Reconstruction, Clustering and Mixing in Random CSPs

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Outline

Based on Current Collaborations

- A. Montanari, Ricardo Restrepo, and P. Tetali. *Reconstruction and Clustering in Random Constraint Satisfaction Problems*. e-print: arXiv:0904.2751; journal version, under revision.

- P. Tetali, J. Vera, E. Vigoda, and Linji Yang. *Phase Transition for the Mixing Time of the Glauber Dynamics for Coloring Regular Trees*. e-print: arXiv:0908.2665; Proc. of ACM-SIAM SODA 2010 (to appear).

Outline

Satisfiability and Clustering

AP Threshold for satisfiability of K-SAT is located at $2^k log2 - O(k)$.

AC Clustering occurs for K-SAT at $(1-\delta)2^k log2$.

AN $G(n, \alpha n)$ is q-colorable w.h.p. if $\alpha < \alpha_s(q)$, where $\alpha_s(q) = q(\log q + o_q(1))$. Vice versa, if $\alpha > \alpha_s(q)(1 + o_q(1))$, such a graph is not q-colorable w.h.p.

AC The set of proper q-colorings of $G(n, \alpha n)$ is clustered w.h.p. if $\alpha > \alpha_d(q)$, where $\alpha_d(q) = \frac{q}{2}(\log q + o(\log q))$.

Gerschenfeld-Montanari, FOCS'07

- If sphericity and balance holds. Then the reconstruction problem is solvable in a sparse random graph if and only if it is solvable in the local limiting tree.

Sly, Bhatnagar-Vera-Vigoda'08

BVV Reconstruction when $2\alpha > q \log q + o(q \log q)$.
Non-reconstruction when $2\alpha < q \log q - o(q \log q)$.

S Reconstruction when $2\alpha > q \log q + o(q \log q)$.
Non-reconstruction when
$2\alpha < q[\log q + \log \log q + 1 - \ln 2 - o(1)]$.

Outline

Model: general formula

For $k \geq 3$, a (generalized) k-clause is a binary fn.
$\phi : \{-1, 1\}^k \to \{0, 1\}$.

Consider Γ : set of such ϕ's and a probab. distr. p over Γ.

A random $\mathrm{CSP}_k(n, \alpha, p)$ is the random binary function over n variables $\underline{x} = (x_1, \ldots, x_n)$,

$$\psi(\underline{x}) = \prod_{a=1}^{\alpha n} \varphi_a\left(x_{i_a(1)}, \ldots, x_{i_a(k)}\right),$$

where φ_a, (for $a = 1, \ldots, \alpha n$) are i.i.d. with distribution p and $i_a(r)$, $a = 1, \ldots \alpha n$, $r = 1, \ldots, k$ are i.i.d uniform over $[n]$.

Model: contd...

Similar to the study of solutions of a k-SAT formula, one might study the set of solutions of ψ, that is the set $\psi^{-1}(1)$: for example,

Satisfiability : Is the solution set nonempty? How many elements does it contain?

Clustering : Is the solution set connected (in the Hamming distance)? How many connected components does it have? What is the diameter of each of them?

Basic Assumptions

Measure θ on $\{-1,1\}^k$: Product measure of the one that assigns the weight $\frac{1+\theta}{2}$ to 1 and $\frac{1-\theta}{2}$ to -1. When $\theta = 0$, is the uniform measure.

The internal product and norm for the associated Hilbert space is denoted by $(\cdot,\cdot)_\theta$ and $\|\cdot\|_\theta$. (when $\theta = 0$ we drop the θ in the notation).

Let $\gamma_Q(x) \stackrel{def}{=} \prod_{i\in Q} x_i$. This is a orthogonal basis for $(\cdot,\cdot)$. (Indeed consists of the algebraic characters, that is why we can call it Fourier basis).

Basic Assumptions

$$\|\varphi\|^2 = (\varphi, \gamma_\emptyset) = \frac{\#\text{ satisfying assignments}}{2^k},$$

$$\left\|\varphi^{(i)}\right\|^2 = \text{Influence of the first variable},$$

$$\left(\varphi^{(i)}, \gamma_Q\right) = \left(\varphi, \gamma_{Q \cup \{i\}}\right).$$

The Bonami-Beckner operator $(T_\theta \varphi)(x) = \|\varphi(xy)\|_\theta$

Basic Assumptions

1. *Permutation symmetry.* $p(\varphi^\pi) = p(\varphi)$ for all $\pi \in \mathcal{S}_k$.

2. *Balance.* $\varphi(x_1, \dots, x_k) = \varphi(-x_1, \dots, -x_k)$ for all $\varphi \in \mathit{supp}\,(p)$.

3. *Feasibility.* Every partial assignment $(x_1, \dots, x_{k-1})$ can be extended to a satisfying assignment $(x_0, x_1, \dots, x_{k-1})$ of φ. (which implies that $\|\varphi\|^2 \geq 1/2$).

4. *Dominance of balanced assignments.* $\mathbb{E}_\varphi \log \|\varphi\|_\theta \leq \mathbb{E}_\varphi \log \|\varphi\|$ for all $\theta \in [-1, 1]$.

Basic Assumptions

We are going to exhibit results asymptotically on k. Some agreement should exists as $k \to \infty$:
'Small weight' Fourier coefficients are small. There $a, C > 0$ (not depending on k), such that for all $\varphi \in supp\,(p)$,

$$\sum_{Q} \left| \left(\varphi^{(i)}, \gamma_Q \right) \right| \leq k^a \,. \tag{1}$$

$$\left\| T_\theta \varphi^{(i)} \right\|^2 \leq e^{-Ck\,\left(1-\theta^2\right)} \left\| \varphi^{(i)} \right\|^2 \,, \; \theta \in [0,1] \,. \tag{2}$$

Basic Assumptions

The above implies in particular, that for any fixed ℓ, there exists $A_\ell > 0$ such that

$$\sum_{1 \le |Q| \le \ell} |\varphi_Q|^2 \le A_\ell \, e^{-Ck/2} \sum_{|Q| \ge 1} |\varphi_Q|^2 \qquad (3)$$

Results...

An ensemble of binary k-CSP's is characterized by the following quantities.

$$\frac{1}{\Omega_k} \stackrel{def}{=} \mathbb{E}_\varphi \frac{2I_i(\varphi)}{\|\varphi\|^2} \left(= \mathbb{E}_\varphi \frac{1}{\|\varphi\|^2} - 1 \right), \qquad \frac{1}{\widehat{\Omega}_k} \stackrel{def}{=} -2\mathbb{E}_\varphi \log\left(\|\varphi\|\right).$$

$$\frac{1}{\widetilde{\Omega}_k} \stackrel{def}{=} \frac{2\mathbb{E}_\varphi I_i(\varphi)}{\mathbb{E}_\varphi \|\varphi\|^2}$$

Notice that $\Omega_k \leq \widehat{\Omega}_k$ and $\Omega_k \approx \widetilde{\Omega}_k$, whenever the influence is relatively small, or equivalently when the norm is close to 1.

Results: Satisfiability and Clustering

Theorem

A random binary constraint satisfaction instance from the $CSP(n, \alpha, p)$ ensemble is satisfiable, with high probability, if $\alpha < \alpha_{\rm s}(k)(1 - o_n(1))$, where

$$\Omega_k \, \log 2 \, \{1 + o_k(1)\} \le \alpha_{\rm s}(k, n) \le \widehat{\Omega}_k \, \log 2 \, \{1 + o_k(1)\} \, .$$

Vice versa, if $\alpha > \alpha_{\rm s}(k)(1 + o_n(1))$, then with high probability, a $CSP(n, \alpha, p)$ instance is unsatisfiable.

Results: Satisfiability and Clustering

Theorem

Consider a $\mathrm{CSP}(n, \alpha, p)$ *ensemble satisfying the above conditions. The set of solutions of a random instance from this ensemble is clustered, with high probability, if* $\alpha > \alpha_{\mathrm{d}}(k)$, *where*

$$\alpha_{\mathrm{d}}(k) = \frac{\widetilde{\Omega}_k}{k} \{\log k + o(\log k)\} .$$

Result on Reconstruction

Theorem

Let $\mu(\underline{x})$ be the uniform measure over solutions of an instance from the $\mathrm{CSP}_k(n, \alpha, p)$ ensemble. The reconstruction problem is solvable, with high probability, for μ if $\alpha > \alpha_{\mathrm{r}}(k)$, where

$$\alpha_{\mathrm{r}}(k) = \frac{\Omega_k}{k} \{\log k + o(\log k)\} .$$

Vice versa, the reconstruction problem is unsolvable, with high probability, if $\alpha < \alpha_{\mathrm{r}}(k)[1 - o_k(1)]$.

Results...

Thus, a key result is that $\alpha_{\mathrm{d}}(k)$ and $\alpha_{\mathrm{r}}(k)$ do *coincide for a large family of ensembles* (up to the slackness, in the second order terms, of our bounds).

Outline

Result on reconstruction

Theorem

(Graph q-coloring reconstruction) Let $\mu(\underline{x})$ be the uniform measure over of proper q-colorings of random graph with n vertices and $n\alpha$ edges. For q large enough, the reconstruction problem is solvable for μ if $\alpha > \alpha_{\rm r}(q)$, where

$$\alpha_{\rm r}(q) = \frac{q}{2}\left[\log q + \log\log q + O\left(1\right)\right] .$$

Vice versa, the reconstruction problem is unsolvable, with high probability, if $\alpha < \alpha_{\rm r}(q)$.

Proof: Verify sphericity condition!

Outline

Spatial mixing vs Dynamical mixing

For colorings of sparse random graphs (and more generally, random CSP's), could there be a connection between the spatial threshold and any dynamical threshold?

A theorem of Berger et al for general spin systems (such as Ising, Independent sets, Colorings etc.) on bdd. degree graphs implies that when reconstruction holds, the relaxation time (i.e., inverse spectral gap) is strictly slower than linear in n, in discrete time, on an n-vertex bounded degree graph. (see next slide)

But, a suitable converse is missing.

Reconstruction implies "slower" mixing of Glauber

[Berger, Kenyon, Mossel, Peres '05]. Let G be bdd. degree graph, and let σ_r be config. on the set G_r of all vertices at distance r from a fixed vertex v. Let the relaxation time of Glauber dynamics on G_r satisfy $\tau_2(G_r) = O(1)$, in *continuous time*. Then the Gibbs distribution on G_r has the following property:

For any fixed set A of vertices, $\exists c_A > 0$, s.t. for r large enough,

$$\mathrm{Cov}[f, g] \leq e^{-c_A r}\sqrt{\mathrm{Var}(f)\mathrm{Var}(g)}\,,$$

provided $f(\sigma)$ depends only on σ_A and $g(\sigma)$ depends only on σ_r.

- Contrapositive gives what was asserted before.

Glauber for Colorings on Trees

Some recent results:
q = no. of colors. T_b : complete b-branching tree.
Consider Glauber dynamics on T_b on n vertices:
pick a vertex u.a.r. and a color u.a.r. *from its available set of colors*, and apply the color to the vertex.

[BKMP '05]. $q \geq b + 2$ implies polynomial (in n) mixing time, irrespective of the boundary condition; need not be regular tree.

[Martinelli-Sinclair-Weitz '04, '07]. $q \geq b + 3$ implies $O(n \log n)$ mixing time, irrespective of the boundary condition.

Glauber for Colorings on Trees: contd...

[Goldberg, Jerrum, Karpinski '08].
On T_b with $b \geq 3$, and $3 \leq q \leq b/(2 \ln b)$:

Lower bound on mixing time: $n^{\Omega(\frac{b}{q \ln b})}$.

Upper bound on mixing time: $n^{O(\frac{b}{\ln b})}$.

[Lucier-Molloy'08], [L-M-Peres'09].
On tree with *max. degree* b for $b \geq 3$, and $3 \leq q$:
Upper bound on mixing time: $n^{O(1+\frac{b}{q \ln b})}$.

Key: what happens when $q = c\frac{q}{\ln q}$, for a constant $c > 0$?

Phase Transition in mixing

[PT.-J.Vera-E.Vigoda-Linji Yang'09].
Refining and building on earlier proofs.
On T_b with b: large, $q = c\frac{q}{\ln q}$, for a constant $c > 0$.
T_{rel} : relaxation time = inverse gap.

(I) For $c \geq 1$:

$$\Omega(n) \leq T_{\mathrm{rel}} \leq O(n^{1+o(1)}),$$

(II) For $c < 1$:

$$\Omega(n^{1/c-o(1)}) \leq T_{\mathrm{rel}} \leq O(n^{1/c+o(1)}),$$

where all the little oh's ($o(1)$) are going to zero as $b \to \infty$.

Mixing and Reconstruction Transitions

Summary:

For q-coloring on regular b-branching trees,
$q = \frac{b}{\ln b}(1 + o_b(1))$ is a critical point for transition
in mixing rate of Glauber dynamics, as well as for a transition in reconstruction-nonreconstruction regimes.

Outline

Open Problems

"You will only be remembered for two things: the problems you solve or the ones you create." Mike Murdock, 101 Wisdom Keys

1. Relaxing some of the Fourier assumptions.
2. Directly going from Reconstruction to Clustering? – the other direction is clearer.
3. Directly : Non-reconstruction to Fast mixing of Glauber? – other direction is known [BKMP'05]
4. Efficiently finding solutions up to the Clustering threshold for *general CSPs*, a la Coja-Oghlan? – "Mick gets some..." how about the rest of us!

Selected References

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The end

THANKS !!