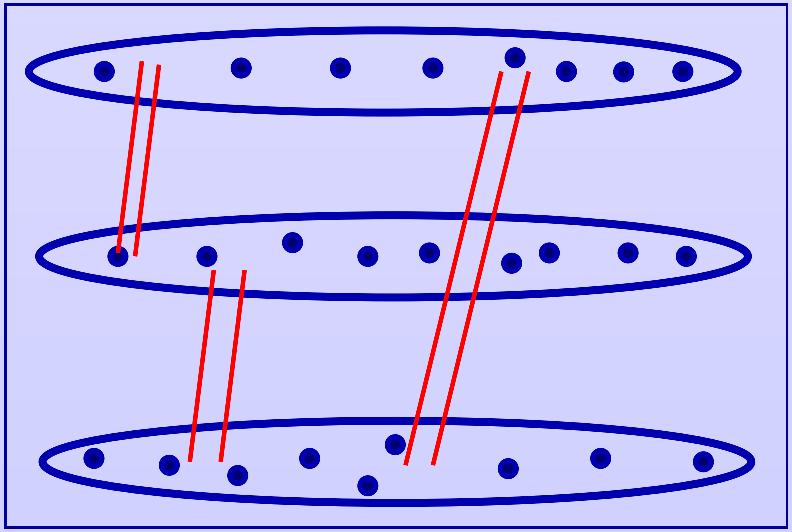
The typical structure of graphs without given excluded subgraphs + related results

Noga Alon
József Balogh
Béla Bollobás
Jane Butterfield
Robert Morris
Dhruv Mubayi
Wojciech Samotij
Miklós Simonovits

The Turán graph

K_p = complete graph on p vertices,
$T_{n,p}$ = p-class Turán graph:

$$\left(1-\frac{1}{p}\right)\binom{n}{2} \leq e(T_{n,p}) \leq \left(1-\frac{1}{p}\right)\frac{n^2}{2}$$



Turán type problems (notation)

Turán type problems (notation)

- $\mathcal{H}$ = family of forbidden graphs. $e(H) > 0$ for each $H \in \mathcal{H}$.

Turán type problems (notation)

- $\mathcal{H}$ = family of forbidden graphs. $e(H) > 0$ for each $H \in \mathcal{H}$.
- G is $\mathcal{H}$**-free** if G contains no $H \in \mathcal{H}$,

Turán type problems (notation)

- $\mathcal{H}$ = family of forbidden graphs. $e(H) > 0$ for each $H \in \mathcal{H}$.
- G is $\mathcal{H}$**-free** if G contains no $H \in \mathcal{H}$,
- not necessarily induced containment.

Turán type problems (notation)

- $\mathcal{H}$ = family of forbidden graphs. $e(H) > 0$ for each $H \in \mathcal{H}$.
- G is $\mathcal{H}$-**free** if G contains no $H \in \mathcal{H}$,
- not necessarily induced containment.

$\mathcal{P}(n, \mathcal{H})$ = the class of $\mathcal{H}$-free graphs on $[n] := \{1, \ldots, n\}$.

$$\mathbf{ex}(n, \mathcal{H}) = \max\{e(G_n) : G_n \quad \text{is} \quad \mathcal{H}\text{–free}\}.$$

The basic Turán type extremal problem

For a given family $\mathcal{H}$, determine or estimate $\mathbf{ex}(n, \mathcal{H})$, and describe the (asymptotic) structure of extremal graphs, as $n \to \infty$.

Erdős Conjecture

Trivially

$$|\mathcal{P}(n,\mathcal{H})| \geq 2^{\mathbf{ex}(n,\mathcal{H})}.$$

Conjecture [Erdős 1965]
For every H containing a cycle

$$|\mathcal{P}(n,H)| = 2^{(1+o(1))\mathbf{ex}(n,H)}.$$

Structure of a.a. H-free graphs

Conjecture suggests: Almost all H-free graphs are subgraphs of an extremal H-free graph.

False! Most triangle-free graphs are not subgraphs of $K_{n/2,n/2}$.

Theorem [Erdős, Kleitman and Rothschild (1976)]
Almost all triangle-free graphs are bipartite.

History

Theorem [Erdős, Kleitman and Rothschild (1976)]

$$|\mathcal{P}(n, K_p)| \leq 2^{(1+o(1))ex(n,K_p)}.$$

Theorem [Kolaitis, Prömel and Rothschild (1987)]
Almost all K_{p+1}-free graphs are p-partite.

Using Szemerédi Regularity Lemma, and the theorem above:

Theorem [Erdős, Frankl, Rödl (1986)]
The number of $\mathcal{H}$-free graphs is

$$|\mathcal{P}(n, \mathcal{H})| \leq 2^{\mathbf{ex}(n,\mathcal{H})+o(n^2)}.$$

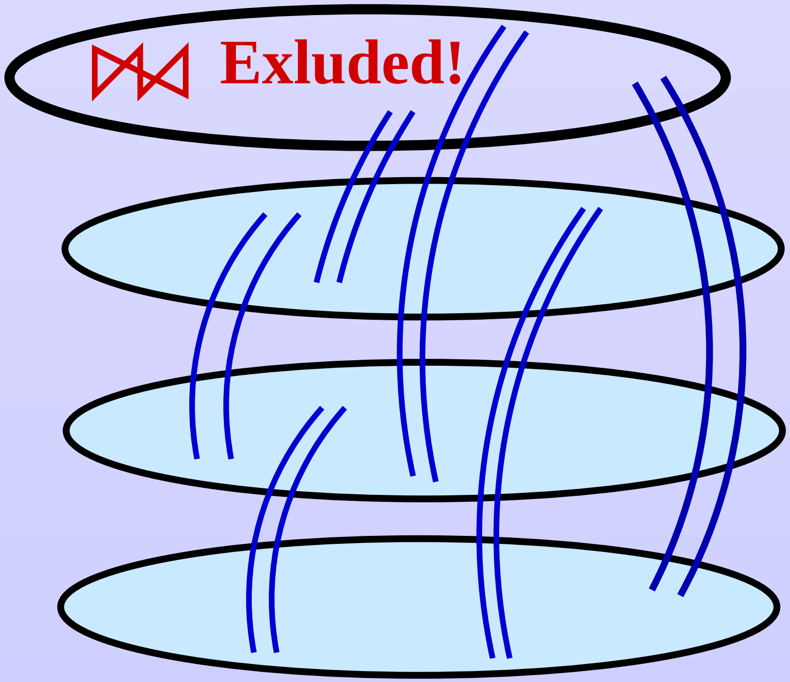
Characterization of a.a. H-free graphs.

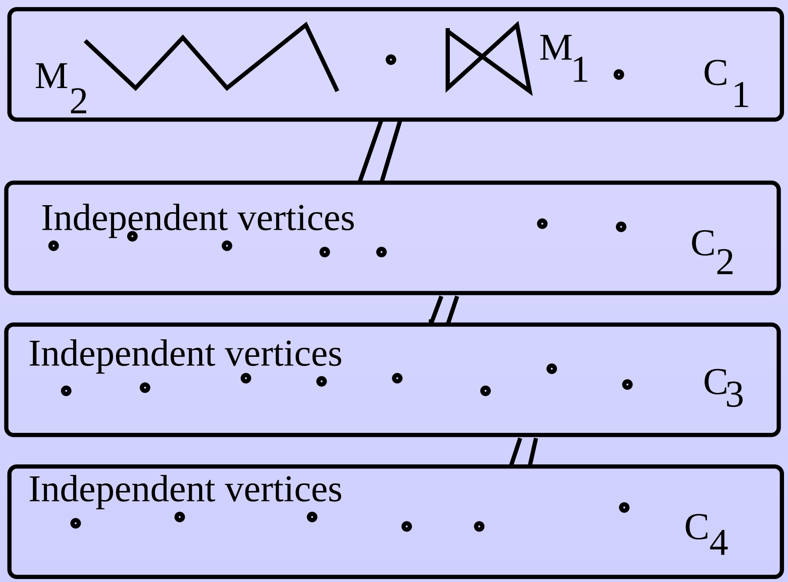
AIM: Most H-free graphs can be regarded as subgraphs of some extremal or almost extremal graphs for H when $\chi(H) > 2$.

Decomposition Family

Given a graph H, let $\mathcal{M} := \mathcal{M}(H)$ be the family of **minimal** graphs M for which there exist a $t = t_H$ such that $H \subseteq (M + I_t) \otimes K_{p-1}(t, \ldots, t)$, where $M + I_t$ is the graph obtained by adding t isolated vertices to M.

$\mathcal{M}$ is the *decomposition family* of H.





Decomposition Family: Examples

Example 1: $\mathcal{M}(K_p) = \{K_2\}$.

Example 2: $\mathcal{M}(C_{2p+1}) = \{K_2\}$.

Example 3: H is weakly edge-critical, i.e., $\exists e \in E(H)$ s.t. $\chi(H - e) = \chi(H) - 1$:

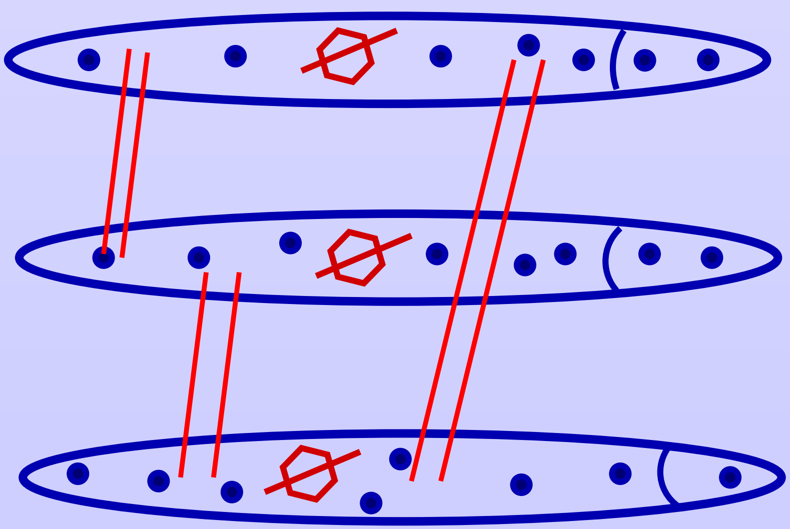
$$\mathcal{M}(H) = \{K_2\}.$$

Example 4: $H = O_6 = K(2,2,2)$, then $\mathcal{M}(H) = \{C_4\}$.

General structure

Many H-free graph can be generated from an $\mathcal{M}(H)$-free graph and $p-1$ vertex-disjoint independent sets.

Theorem [Balogh-Bollobás-Simonovits 2004, 2009].
Let $\mathcal{H}$ be an arbitrary finite family of graphs. Then there exists a constant $h_{\mathcal{H}}$ such that for almost all $\mathcal{H}$-free graphs G_n we can delete $h_{\mathcal{H}}$ vertices of G_n and partition the remaining vertices into p classes $(U_1, \ldots, U_p)$ so that each $G[U_i]$ $(i \le p)$ is $\mathcal{M}$-free



Comments

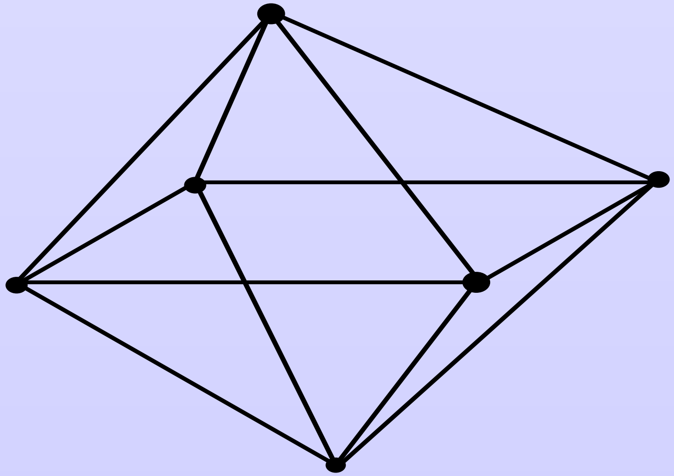
Remark. There is an infinite family $\mathcal{H}$ for which the statement of the theorem would be false.

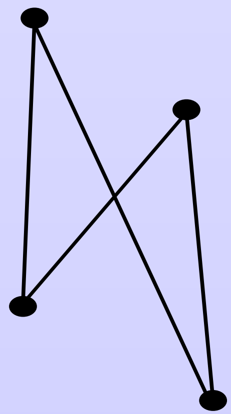
Remark. For $H = K_{19,19} + 8$ independent edges in one class the statement of the theorem would be false for $h_H = 0$.

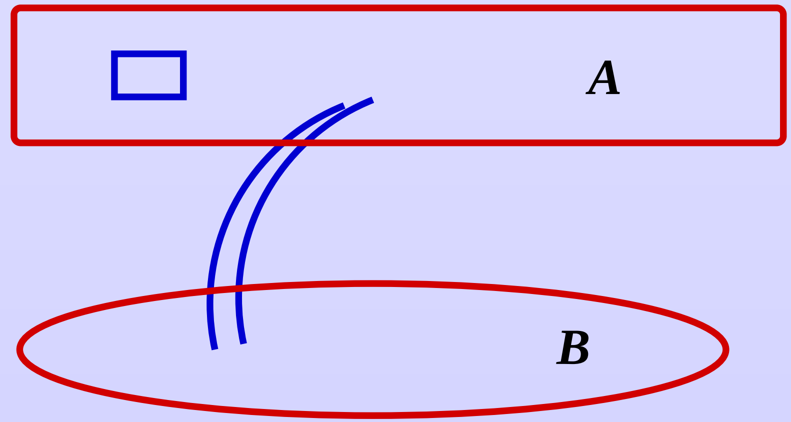
Remark. Complete p-partite graph + independent edges in one class is a counterexample of the natural “theory":
“a.e. H-free is a subgraph of an extremal type of graph."

The Octahedron $O_6 = K(2, 2, 2)$

Decomposition: $C_4 \otimes K_2 = O_6$.
Decomposition family is C_4.
Note: $O_6 \subseteq P_3 \otimes P_3$.

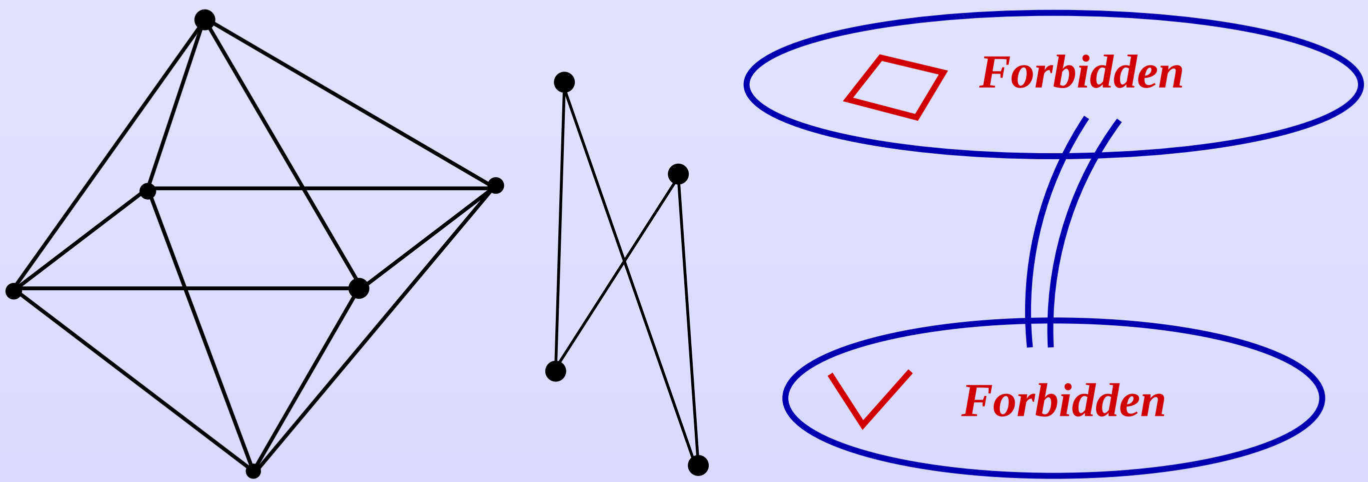






If B contains a C_4 then G_n contains an octahedron: K(2,2,2).

Octahedron Theorem:



Theorem [Erdős-Simonovits 1971].
The vertices of an O_6-extremal G_n can be partitoned into (A, B) so that

- $G[A]$ is C_4-extremal,
- $G[B]$ is P_3-extremal,
- A, B are completely joined.

Octahedron Theorem:

Theorem [Balogh-Bollobás-Simonovits 2009+].

Almost every O_6-free G_n is such that $V(G_n)$ can be partitioned into (A, B) so that

- $G[A]$ is C_4-extremal,
- $G[B]$ is P_3-extremal.

Ideas of the Proof:

- Szemerédi's Regularity Lemma,

- Simonovits' Stability Theorem,

Almost every H-free graph has structure similar to an H-free extremal graph.

- Ad hoc cleaning. (can be hard)

Ad hoc cleaning: Octahedron

Theorem [Füredi (1994)]
Almost every C_4-free graph on n vertices has at least $n^{1.5}/100$ edges.

Remark. Octahedron= $K(2,2,2)$. To generalize result to $K(a_1,\ldots,a_p)$ for $a_1 \le \ldots \le a_p$ one needs:

Theorem [Balogh-Samotij (2009+)]

Assume that $a_1 \le a_2$ such that $\Theta(ex(n,K(a_1,a_2)))$ is known. Then almost every $K(a_1,a_2)$-free graph has at least $c \cdot ex(n,K(a_1,a_2))$ edges for some $c = c(a_1,a_2)$.

Bipartite case

Erdős' Conjecture is open for all bipartite graphs!

Theorem [Balogh-Samotij (2009+)]

$$\mathcal{P}(n, K(s,t)) = 2^{O(n^{2-1/s})}.$$

Sharp for $s \leq t$ when $\Theta(ex(n, K(s,t)))$ is known.

Similar statements were only known for C_4 (Kleitman, Winston 1982), C_6, C_8 (Kleitman, Wilson 1996, Kohayakawa, Kreuter, Steger 1996).

Bipartite case: Application

Corollary [Balogh-Samotij (2009+)]
For all $2 \le s \le t$, there exist an integer $u = u(s,t) > t$ and $c = c(s,t) > 0$, such that for all large enough m, there exists a $K(s,u)$-free graphs G with m edges, whose largest $K(s,t)$-free subgraph has only m^{1-c} edges. In particular, $u(3,3) = 4$.

Was known for $K(2,2)$ with $u(2,3) = 3$ by Füredi (1994).

Induced case, general

Alekseev (1992), Prömel, Steger (1993), Bollobás, Thomason (1995):
Defined *coloring number* of $\mathcal{H}$. Solved a variant of Erdős' conjecture:

Definition. $\chi_c(H)$ is the minimum r such that for every $s+t=r$, $V(H)$ can be covered by s complete graphs and t independent sets.

SAME: There are $s+t=r-1$ such that $V(H)$ cannot be covered by s complete graphs and t independent sets.

Remark. Many = $2^{(1-1/(\chi_c(H)-1))n^2/2}$ H-free graphs can be generated this way.

Coloring number: Application

Important parameter for computing edit distance!

Axenovich, Kézdy, Martin;

Alon, Stav;

Induced case, general

Theorem [Alekseev (1992), Prömel, Steger (1993), Bollobás, Thomason (1995):]

$$\mathcal{P}^{(i)}(n,H) = 2^{(1-1/(\chi_c(H)-1)+o(1))n^2/2}.$$

STRUCTURE?

Theorem [Prömel, Steger (1991)]
Almost every induced C_4-free graph can be partitioned into a clique and an independent set.

Prömel, Steger (1993) handled C_5 as well.

Structural results:

$U(k)$:= bipartite graph with classes $([k], 2^{[k]})$, edges

$$\{(i, A):\ i \in A \subset [k]\}.$$

G is $U(k)$-free, if no $A, B \subset V(G)$ exists with $G[A, B] = U(k)$ (only cross-edges matter).

Theorem [Alon, Balogh, Bollobás, Morris (2009+)]:
For every H there is a $c > 0$ and a k such that the vertex set of almost every H-free graph can be partitioned into
$U, V_1, \ldots, V_{\chi_c(H)-1}$ such that
$|U| < n^{1-c}$ and
V_i is $U(k)$-free.

Sharpness:

Lemma [Alon, Balogh, Bollobás, Morris (2009+)]:
For every k there is a c such that the number of $U(k)$-free graphs is at most

$$2^{n^{2-c}}.$$

Quantitative improvement of the main result:

$$2^{(1-1/r+o(n^{-c}))n^2/2}.$$

Sharpness:

Lemma [Alon, Balogh, Bollobás, Morris (2009+)]:
The number of $U(k)$-free bipartite graphs on $([n],[n])$ is at most

$$2^{n^{2-1/(k-1)}\log^k n}.$$

Comparable with

Lemma [Alon, Krivelevich, Sudakov (2003+)]:

$$ex(n, U(k)) = O(n^{2-1/(k-1)}).$$

Application: Improves the bounds on the number of String graphs (and d-rank string graphs) [Pach, G. Tóth (2004)].

Critical graphs:

Theorem [Prömel, Steger (1991)]
Almost every induced C_4-free graph can be partitioned into a clique and an independent set.

Theorem [Prömel, Steger (1992)]
The following true iff H is a **weakly edge-color-critical** $(p+1)$-chromatic graph. Then almost all H-free (non-induced containment) graphs have chromatic number p.

Recall: H is a **weakly edge-color-critical** if there is an edge whose removal decreases $\chi(H)$.

Natural generalization for the induced case: H is a **critical** if there is a pair of vertices whose flipping in H decreases $\chi_c(H)$.

Critical graphs (induced case):

Theorem [Balogh, Butterfield (2009+)]
Let H be a graph with $\chi_c(H) = p + 1$. The following holds iff H is critical: Then the vertex set of almost every H-free graph can be covered with p cliques/independent sets.

Remark. The proper definition of 'critical' is different, K_4^- is critical only by our definition.

Remark. C_{2k+1} is critical for $k \geq 3$.

Remark. The vertex set of almost every C_7-free graph can be covered with 3 cliques or 2 cliques and an independent set.

Hypergraphs: Triple Systems.

Theorem [Bollobás and Thomason (1995)]
For every k-hypergraph H there is a constant c s.t.

$$\mathcal{P}(n, H) = 2^{cn^k + o(n^k)}.$$

Theorem [Nagle and Rödl (2001)]
For every 3-hypergraph H

$$\mathcal{P}(n, H) = 2^{ex(n,H) + o(n^3)}.$$

Strong Hypergraph Regularity Lemma is used.

Hypergraphs: Fano planes, linear hyp graphs.

Theorem [Keevash- Sudakov, Füredi- Simonovits (2006)]
The extremal Fano-plane-free triple system is bipartite.

Theorem [Person and Schacht (2009)]
Almost all hypergraphs without Fano planes are bipartite.

Remark. Using embedding result of [Kohayakawa, Nagle, Rödl, Schacht (2009)] argument works for 'nice' linear hypergraphs.

Weak Hypergraph Regularity Lemma is used.

Hypergraphs: Cancellative triple syste

A triple system is **cancellative** if no three of its distinct edges satisfy $A \cup B = A \cup C$.

Theorem [Bollobás (1974)]
The extremal cancellative triple system is tripartite.

Note that a cancellative system does not contain $F_5 = \{abc, abd, cde\}$. (extended triangle?)

Theorem [Frankl, Füredi (1983)]
The extremal F_5-free triple system is tripartite.

Theorem [Balogh, Mubayi (2009+)]
Almost all F_5-free triple system is tripartite.

Hypergraphs: Independent neighborho

$T_5 = \{abc, abd, abe, cde\}$. In a T_5-free hypergraph the neighborhood of any pair of vertices is an independent set.

A semi-bipartite hypergraph is T_5-free.

Theorem [Balogh, Mubayi (2009+)]
Almost all T_5-free triple system is semi-bipartite.

Ideas of the Proof:

- Apply Strong Hypergraph Regularity Lemma.

- Apply Stability result to cluster hypergraph (if it exists.)

- There are many cluster hypergraphs!

- Apply stability to each, and then prove that their structure are similar.

- Gives: a.a. T_5-free almost semi-bipartite.

- ad hoc methods (some new pseudo-random terminology)...