ETH
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On the Evolution of K_ℓ-free Graphs

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joint work with Lutz Warnke

Definition: Random graph

$G_{n,m}$: uniformly sampled from all n-vertex graphs with exactly m edges
$G_{n,p}$: each of the $\binom{n}{2}$ edges appears independently with probability p

Both are essentially equivalent:

- $G_{n,m} \approx G_{n,p}$ for $m \approx \binom{n}{2} p$

Classical results/thresholds:

- Containment of a fixed subgraph, e.g. K_3 or K_4
- Chromatic number

Key property of $G_{n,p}$:

- Independence of edges

Random H-free Graphs

General Problem

Consider random graphs with constraints (e.g. forbidden subgraphs)

Definition

Random H-free graph on n vertices
$\cong$ $G_{n,1/2}$ conditioned on not having H as a subgraph

Difficulty:

- Edges are not independent

Erdős-Kleitman-Rothschild (1976)

A random K_3-free graph on n vertices is whp 2-colorable:

$$\mathbb{P}\left[G_{n,1/2} \text{ is 2-colorable} \mid K_3\text{-free}\right] \rightarrow 1$$

Random H-free Graphs

Erdős-Kleitman-Rothschild (1976), Kolaitis-Prömel-Rothschild (1987)

A random $K_{\ell+1}$-free graph on n vertices is whp ℓ-colorable:

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Remark

- Erdős-Frankl-Rödl (1986): If $\chi(H) \geq 3$, then

$$\# \; H\text{-free graphs on } n \text{ vertices} = 2^{(1+o(1))\cdot \mathrm{ex}(n,H)}$$

- Prömel-St. (1992): If H has a color-critical edge, then a random H-free graph is whp $(\chi(H)-1)$-colorable
- recent progress on $o(\cdot)$-term by Balogh-Bollobás-Simonovits (2009+)
- corresponding problem for bipartite graphs is wide open

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Random H-free Graphs - Evolution

What is a suitable model for the "evolution" of H-free graphs?

Definition - Model 1

Random H-free graph on n vertices with m edges
$\cong$ $G_{n,m}$ conditioned on not having H as a subgraph

$\Longrightarrow$ **this talk**

Definition - Model 2

Order edges of complete graph uniformly at random,
start with empty graph and process edges sequentially,
insert edge if and only if it does not close a copy of H.

$\Longrightarrow$ **recent progress by Bohman (2009), Bohman-Keevash (2009+), e.g. for $H = K_3$ final graph has $\Theta(n^{3/2}\sqrt{\log n})$ edges**

Random H-free Graphs - Evolution

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Remark

- typical K_3-free graphs are dense, i.e. have $m \approx n^2/8$ edges
- $\exists$ sparse K_3-free graphs with 'high' chromatic number (Erdős, 1961)

What about K_3-free graphs with $m = o(n^2)$ edges?

Previous Results

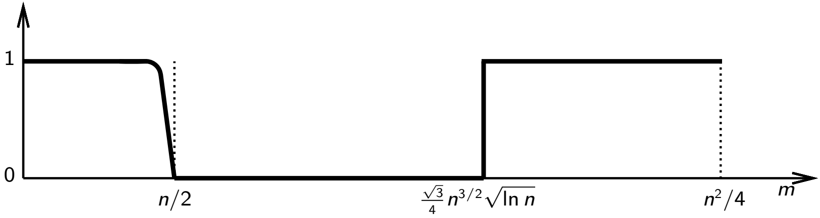
Prömel-St., St., Osthus-Prömel-Taraz (1996–2005)

$$\mathbb{P}\left[G_{n,m} \text{ is 2-colorable} \mid K_3\text{-free}\right] \to \begin{cases} 1 & \text{if } m = o(n) \\ 0 & \text{if } n/2 \le m \le (1-\varepsilon)t_2 \\ 1 & \text{if } m \ge (1+\varepsilon)t_2 \end{cases},$$

where

$$t_2 = t_2(n) = \sqrt{3}/4 \; n^{3/2}\sqrt{\ln n} \; .$$

Two phase transitions: (wrt. being 2-colorable)



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Similar result for odd cycles (Osthus-Prömel-Taraz, 2003)

- note: $K_3 = C_3$

What about larger cliques?

- for which m are random $K_{\ell+1}$-free graphs whp ℓ-colorable?

Our Results: Evolution of $K_{\ell+1}$-free Graphs

St.-Warnke (2009+)

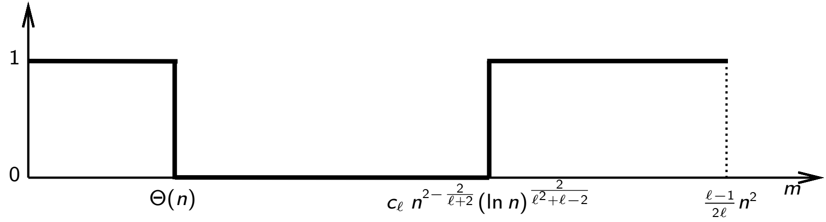
$$\mathbb{P}\left[G_{n,m} \text{ is } \ell\text{-colorable} \mid K_{\ell+1}\text{-free}\right] \to \begin{cases} 1 & \text{if } m \le (d_\ell - \varepsilon)n \\ 0 & \text{if } (d_\ell + \varepsilon)n \le m \le (1-\varepsilon)t_\ell \\ 1 & \text{if } m \ge (1+\varepsilon)t_\ell \end{cases},$$

where

$$t_\ell = t_\ell(n) = c_\ell\, n^{2-\frac{2}{\ell+2}} (\ln n)^{\frac{2}{\ell^2+\ell-2}} .$$

(second 1-statement holds provided the KŁR-Conjecture is true for $K_{\ell+1}$)

Two phase transitions: (wrt. being ℓ-colorable, $\ell \ge 3$)



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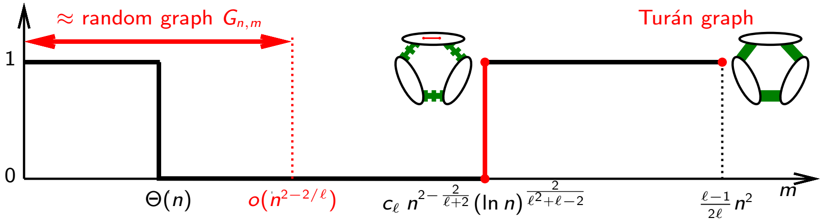

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KŁR-Conjecture: (Kohayakawa-Łuczak-Rödl, 1997)

- important open question in the theory of random graphs
- *verified for* K_3–K_5
- partial results for $K_{\ell+1}$ (valid for 'large' m)

Proof Outline for K_4-free Graphs

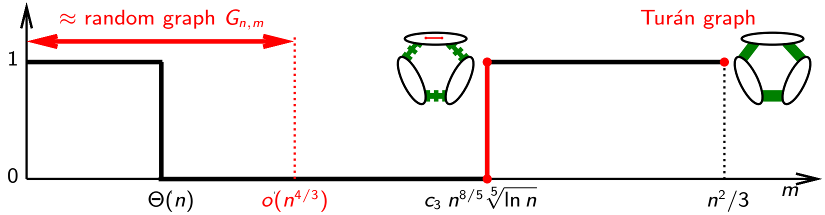
St.-Warnke (2009+)

$$\mathbb{P}\left[G_{n,m} \text{ is 3-colorable} \mid K_4\text{-free}\right] \to \begin{cases} 1 & \text{if } m \le (d_3 - \varepsilon)n \\ 0 & \text{if } (d_3+\varepsilon)n \le m \le (1-\varepsilon)t_3 \\ 1 & \text{if } m \ge (1+\varepsilon)t_3 \end{cases},$$

where

$$t_3 = t_3(n) = \sqrt[5]{8/135}\; n^{8/5} \sqrt[5]{\ln n}\ .$$

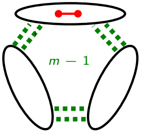
Intuition for the two phase transitions: (wrt. being 3-colorable)



Below the second Threshold

Theorem:

$$\mathbb{P}\left[G_{n,m} \text{ is 3-colorable} \mid K_4\text{-free}\right] \to 0 \qquad \text{if} \quad n \ln n \ll m \leq (1-\varepsilon) t_3$$



'Almost' 3-colorable:

- 3-colorable graph + edge inside one class

Counting-Lemma:

'Almost' 3-colorable K_4-free graphs $\gg$ 3-colorable graphs (K_4-free)

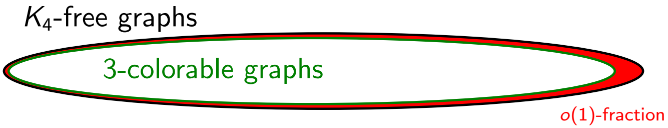
$$\mathbb{P}\left[G_{n,m} \text{ is 3-colorable} \mid K_4\text{-free}\right] = \frac{\#\text{ 3-colorable graphs}}{\#\ K_4\text{-free graphs}} = o(1)$$

Above the second Threshold

Theorem:

$$\mathbb{P}\left[G_{n,m} \text{ is 3-colorable} \mid K_4\text{-free}\right] \rightarrow 1 \qquad \text{if} \quad m \geq (1+\varepsilon)t_3$$

We count all K_4-free graphs that are not 3-colorable:



$\Rightarrow$ we intend to show that *almost all K_4-free graphs are 3-colorable*

Setup of Proof

We consider an 'optimal' 3-partition: (of a K_4-free graph)

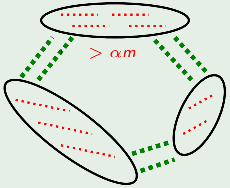
- minimizes # edges inside classes

Case distinctions

- (1) more than αm edges inside color classes
- (2) color classes unblanced
- (3) assume we have $k_A \geq k_B \geq k_C$ edges inside color classes, case distinction according to the structure of E_A:
  (3a) "many" vertices of "not too large" degree
  (3b) "many" vertices of "large" degree

Proof: Cases (1) & (2)

Counting K_4-free graphs that are not 3-colorable:



Graphs 'far' from 3-colorable: $o(1)$-fraction

- optimal 3-partition has $> \alpha m$ edges inside

'Unbalanced' graphs: $o(1)$-fraction

- class-sizes deviate much from $n/3$

Łuczak (2000) & Truth of KŁR-Conjecture for K_4:

For $m = \Omega(t_3)$ almost every K_4-free graph admits a 3-partition with $\leq \alpha m$ edges inside, where α is an arbitrary small constant.

Proof: Cases (1) & (2)

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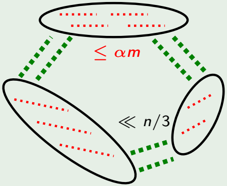

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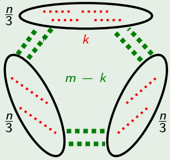
Counting-Lemma:

For $m = \Omega(t_3)$ almost every K_4-free graph with an optimal 3-partition having $\leq \alpha m$ edges inside is balanced, i.e. satisfies $|V_i| \approx n/3$.

Proof-Idea for Case (3):

It suffices to count K_4-free graphs with $1 \le k \le \alpha m$ inside edges:



We construct all such graphs by:

1. fixing a *balanced partition*
2. fixing the *k edges inside the vertex classes*
3. choosing the *remaining $m-k$ edges* (without creating a K_4)

Counting the number of choices in our construction:

$$\le 3^n \cdot \sum_{1 \le k \le \alpha m} \underbrace{\binom{n^2}{k}}_{\le n^{2k}} \cdot \binom{\frac{n^2}{3}}{m-k} \vartheta_k$$

$$\le \cdots \le \# K_4\text{-free graphs} \cdot \sum_{1 \le k \le \alpha m} n^{2k} \cdot \vartheta_k$$

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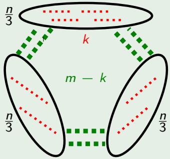

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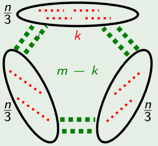
PROOF-IDEA FOR CASE (3):

Alternative point of view:

Insert 'random graph' between vertex classes:

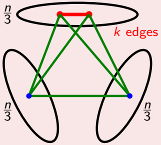
- $m - k$ random edges between the classes
- here: each appears independently with $p \approx 3m/n^2$



Approximating ϑ_k: **** Ignoring All Dependencies!!! ****

- ϑ_k = Probability of closing no K_4

$$\vartheta_k \approx \underbrace{\left[\left(1 - p^5\right)^{\left(\frac{n}{3}\right)^2}\right]^k}_{\leq \mathrm{e}^{-c\, kp^5 n^2}} \leq \mathrm{e}^{-c\, k \frac{m^5}{n^8}}$$



Above the second threshold: $\vartheta_k \leq n^{-4k}$

- Threshold: $t_3 = \Theta(n^{8/5} \sqrt[5]{\ln n})$

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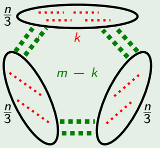

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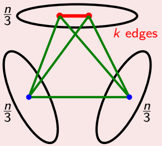

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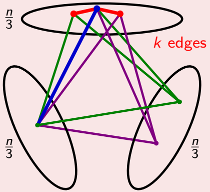

Bounded maxdegree:

- controls dependencies (→ Janson-Ineq.)

Calculation shows:

- suffices if 'most' vertices satisfy maxdegree condition

If this approach fails:

∃ 'many' vertices with 'high' degree inside class

And now the real work starts: show...

- ... this case is also unlikely !!

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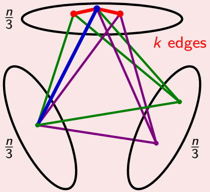

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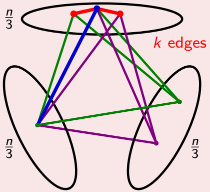

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