 - 6f_3 + 10f_4) + 2(f_2 - f_3 + f_4)$$

$$= -f_2 + 4f_3 - 8f_4 = -f_2 + 10f_4 - 8f_4 = 2f_4 - f_2 \geq 0.$$

$\square$
$d = 4$ case: clouds are gathering

♠ The 4-cube does not have an acute triangulation. (KPP)
♠ There is no periodic acute partition of $\mathbb{R}^4$. (KPP)

Main Theorem [KPP]:
For every $\varepsilon > 0$, there is no partition of $\mathbb{R}^4$ into simplices with all dihedral angles $< \pi/2 - \varepsilon$. 
Proof of corollaries:

1) Acute triangulation of a 4–cube can be repeatedly reflected to make a periodic acute partition of the whole space $\mathbb{R}^4$.

2) A periodic acute partition of $\mathbb{R}^4$ gives a rich triangulation of a 4-torus, a contradiction.
$d = 4$ case continued:

♠ The 600–cell has an acute triangulation. (easy)

♠ What about other regular polytopes: the 16–cell (cross-polytope),
    the 24–cell, the 120–cell? (open*)

Conjecture**: Space $\mathbb{R}^4$ has an acute partition.

* I am willing to bet $100 that the answer is NO for the regular cross-polytope.

Note: space $\mathbb{R}^4$ can be tiled with regular cross-polytopes.

Outline of the proof of Main Theorem:

**Lemma 1.** If all dihedral angles are \( < \pi/2 - \varepsilon \), then the simplices have *bounded geometry* (the ratio of the edge lengths in every tetrahedron is bounded).

**Lemma 2.** Let \( M \) be a compact 4-manifold which is a subcomplex of a rich partition of \( \mathbb{R}^4 \). Then \( f_0 \leq 1 + f_2^\partial + f_1^\partial / 2 \).

**Lemma 3.** Let \( G \) be the graph (1-skeleton) of a rich partition of \( \mathbb{R}^4 \). Then: \( |X| \leq C|\partial X| \) for all \( X \subset G \), \( |X| < \infty \), and some \( C = C(\varepsilon) \).

**Proofs:** L1 is easy. L2 uses D-S equations (boundary version) + homology calculations. L3 follows directly from L1 and L2.
Definition \([p\text{-parabolicity of graphs}]\)

\(G = (V, W)\) is a locally finite infinite graph, \(\Gamma(v)\) be the set of all semi-infinite self-avoiding paths \(\gamma\) in \(G\) starting from \(v \in V\).

\(L^p(V)\) is the \(L^p\) space of functions \(f : V \to \mathbb{R}_+\) on vertices.

The length of a path \(\gamma\) in \(G\) is defined by \(\text{Length}_f(\gamma) = \sum_{w \in \gamma} f(w)\).

If graph \(G\) is \(p\text{-parabolic}\), then

\[
\text{EL}(G) = \sup_{f \in L^p(V)} \inf_{\gamma \in \Gamma(v)} \frac{\text{Length}_f(\gamma)^p}{\left(\|f\|_p\right)^p} = \infty.
\]

Note: This definition does not depend on the choice of \(v \in V\).
Theorem. [KPP, variation on Bonk–Kleiner (2002), etc.]

Graph of a triangulation of $\mathbb{R}^d$ with bounded geometry is $d$-parabolic.

Lemma. [Benjamini–Curien (2009)]

Let $G = (V, E)$ be a $d$–parabolic infinite locally finite connected graph, and $\mu : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\mu(|X|) \leq |\partial \Omega|$ holds for every finite $X \subset V$. Then for $p > d$, we have:

$$\sum_{k=1}^{\infty} \frac{1}{\mu(k)^{\frac{p}{p-1}}} = \infty.$$

Note: this is an extension of the Benjamini–Schramm inequality for impossibility of certain kissing sphere configurations in higher dimensions.