Hamilton Cycles in Random Graphs

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C.M.U.

$G_{n,m}$ and $G_{n,p}$

$\downarrow$

vertex set $[n]$

m random edges

$\downarrow$

vertex set $[n]$

Each of $N = \binom{n}{2}$ edges independently included with probability p.

If $m = Np \to \infty$

then $G_{n,m}$ and $G_{n,p}$

are similar.

Aim:

Estimate

$$\Pr(G_{n,m} \text{ is Hamiltonian}).$$

Pósa's breakthrough:

$G_{n,p}$ is Hamiltonian whp

if $p \geq \frac{k \log n}{n}$

Pósa 1976

whp $\equiv$ Prob. $1 - o(1)$.

THEOREM Komlós, Szemerédi 1983

Let $m = \frac{n}{2}(\log n + \log\log n + c_n)$.

$$\lim_{n\to\infty} \Pr(G_{n,m} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

Prob. $\delta(G_{n,m}) \geq 2$

Graph Process:

Let $e_1, e_2, \ldots, e_N$ be a random ordering of the edges of K_n.

$$G_m = ([n], \{e_1, e_2, \ldots, e_m\})$$

$G_0, G_1, \ldots, G_N = K_n$; Graph Process

$$G_m \sim G_{n,m}$$

For property $\mathcal{P}$,

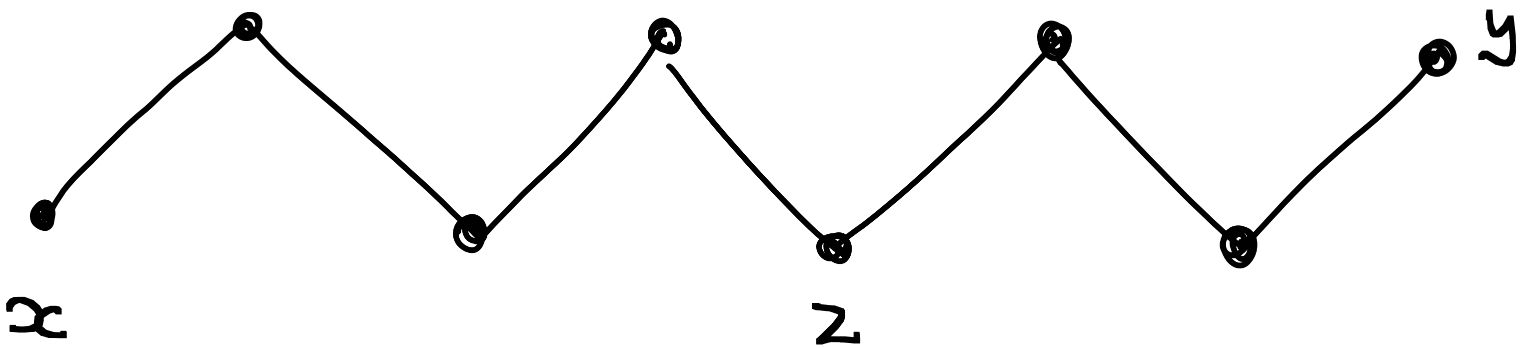
$$\tau_{\mathcal{P}} = \min\{m : G_m \in \mathcal{P}\}$$

THEOREM Ajtai, Komlós, Szemerédi 1985
Bollobás 1984

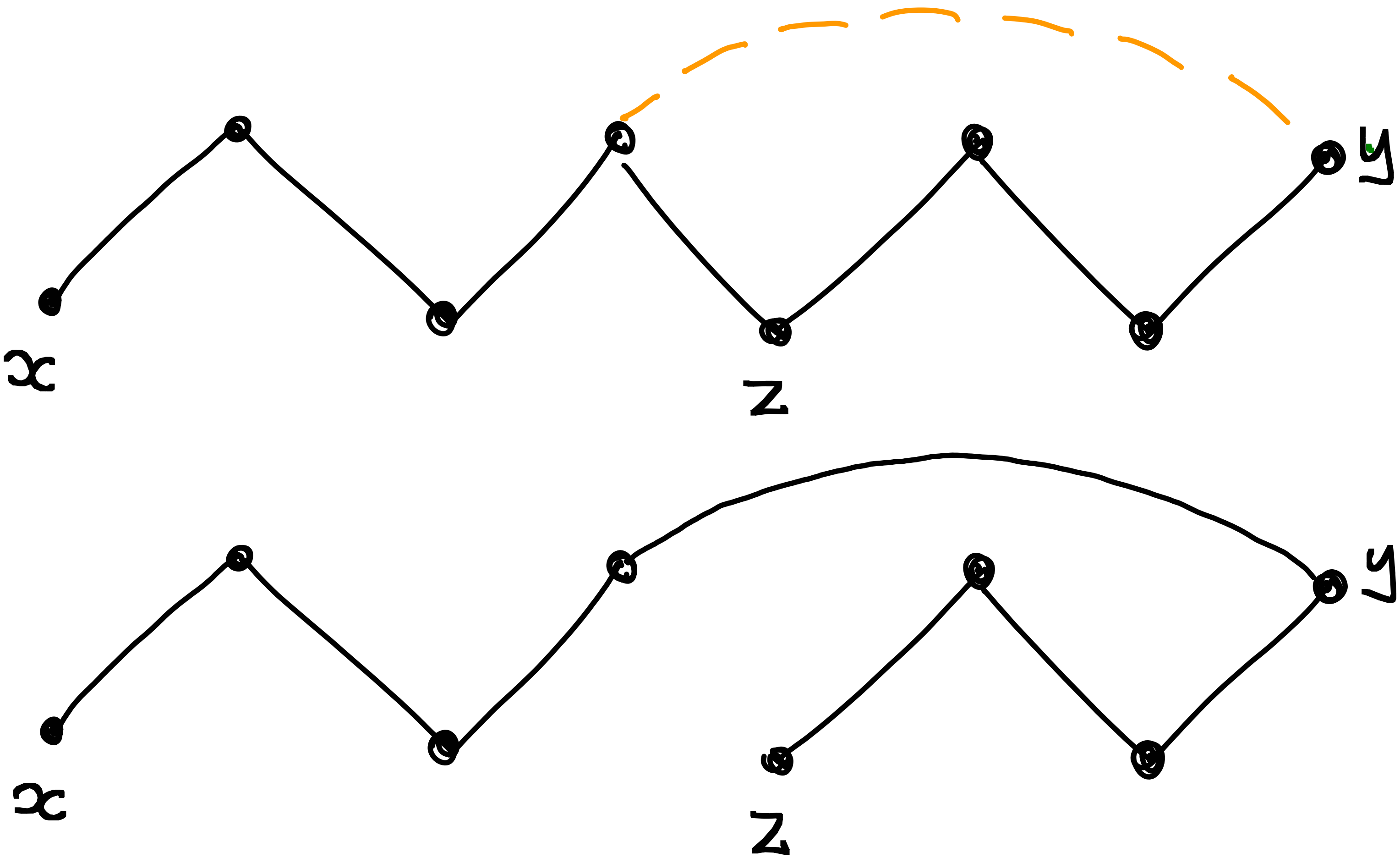
$$\tau_{\text{Hamiltonian}} = \tau_{\delta \geq 2} \quad \text{whp}$$

i.e. when randomly adding edges one by one, G_m becomes Hamiltonian when it first has minimum degree two.

Posá Rotations

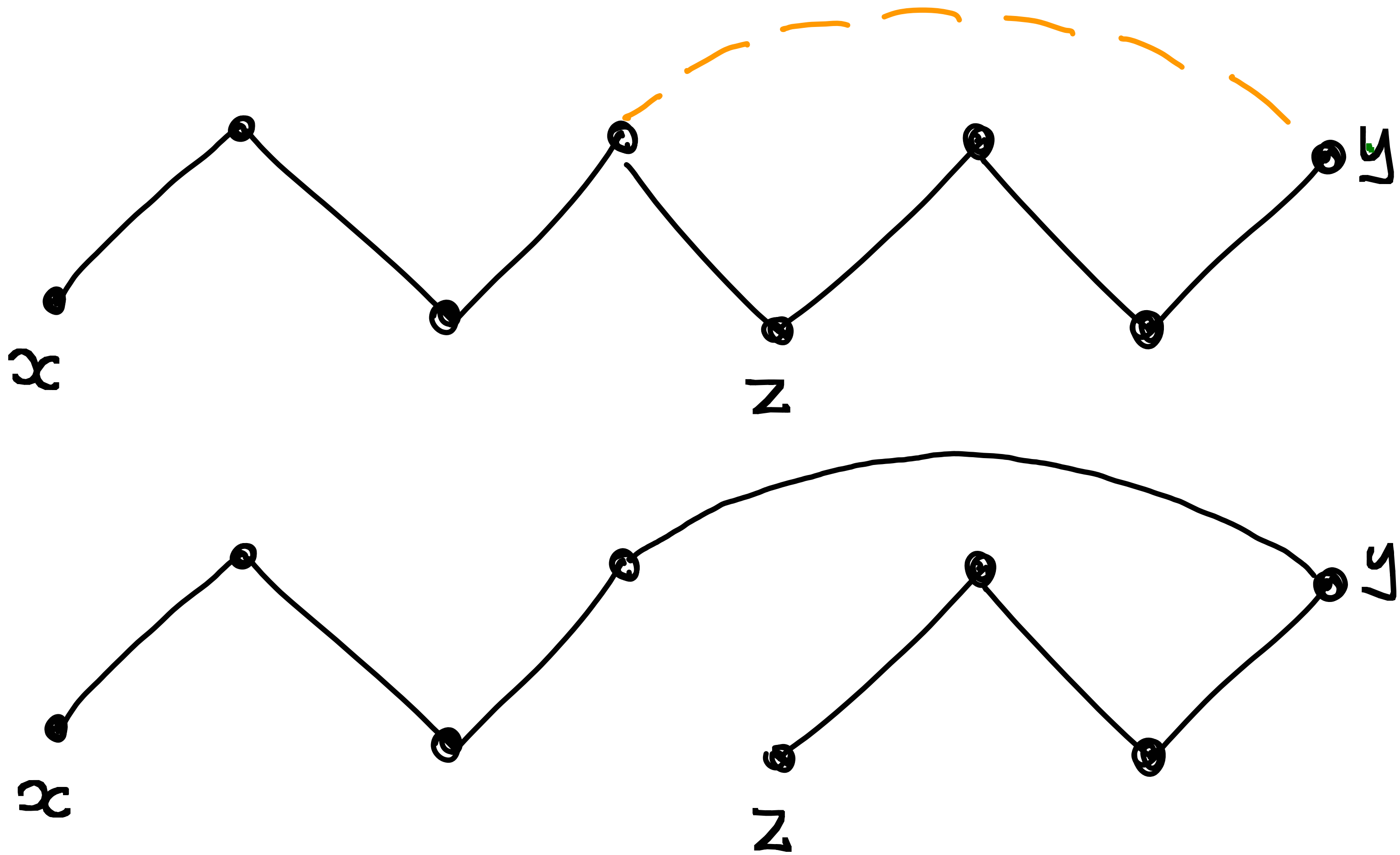


Longest Path in G



New longest path.

z replaces y as endpoint



Rotation with x as fixed endpoint.

Algorithm HAM: Bollobás, Fenner, Frieze 1987

HAM runs in $n^{3+o(1)}$ time

It is deterministic

$$\lim_{n\to\infty} \Pr(\text{HAM succeeds}) = \lim_{n\to\infty} \Pr(G_{n,m} \text{ is Hamiltonian})$$

Suppose that

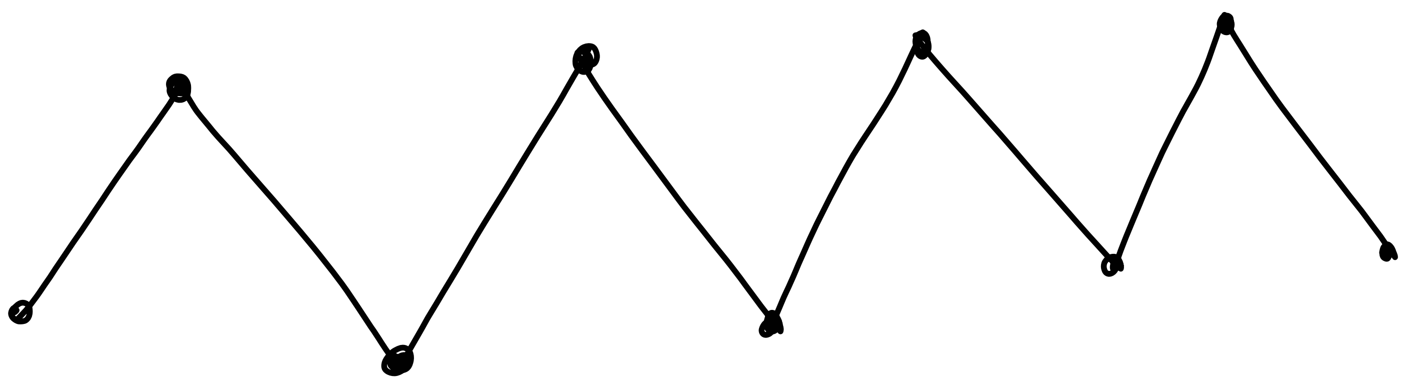
$$m = \frac{n}{2}\left(\log n + \log\log n + c_n\right)$$

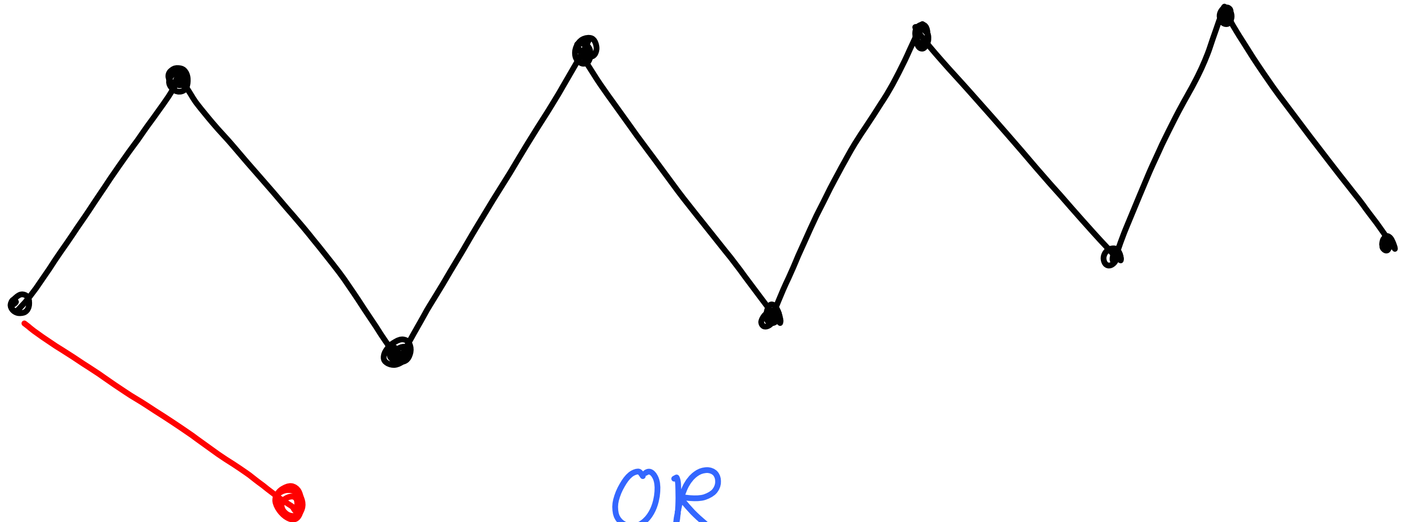
$c_n \not\to -\infty$

$G_{n,m}$ is connected whp

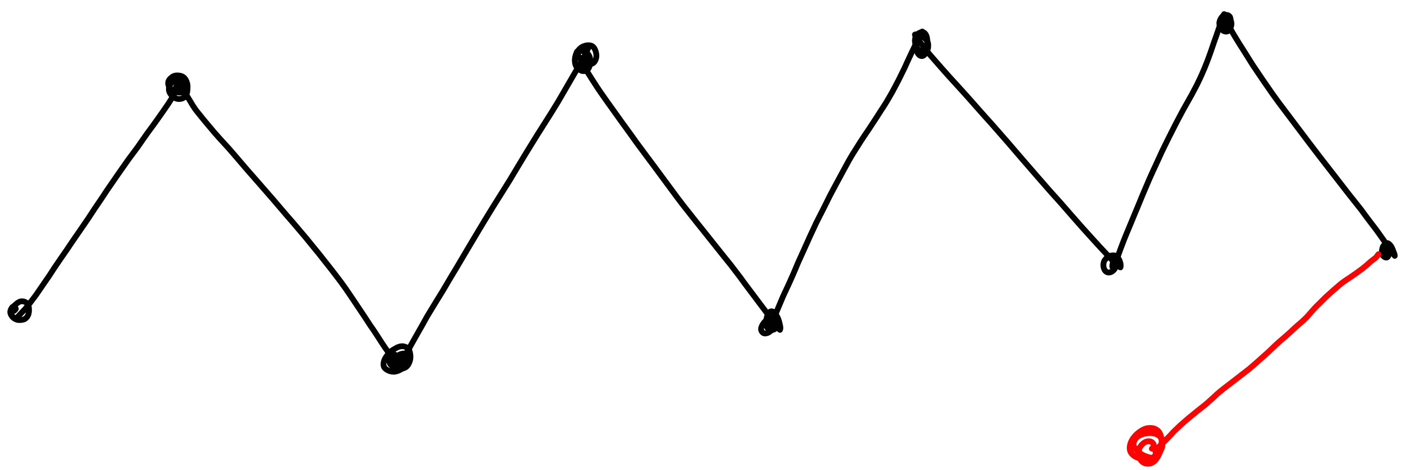
Stage k

Currently have path of length k.

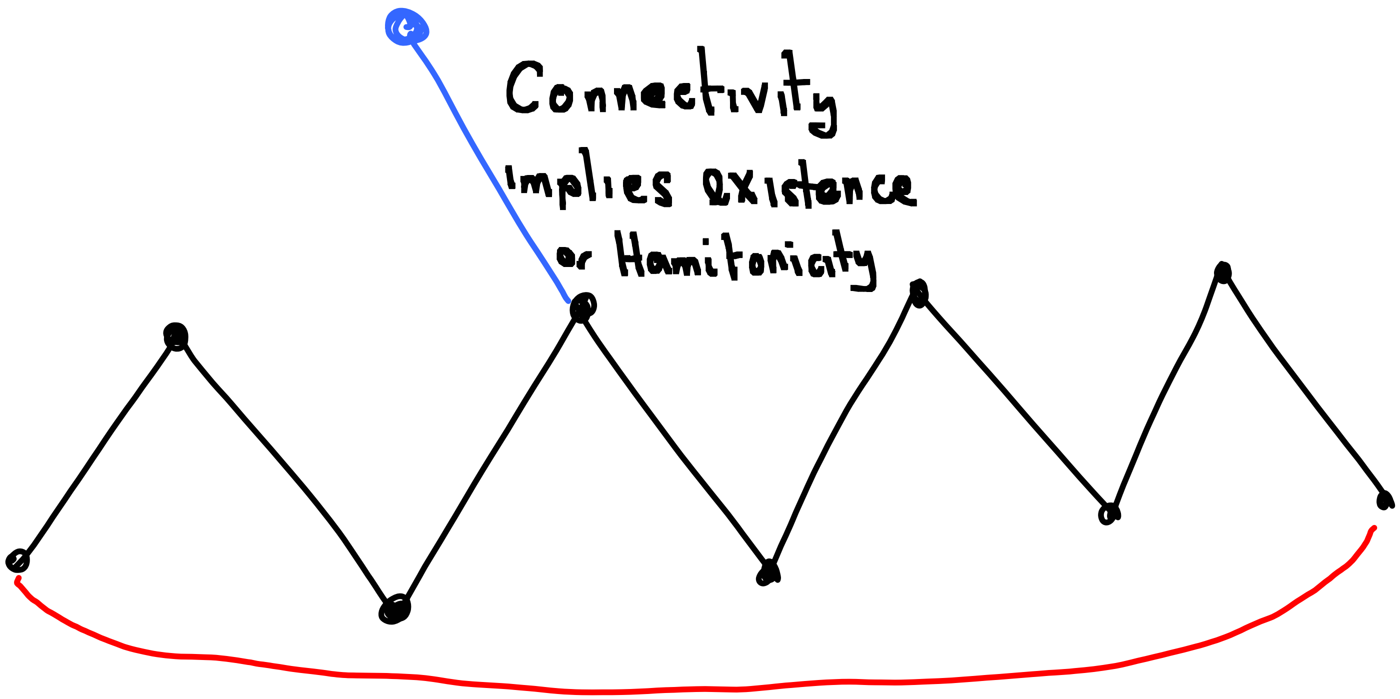




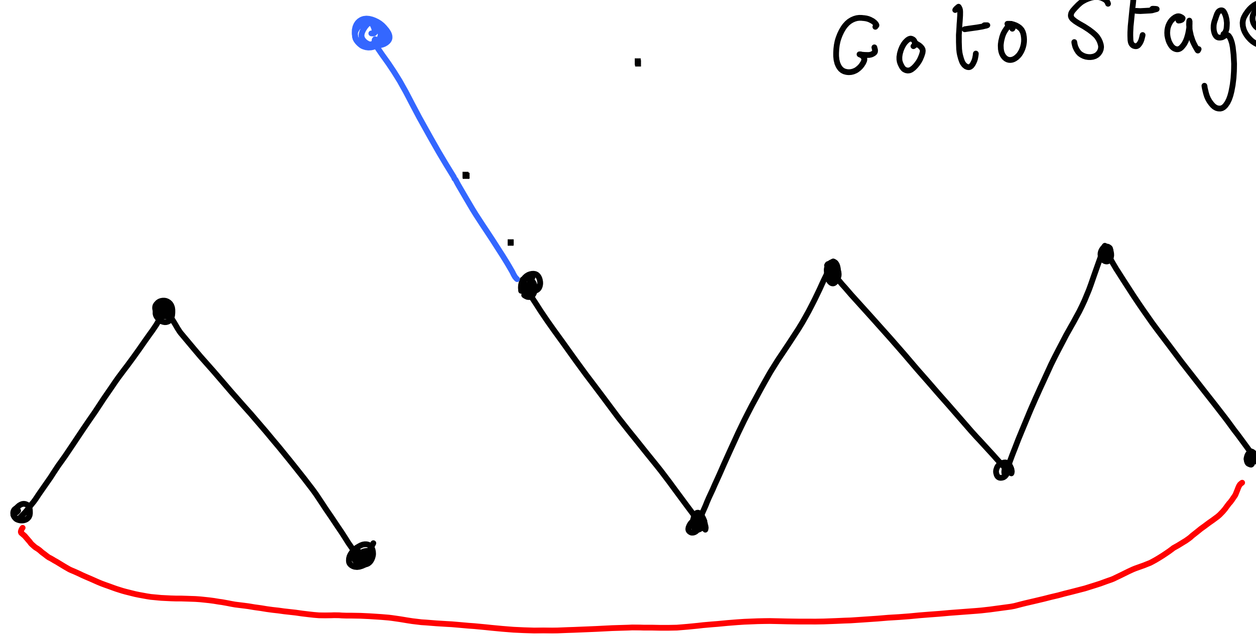
OR

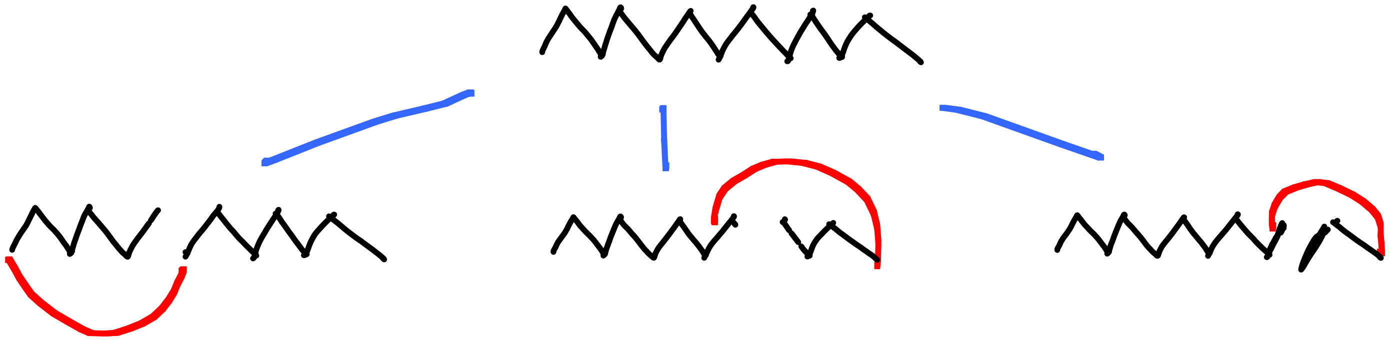


GO TO Stage $k+1$



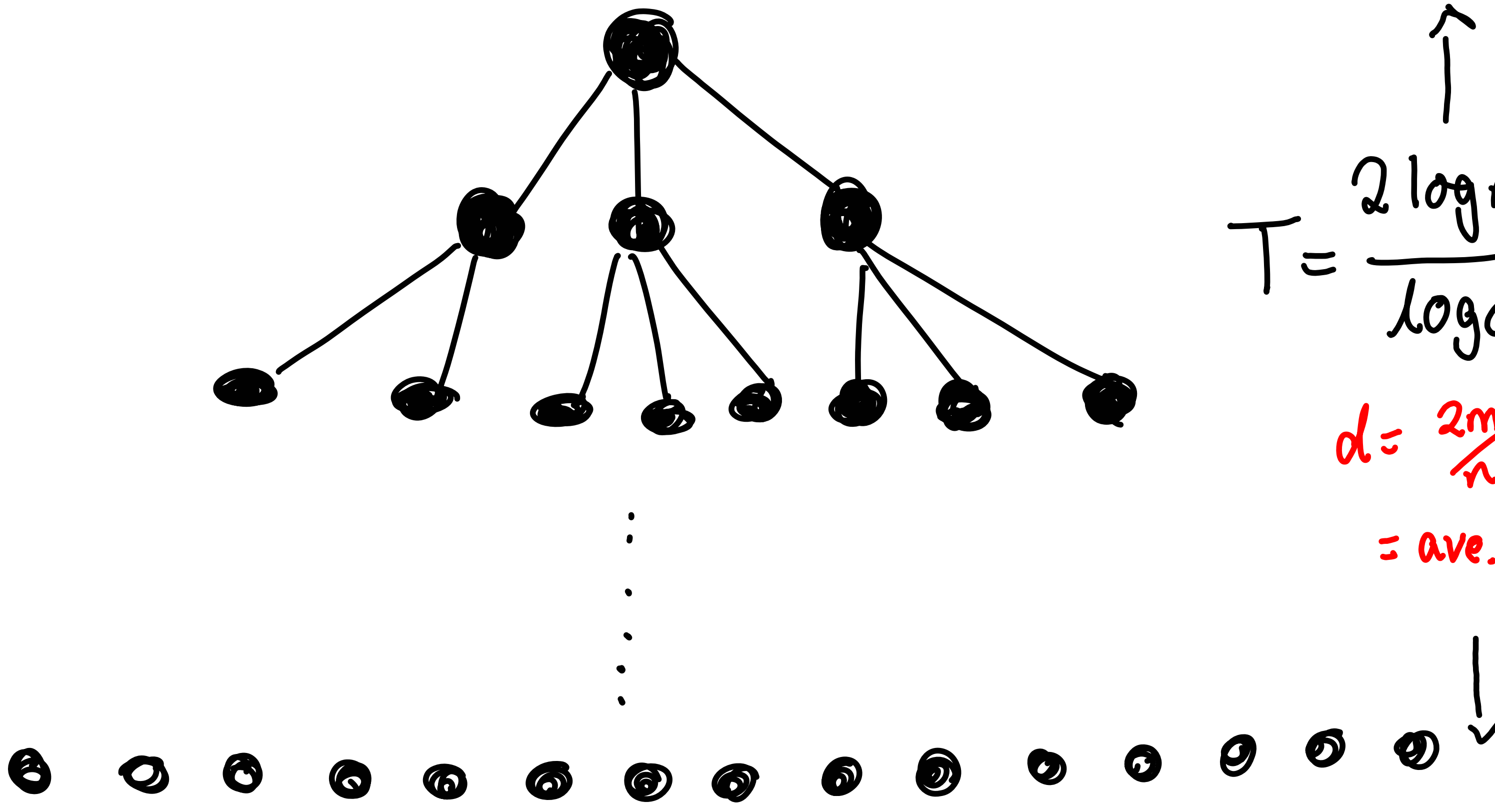
Go to Stage k+1





ROTATE FROM

BOTH ENDS

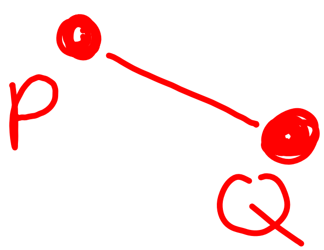


$$T = \frac{2\log n}{\log d}.$$

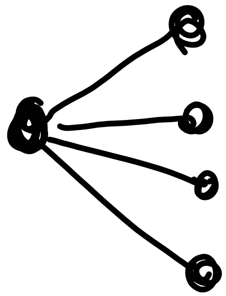
$d = 2m/n$

$=$ ave. deg.

Each ● is a path



Q obtained from P by a rotation.

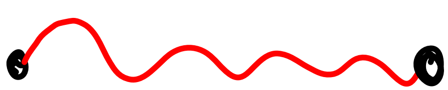
(i)  $\leftarrow \leq 5d \Rightarrow$ construction is poly time

(ii) If no longer path found then at bottom of tree

$\exists\, cn$ vertices END

and for each $x \in$ END

there are $\geq cn$ y with a

path x  y

$\text{END}(x) = \{y\}$

$\boxed{\text{BIG TREE PROPERTY}}$

Argument for

$$m = \frac{n}{2}(\log n + \log\log n + \omega)$$

Let $m' = m - \frac{\omega n}{4}$.

$G_{n,m'}$ has big tree property

Now whenever HAM gets stuck on $G_{n,m'}$ add random edges, from $m-m'$ not used, until we find edge (x,y) where $y \in END(x)$

$Pr[\text{next edge OK}] \geq c^2$.

$\Pr[$ HAM runs out of OK edges $]$

$$\leq \Pr\left[\operatorname{Bin}\left[\frac{\omega n}{4}, c^2 \right] < n \right]$$

$$= o(1).$$

Previous argument cannot work for $c_n \to c$.

Also, algorithm is a bit artificial.

Now suppose we run HAM as is:

$$m = \frac{n}{2}(\log n + \log\log n + c)$$

and condition: $\delta \geq 2$.

Suppose that we randomly choose $\omega = \log n$ random edges X.

$H = G - X$.

$\mathcal{A} = \{$ HAM fails on G + expansion $\}$

$\mathcal{B} = \{$ HAM does as well on H as on G + expansion $\}$

$$\Pr(\mathcal{B} \mid A) \geq (1 - o(1))^{\omega}$$

Just avoid $o(n)$ important edges

$$\Pr(\mathcal{B}) \leq (1-\epsilon)^{\omega}$$

Have to avoid lots of choices

So

$$\Pr(\mathcal{A}) \leq \left(\frac{1-\epsilon}{1-o(1)}\right)^{\omega} = o(1).$$

Suppose HAM fails in Stage k.

Let P be path of length k constructed.

$W = \{$edges <u>used</u> to make $P\}$

$|W| \leq \underset{\#\text{stages}}{n} \times \underset{\substack{\#\text{used per stage} \\ \text{for } P}}{2T}$

If edges $X \cap W = \emptyset$ are deleted then HAM will begin Stage k with P and fail in Stage k.

$(*)$

If edges X are deleted

and

(i) X is not incident with low degree vertices

(ii) No vertex is incident with $\geq d/1000$ edges of X

then

G-X will have the big tree property

(**)

A set of edges X,

$|X| = \log n \leftarrow \ell$

is deletable if

$(*)$, $(**)$ hold and

HAM gets no further in G

than in $G - X$.

$$a(G,X) = \begin{cases} 1 & G \text{ normal}^{*} \\ & G \ \neg \text{Hamiltonian} \\ & X \text{ deletable} \\ 0 & \text{otherwise} \end{cases}$$

$X \subseteq E(G)$

normal $\equiv$ some $1-o(1)$ properties.

(1) G normal & non-Hamiltonian

$$\Rightarrow \sum_{X} a(G, X) \geq (1-o(1)) \binom{m'}{\ell}$$

$$m' = m - 2\tau n = (1-o(1))\, m$$

(ii) For fixed H with $m-\ell$ edges

$$\sum_X a(H+X, X) \leq \binom{N'-m+\ell}{\ell}$$

where $N' = \binom{n}{2} - \binom{cn}{2}$

Not allowed to join x to $END(x)$, else HAM gets further in $H+X$ than in H.

$$\Pr[G_{n,m} \text{ not Hamiltonian} \mid \delta \geq 2]$$

$$\leq o(1) \qquad \text{abnormal}$$

$$+ \frac{\binom{N}{m-\ell}\binom{N'-m+\ell}{\ell}}{(1-o(1))\binom{m'}{\ell}\binom{N}{m}}$$

$$= o(1).$$

Open Question

Is there a polynomial expected time algorithm that correctly determines Hamiltonicity of $G_{n,m}$, for all m?

Property $\mathcal{A}_k$

A graph has property $\mathcal{A}_k$ if it contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles + disjoint $\lfloor n/2 \rfloor$ matching if k is odd.

Whp

$$\tau_{\mathcal{A}_k} = \tau_{\delta \geq k}$$

Bollobás, Frieze 1985

$$k = O(1)$$

$$m = (1+o(1))\, n \log n / 2$$

$$G_{n,m} \in \mathcal{A}_{\delta(G_{n,m})}$$

whp

Frieze, Krivelevich 2008

Conjecture

$G_m \in \mathcal{A}_{\delta(G_m)}$ whp

throughout graph process.

$G_{n,1/2}$ contains

$$\frac{n}{4} - O(n^{5/6} \ln^{1/6} n)$$

what is correct error term?

edge disjoint Hamilton cycles whp.

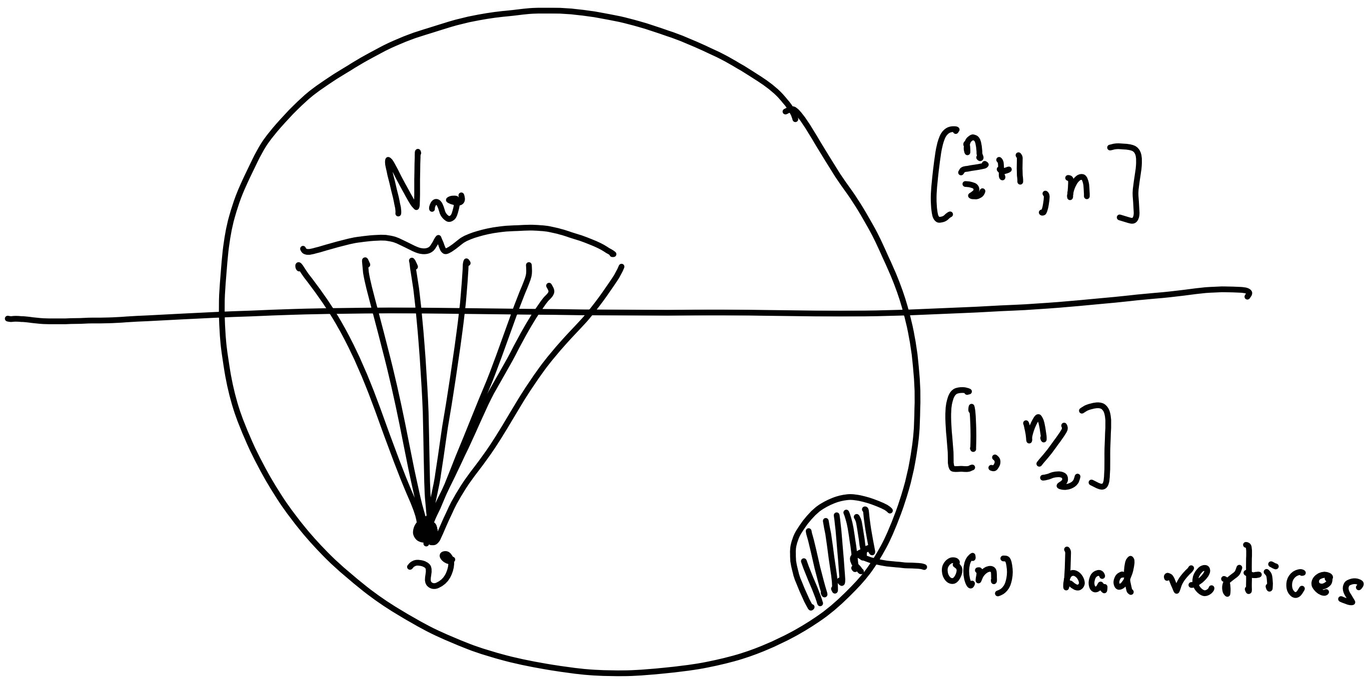
Frieze, Krivelevich 2005

If $m = \tau_{\delta \geq 2}$ then whp

G_m contains $(\log n)^{n-o(n)}$

distinct Hamilton cycles

Cooper, Frieze 1989



For each v, construct $s = (\log n)^{2-o(1)}$ sets $X_1, X_2, \dots, X_s \subseteq N_v$ where each $|X_i| = \Theta((\log\log n)^2)$ and $|X_i \cap X_j| \le 1$.

For each of $n/2$ v's choose one X_j: # choices is $s^{n/2 - o(n)} = (\log n)^{n - o(n)}$.

For almost all choices + edges up top, get a Hamiltonian subgraph.

By construction H-cycles found are distinct

Random Bipartite Graphs

$B_{n,p}$ = random subgraph of $K_{n,n}$ where we keep edge with probability p.

Let $p = \frac{1}{n}(\log n + \log\log n + c_n)$

$$\lim_{n\to\infty} \Pr[B_{n,p} \text{ is Hamiltonian}] = \begin{cases} 0 & c_n \to -\infty \\ e^{-2e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

Frieze 1985; Bollobás, Kohayakawa 1991

(shorter proof)

Resilience

How resilient is Hamiltonicity to adversarial edge deletion:

(i) $m = \Omega(n \log^4 n) \Rightarrow$ adversary can delete up to $(\frac{1}{2}-\epsilon)\, d(v)$ edges at degree v and what is left of $G_{n,m}$ is Hamiltonian

Sudakov, Vu 2008

(ii) $m = Kn\log n \Rightarrow$ $\cancel{\frac{1}{2}-\epsilon}$ ϵ

Frieze, Krivelevich 2008

Conjecture: result of (i) true given hypothesis of (ii)

Directed Graphs

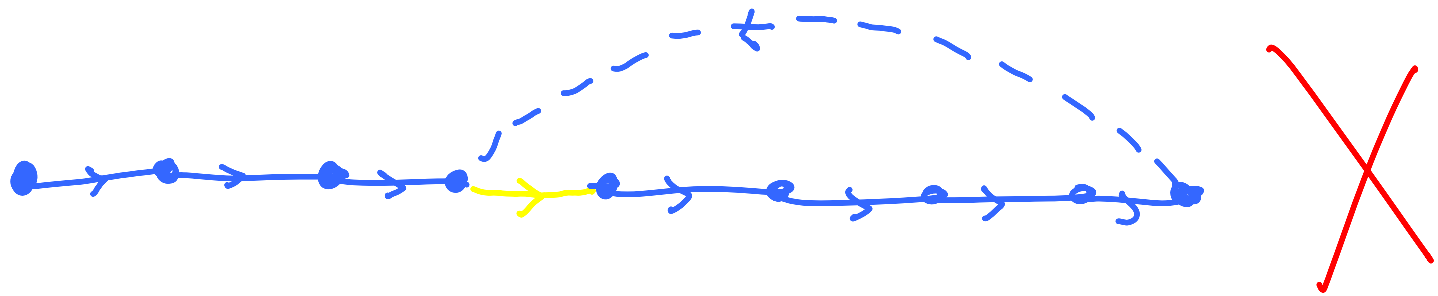
$D_{n,m}$ is random digraph with vertex set $[n]$ and m random directed edges.

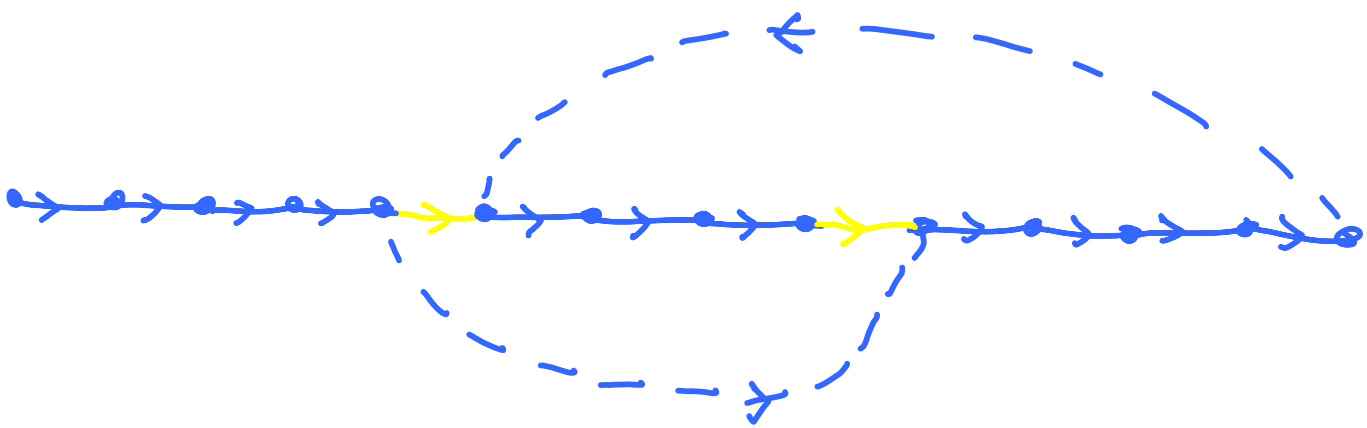
$$m = n(\log n + c_n)$$

$$\lim_{n\to\infty} \Pr(D_{n,m} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-2e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

Prob. $\delta^+, \delta^- \geq 1$

Frieze 1988



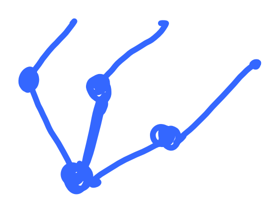


$G_{n,m}^{(\delta \geq k)}$ is uniformly chosen from graphs with vertex set $[n]$, m edges and $\delta \geq k$.

Let $m = \frac{n}{6}\left(\log n + 6\log\log n + c_n\right)$.

$$\lim_{n\to\infty} \Pr\left(G_{n,m}^{(\delta \geq 2)} \text{ is Hamiltonian}\right) = \begin{cases} 0 & c_n \to -\infty \\ e^{-3c/6\times 3^6} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

Bollobás, Fenner, Frieze 1990



Sparse Graphs

$$m \geq C_k n, \quad C_k = 2(k+1)^3, \quad k \geq 3$$

$$G_{n,m}^{(\delta \geq k)} \in \mathcal{A}_{k-1} \quad \text{whp} \qquad (*)$$

Bollobás, Cooper, Fenner, Frieze 2000

Conjecture

$(*)$ holds with C_k replaced by $k/2$.

Random Regular Graphs

$G_{n,r}$ is chosen uniformly from the set of r-regular graphs with vertex set $[n]$.

$G_{n,r}$ is Hamiltonian whp

$3 \leq r = O(1)$

Robinson, Wormald 1994

Use configuration model

[Frieze, Jerrum, Molloy, Robinson, Wormald 1996]

Z_H = # Hamiltonian pairings.

$$E(Z_H) \approx \sqrt{\frac{\pi}{2n}}\left((r-1)\left(\frac{r-2}{r}\right)^{(r-2)/2}\right)^n$$

$$E(Z_H^2) \approx \frac{r}{r-2}\,E(Z_H)^2$$

$$\Omega_c = \{ F : C_{2k-1} = c_k ,\ 1 \le k \le b \}$$

of (2k-1)-cyles

large integer

$$E(Z_H^2) = \sum_c \pi_c \overset{\substack{\text{small} \\ Var(Z_H | \Omega_c)}}{V_c} + \underbrace{\sum_c \pi_c \overset{\substack{\text{large} \\ E(Z_H|\Omega_c)}}{E_c^2}}_{\approx \frac{r}{r-2} E(Z)^2}$$

Follows that whp $Z_H \ge \dfrac{E(Z_H)}{n}$

Finding a Hamilton Cycle

Let Z_F = # of 2-Factors in $G_{n,r}$.

$$E(Z_F) \le 2n^{1/2} E(Z_H)$$

So whp

$$\frac{Z_H}{Z_F} \ge \frac{1}{2n^{5/2}}$$

Algorithm: Use MCMC to choose a (near) random 2-factor of $G_{n,r}$. Repeat until a hamilton cycle is found.

$r \to \infty$

Cooper, Frieze, Reed 2002

Krivelevich, Sudakov, Vu, Wormald 2001

$G_{k\text{-out}}$: Random graph with vertex set $[n]$ in which each vertex independently chooses k random neighbors.

[Graph underlying $D_{k\text{-out}}$]

THEOREM Bohman, Frieze 2009

$G_{3\text{-out}}$ is Hamiltonian whp

$G_{5\text{-out}}$: Frieze, Łuczak 1990

$$\begin{gathered} G_{5\text{-out}} \\ \| \\ G_{2\text{-out}} + G_{2\text{-out}} + G_{1\text{-out}} \end{gathered}$$

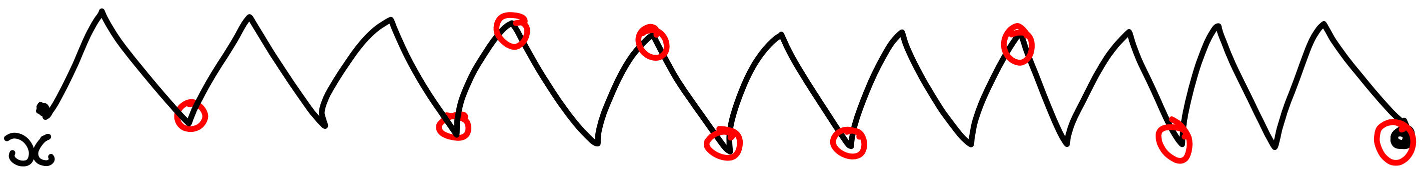
Has a perfect matching whp

So whp

$$G_{2\text{-out}} + G_{2\text{-out}} = G_{4\text{-out}}$$

contains a collection of vertex disjoint cycles that cover $[n]$.

Re-call Posá's Lemma:



$\text{END}(x) = \{ \circ \}$

Lemma: $|N(\text{END}(x))| < 2|\text{END}(x)|$.

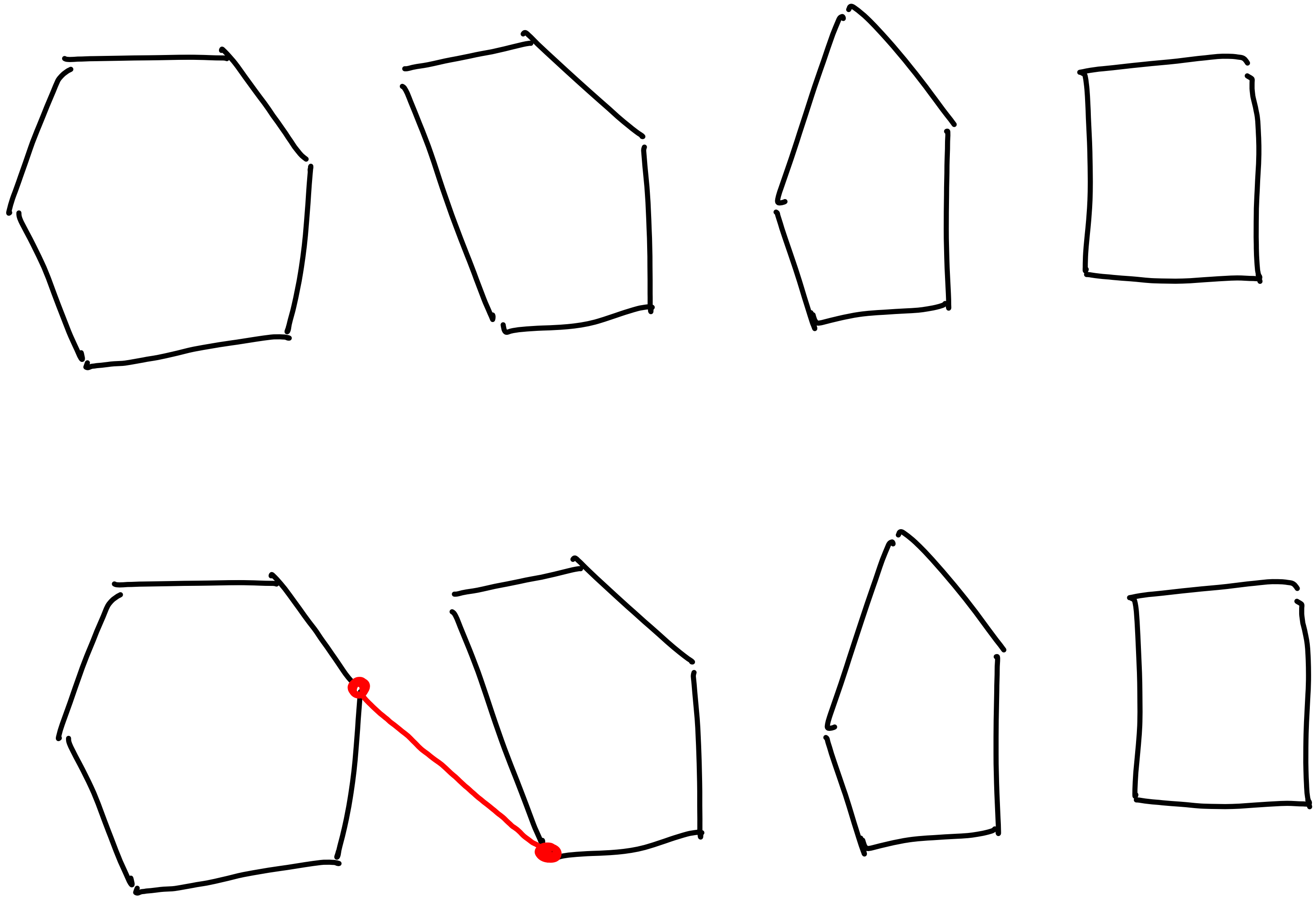
Lemma

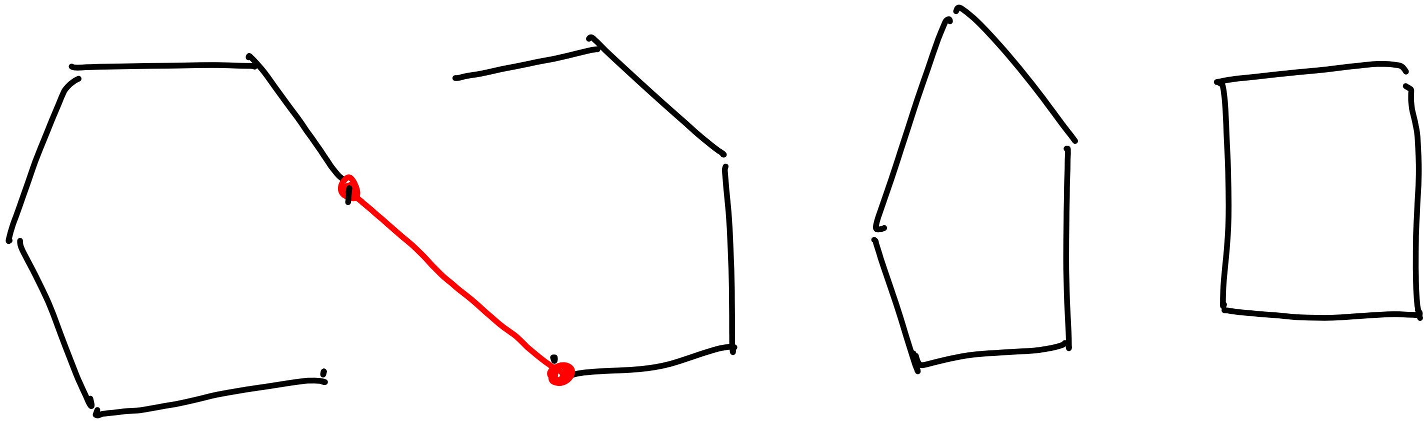
Whp, $S \subseteq [n]$, $|S| \le \frac{n}{100} \Rightarrow |N(S)| \ge 2|S|$

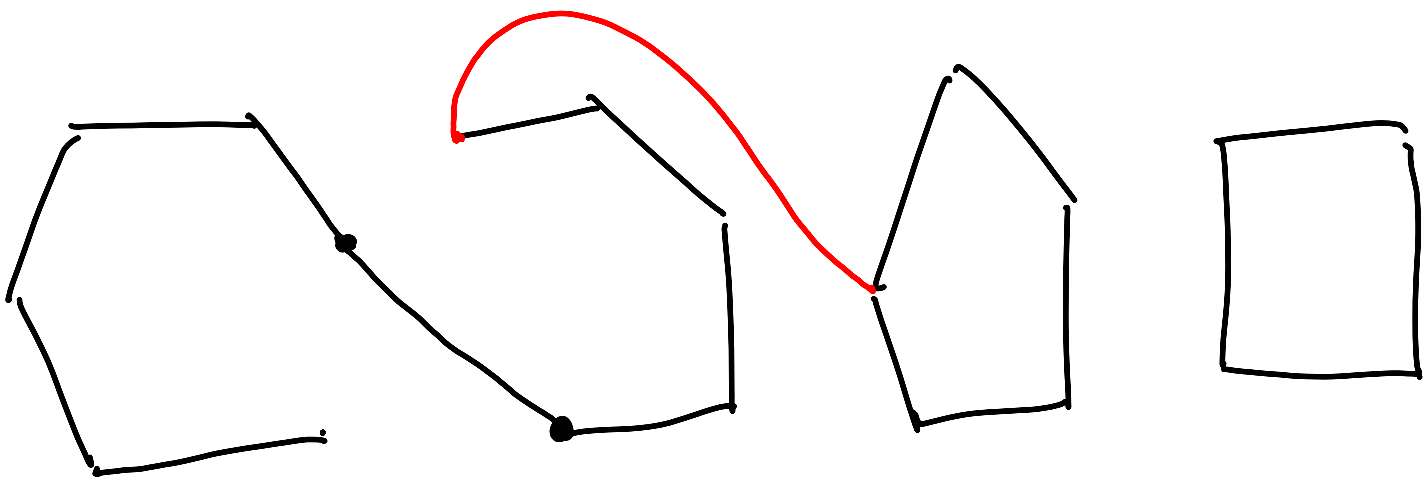
in $G_{4\text{-out}}$

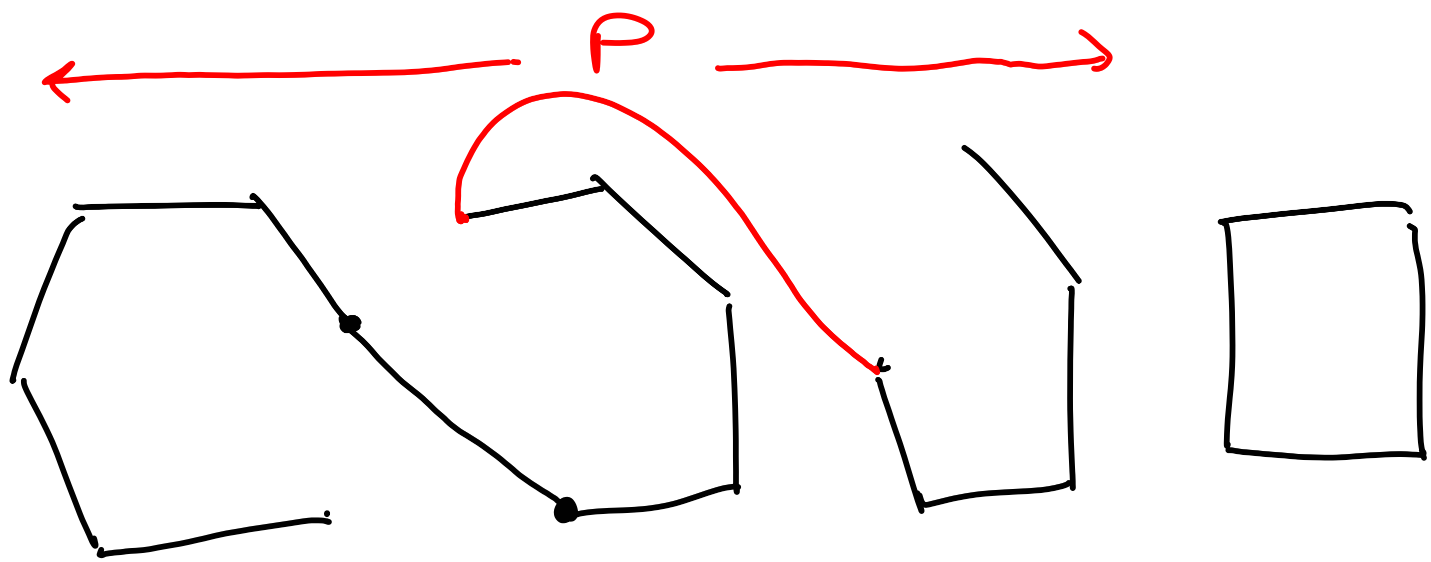
So whp if $H \supseteq G_{4\text{-out}}$ and P is a longest path in H then

$$|END(x)| > \frac{n}{100} \quad , \forall x \in END$$









Repeat until path cannot be extended.

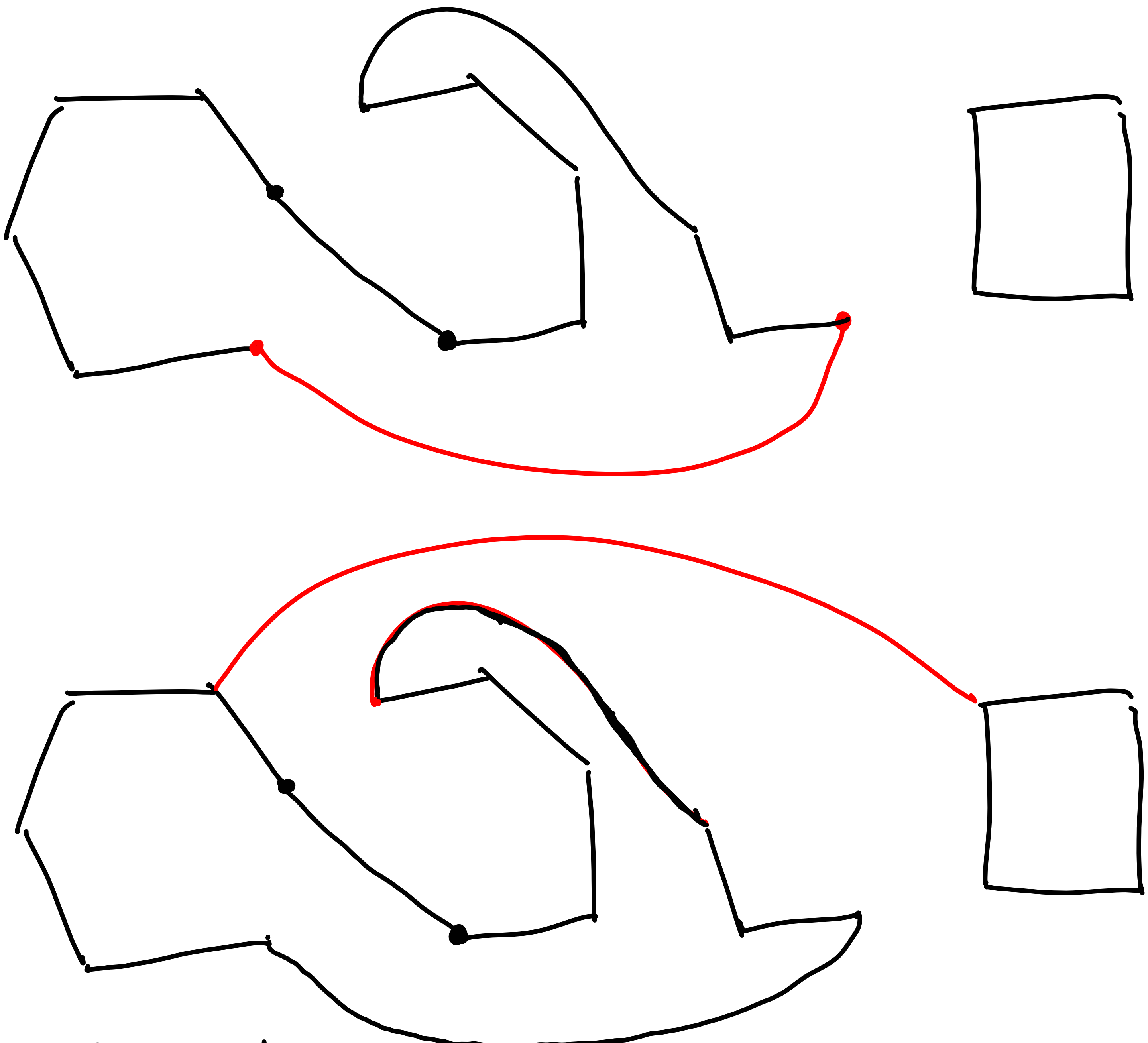
Do rotations; construct END(x), $x \in$ END

where $|END(x)| \geq \frac{n}{100}$, $\forall x$.

Go through END; check if $G_{1\text{-out}}$

connects x — END(x).

Whp succeed after $O(\log n)$ steps



One less cycle

$O(\log n)$ cycles.

$O(\log^2 n)$ $G_{1\text{-out}}$ edges are enough to find Hamilton cycle.

$G_{3\text{-out}}$

(i) Randomly chose $K \subseteq [n]$, $|K| = \frac{n}{\sqrt{\log n}}$

(ii) Remove 3^{rd} choice for each $v \in K$

(iii) Put back 3^{rd} choice for $v \in K$ which would otherwise be of degree 2.

Call resulting graph G_2

(a) Whp G_2 contains a simple 2-matching [collection of vertex disjoint paths & cycles] with $O(n/\log n)$ pieces

(b) Whp $|END(x)| \geq \frac{n}{100}$ for any path that cannot be extended.

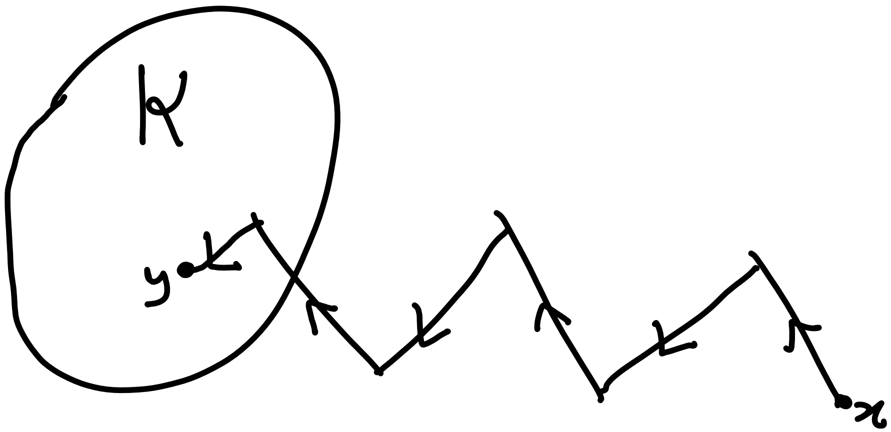
(c) Now use $\sim \frac{n}{\sqrt{\log n}}$ missing edges to join together $O(n/\log n)$ pieces

THERE IS A CATCH!

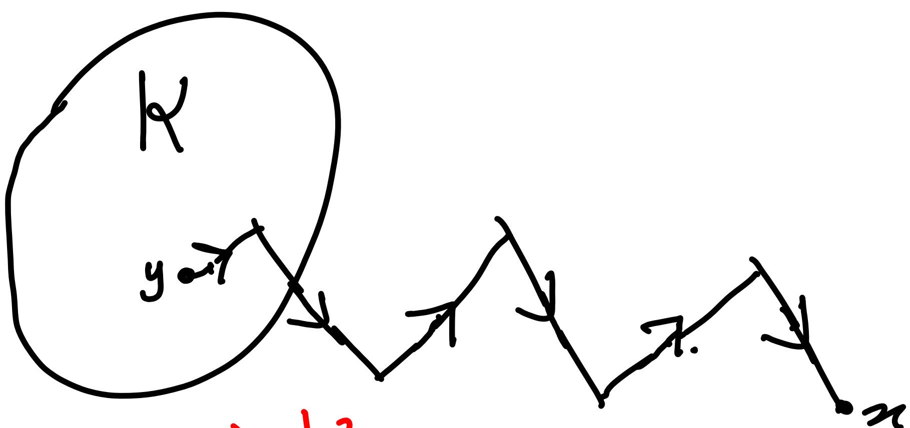
What if conditioning makes most of END outside K.

Then there will not be any extra edge to "aim" at END(x).

We solve this problem by considering underlying graph G to be fixed and probability space $\equiv$ space of digraphs D with underlying graph G.



is about as likely as



Now $y \notin K$

This requires checking that the numbers of paths of some fixed length l, are usually close enough to expectation

So END must be distributed almost independently with respect to K.

Conjecture: Whp

$$G_{k\text{-out}} \in \mathcal{A}_{k-1}, \quad k \geq 3$$

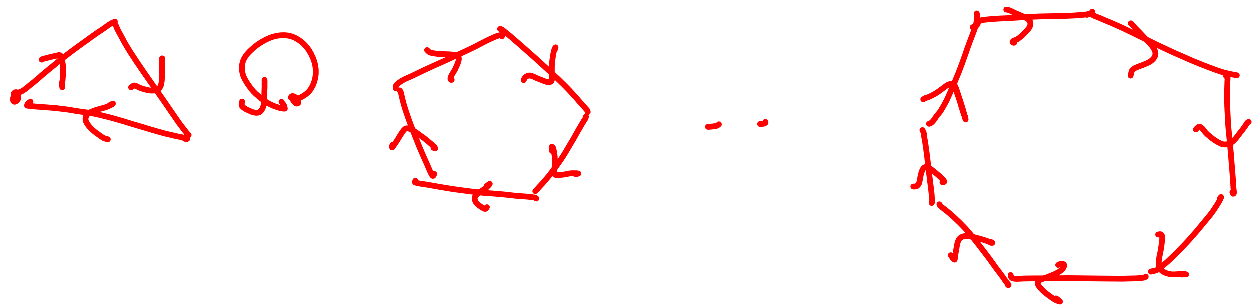
$D_{k\text{-in},k\text{-out}}$: Random digraph with vertex set $[n]$ in which each vertex independently chooses k random in-neighbors and k random out-neighbors.

THEOREM Cooper, Frieze 2000

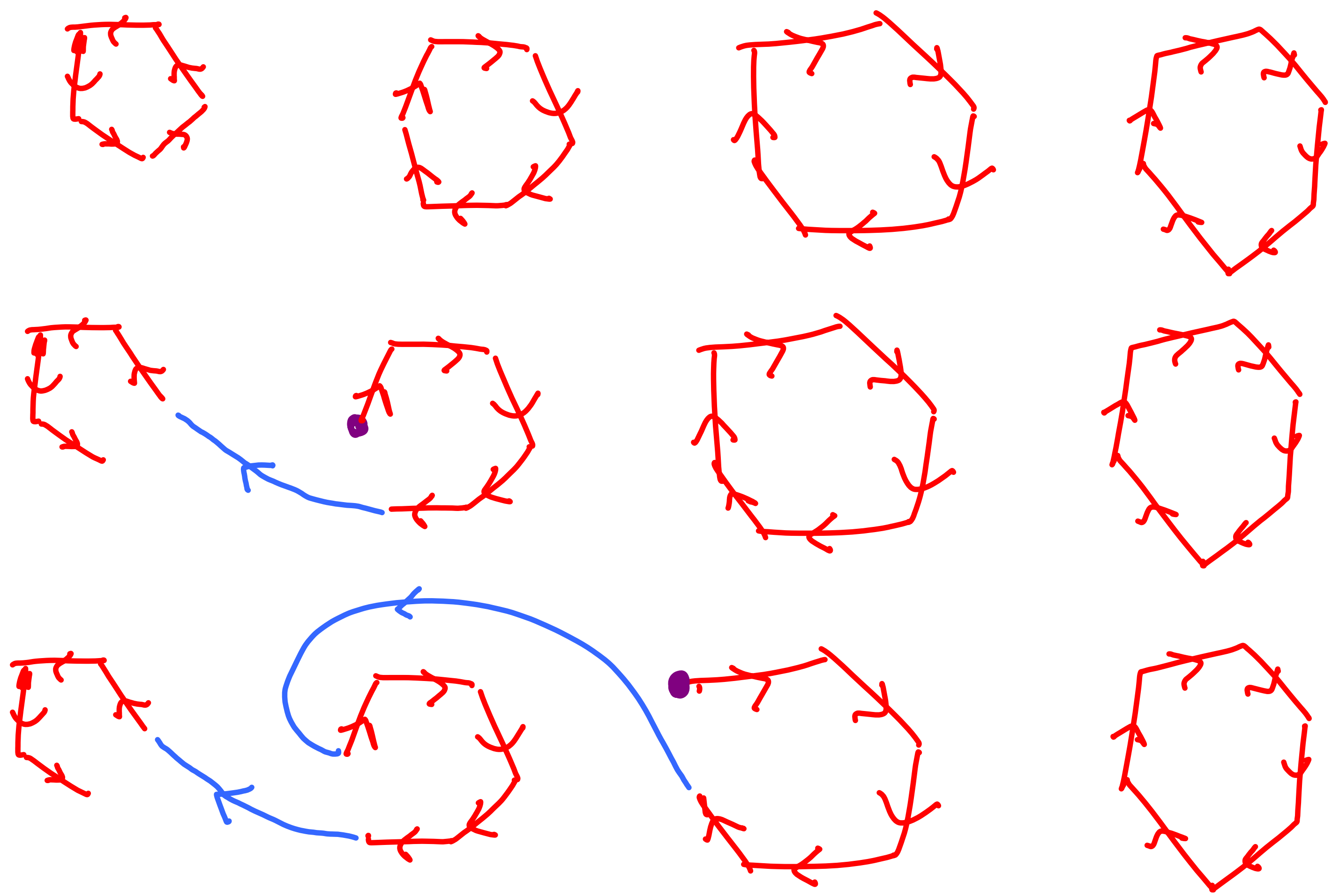
$D_{2\text{-in},2\text{-out}}$ is Hamiltonian whp

$$\underline{3\text{-in}, 3\text{-out}} = \underbrace{2\text{-in}, 2\text{-out}}_{D_1} + \underbrace{1\text{-in}, 1\text{-out}}_{D_2}$$

1) Construct random cycle cover using D_1 – $O(\log n)$ cycles

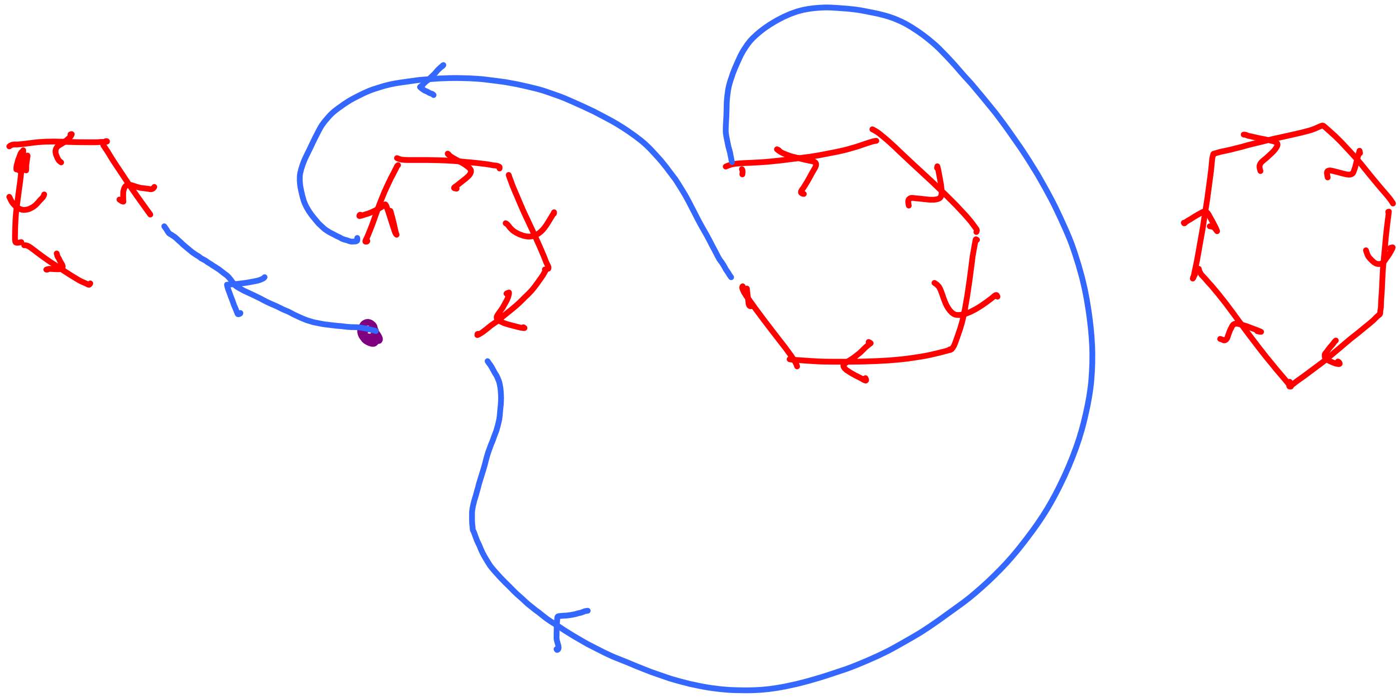


2) Absorb small cycles into rest so that each cycle is $\Omega(n/\log n)$ in size

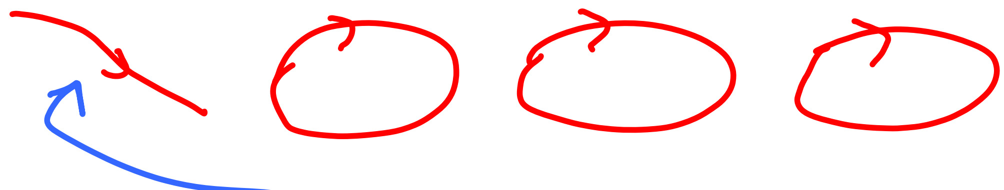


Blue ≡ D_2 edges

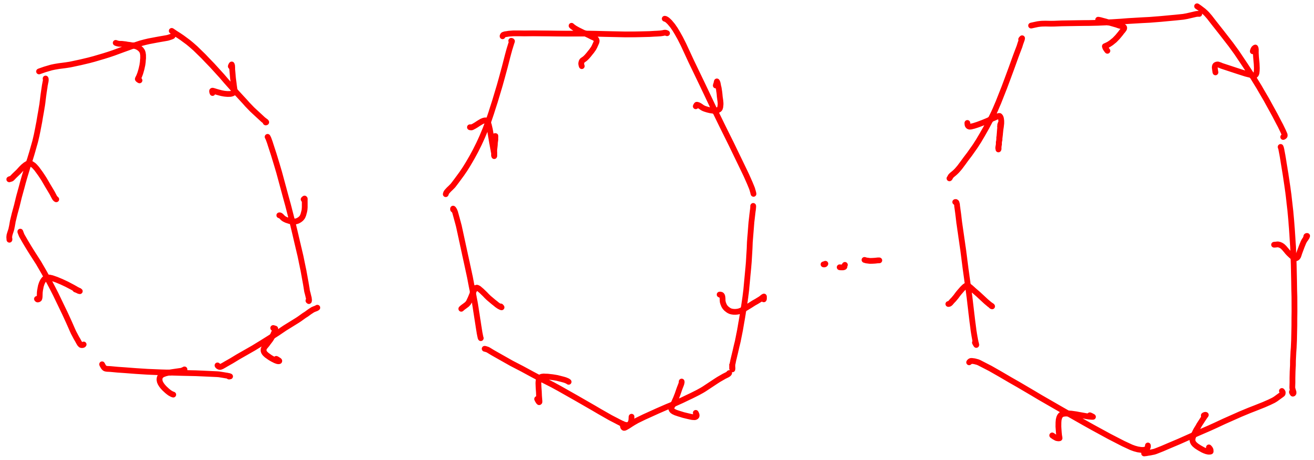
Follow up all choies



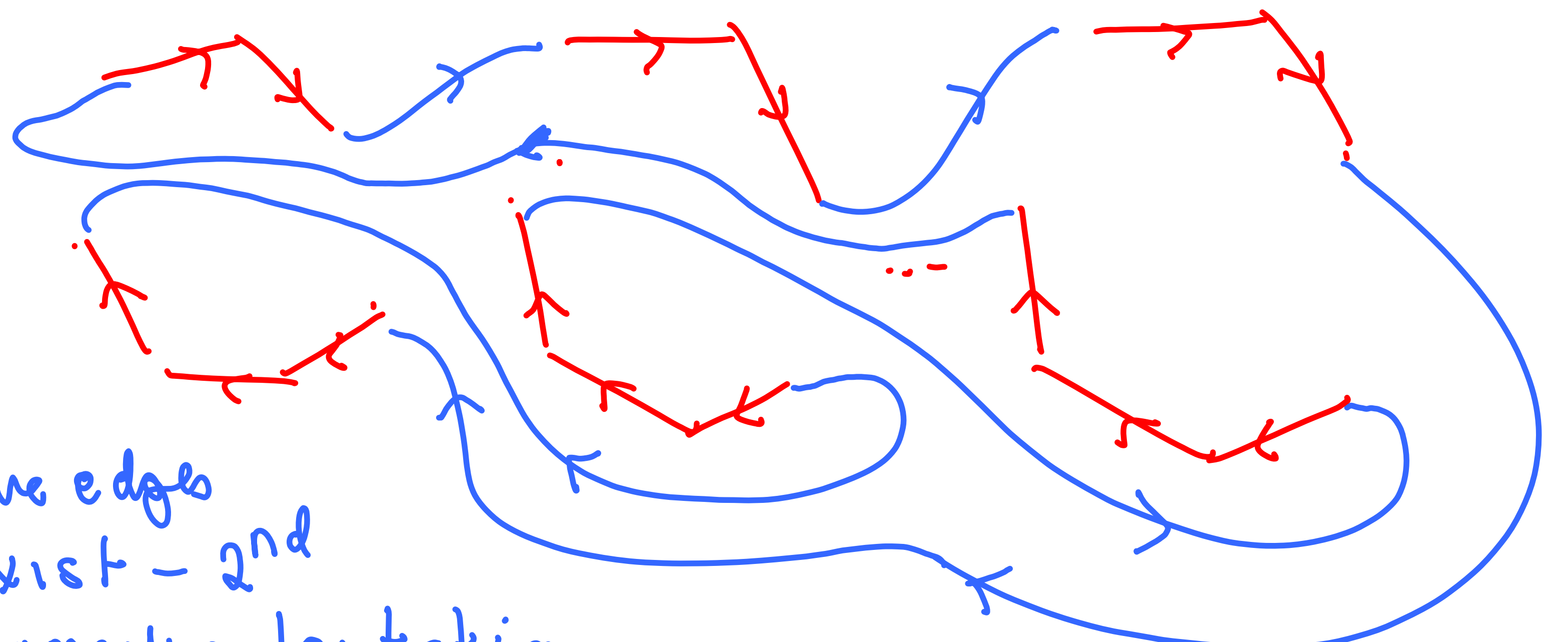
Create many



Use D_2 to close one of paths



All of size $\Omega(n/\log n)$



Blue edges exist – 2nd moment calculation – problem with one term; consider special class of re-arrangements.

Random Regular Digraphs.

$D_{n,r}$ has vertex set $[n]$ and indegree/outdegree r.

Theorem Cooper, Frieze, Molloy 1994

$D_{n,r}$ is Hamiltonian whp, $r \geq 3$

$D_{n,2}$ is not Hamiltonian whp.

Union of random permutations

Union of 2 random permutations contains undirected Hamilton cycle whp — Frieze 2001; Greenhill, Janson, Kim, Wormald 2002

Union of 3 random permutations contains directed Hamilton cycle whp — Frieze 2001

Union of 2 random permutations whp does not contain a directed Hamilton cycle whp — Cooper 2001

Multi-Colored Hamilton Cycles Cooper, Frieze 2002

Suppose that $G_{n,m}$ is randomly edge coloured with s colours. When does it have a Rainbow Hamilton cycle - each edge a different colour.

$m \geq Kn\log n$ & $s \geq Kn$ implies yes, whp

Question

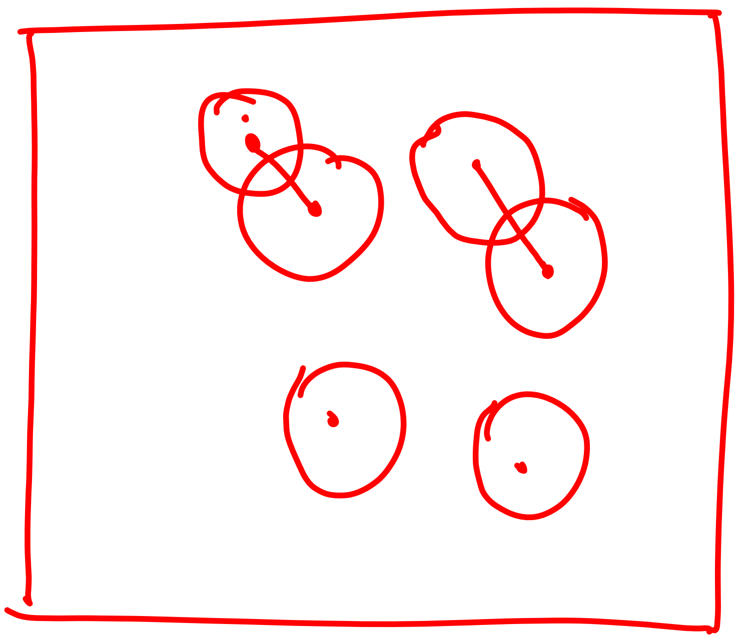
1) If $s = n$, how large should m be?

2) If $m = \frac{n}{2}(\log n + \log\log n + \omega)$, how large should s be?

Random Geometric Graphs

$X = \{X_1, \dots, X_n\}$ chosen uniformly from $[0,1]^2$.

$G_{n,\rho} = (X, \{(X_i, X_j) : |X_i - X_j| \le 2\rho\})$



Suppose $\pi p^2 = \frac{1}{n}(\log n + \log\log n + c_n)$.

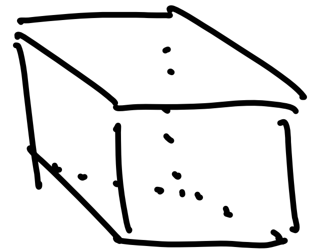
$$\lim_{n\to\infty} \Pr(G_{n,p} \text{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

Balogh, Bollobás, Krivelevich, Muller, Walters 2009
Pérez, Wormald 2009

Improved earlier result of Diaz, Mitsche, Pérez for case $\pi p^2 = \frac{(1+\epsilon)}{n}\log n$.

$Q_{n,p}$ = random sub-graph of the n-cube.

Each edge included with probability p.



$n=3$

Connectivity occurs around $p=\frac{1}{2}$ Erdős, Spencer 1979

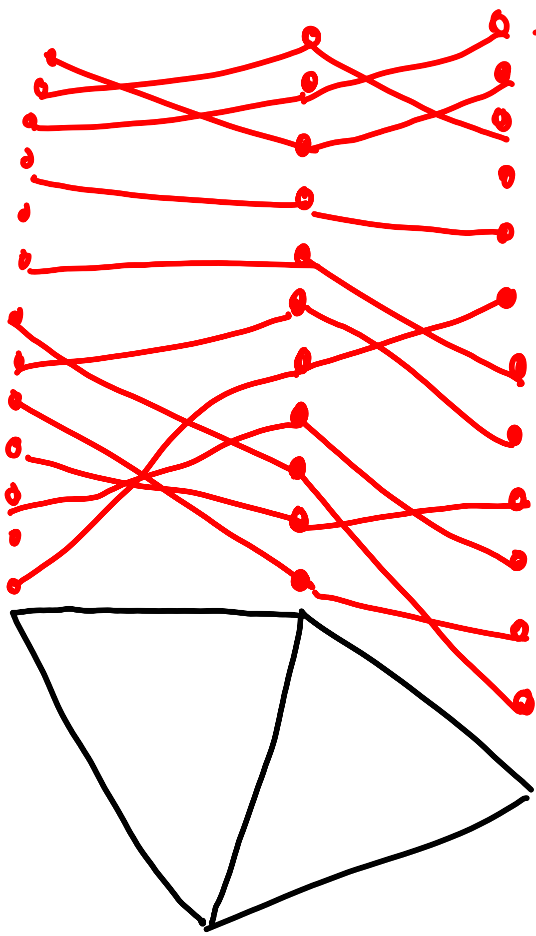
matching around $p=\frac{1}{2}$ Bollobás 1990

Hamilton cycle around $p=\frac{1}{2}$???

Random Lifts

one random perfect matching for each edge of H.

Amit, Linial 2002



n vertices for each vertex of base graph H

When is a random lift hamiltonian whp:

$H = K_h$ or $K_{h,h}$ and h is large: Burgin, Chebolu, Cooper, Frieze 2006

$H = \vec{K}_h$ (directed lift) Chebolu, Frieze 2008

Problem: Is a random lift of K_4 Hamiltonian whp?

THANK
YOU