# Partition bijections

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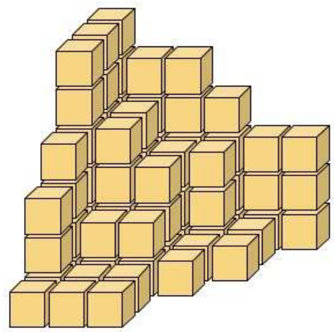


# Integer partitions

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a *partition* of $n$ if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$.

$\lambda_i$ – *parts of the partition* $\lambda$
$\ell = \ell(\lambda)$ – *the number of parts* in $\lambda$

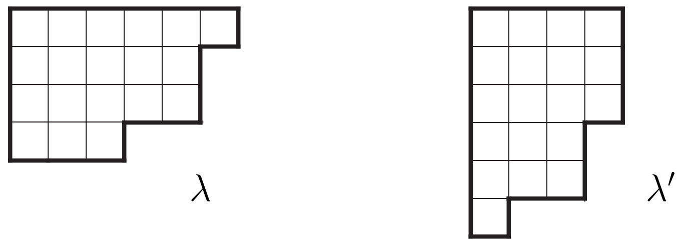
$p(n)$ – the number of partitions of $n$

**Example.** $n = 5$, $p(5) = 7$. Here are all partitions of 5 :
$(5)$, $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, $(1, 1, 1, 1, 1)$

**Euler's Theorem:**

$$\prod_{i=1}^{\infty} \frac{1}{1 - t^i} = 1 + \sum_{n=1}^{\infty} p(n)\, t^n,$$

## Young diagrams




Young diagrams of partitions $\lambda = (6, 5, 5, 3)$ and $\lambda' = (4, 4, 4, 3, 3, 1)$.

## Classical Theorem:

*The number of partitions of $n$ with largest part $k$ is equal to the number of partitions of $n$ into exactly $k$ parts.*

(Encyclopedia Britannica)

# Partition identities

$$1 \,+\, \sum_{n=1}^{\infty} \frac{(1+at)(1+at^3)\cdots(1+at^{2n-1})\, z^n t^{2n}}{(1-bt^2)(1-bt^4)\cdots(1-bt^{2n})}$$
$$= \sum_{r=0}^{\infty} \frac{(1+azt^{4r+3}) b^r z^r t^{2r(r+1)}}{(1-zt^{2(r+1)})} \prod_{i=1}^{r} \frac{(1+at^{2i-1})(1+ab^{-1}zt^{2i+1})}{(1-bt^{2i})(1-zt^{2i})}\,.$$

(Rogers-Fine identity)

$$1+\sum_{n=1}^{\infty} q^{n(n+1)/2} \frac{(1+q)(1+q^2)\cdots(1+q^n)}{(1-q)(1-q^2)\cdots(1-q^n)} \,=\, \prod_{n=1}^{\infty} (1+q^n)(1-q^{2n})\,.$$

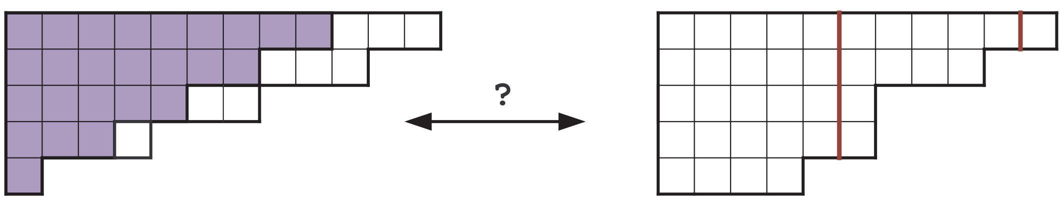
(Ramanujan's identity)

# Rogers–Ramanujan identities

$$1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} = \prod_{i=0}^{\infty} \frac{1}{(1-t^{5i+1})(1-t^{5i+4})}$$

**R-R theorem:**

*The number of partitions of $n$ into parts which differ by at least 2 is equal to the number of partitions of $n$ into parts which are $\pm 1 \mod 5$.*



**G. H. Hardy** (1936):

"None of the proofs of Rogers–Ramanujan identities can be called 'simple' and 'straightforward', since the simplest are essentially verifications; and no doubt it would be unreasonable to expect a really easy proof."

**G. E. Andrews** (1978):

"Hardy's comments about the nonexistence of a really easy proof of the Rogers–Ramanujan identities are still true today."

**A. M. Garsia** (1989):

"Schur independently discovers the Rogers identities, and (unlike Ramanujan) is also able to provide a proof. We may add that it is really a great historical injustice (mostly due to the tabloid sensationalism of G. H. Hardy) to refer to as the Rogers-Ramanujan identities."

## Ramanujan's congruences

**mod-5 congruence:** $p(5n-1) \equiv 0 \mod 5$

*¿¿¿ Combinatorial proof ???*

$r(\lambda) = \lambda_1 - \ell(\lambda)$ – *rank* of $\lambda$

$p(n,i)$ – the number of partitions $\lambda \vdash n$ with $r(\lambda) \equiv i \mod 5$.

**Theorem.** (Dyson, Atkin & Swinnerton-Dyer)
$p(5n-1,0) = p(5n-1,1) = p(5n-1,2) = p(5n-1,3) = p(5n-1,4)$.

**Oliver Atkin:** *"it is probably bad advice to a young man to look for a true combinatorial proof* [of Ramanujan's congruences]*."*

**Ramanujan's identity:**

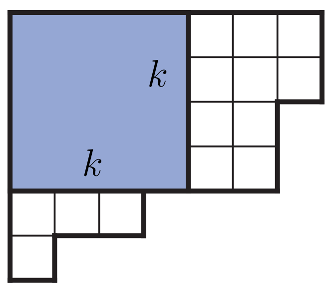
$$\sum_{k=1}^{\infty} p(5k-1)\, t^k \;=\; 5 \prod_{i=1}^{\infty} \frac{(1-t^{5i})^5}{(1-t^i)^6}\,.$$

# Euler's identity:

$$\prod_{i=1}^{\infty} \frac{1}{1-t^i} = 1 + \sum_{r=1}^{\infty} \frac{t^{r^2}}{(1-t)^2(1-t^2)^2\cdots(1-t^r)^2}\,,$$

**Theorem:** *The number of pairs of partitions with at most $k$ parts and combined size $n$ is equal to the number of partitions of $n+k^2$ with the largest inscribed square of size $k$.*

**Proof:** Durfee bijection (1850's).

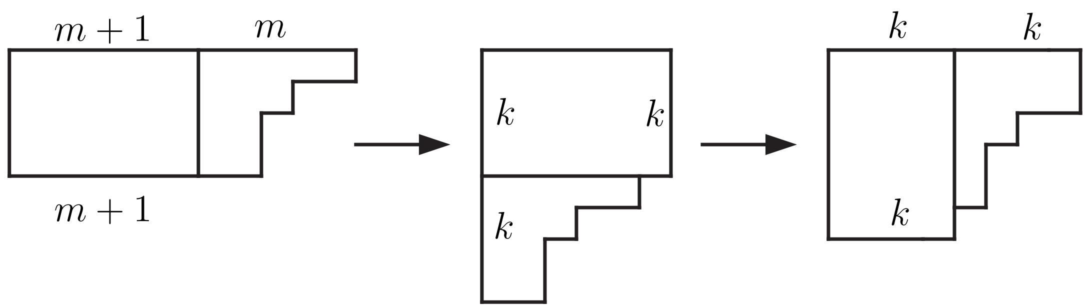


# Ramanujan's identity

$$\sum_{m=0}^{\infty} \frac{t^{2m+1}}{(1-t^{m+1})\cdots(1-t^{2m+1})} = \sum_{m=0}^{\infty} \frac{t^{m}}{(1-t^{m+1})\cdots(1-t^{2m})}$$

**Theorem:** *Among partitions of $n$ with the smallest part at least half the largest part, the number of those withe the largest part odd is equal to the number of those where the smallest part is unique.*

**Proof:** Andrews's bijection (1968).



# Euler's Theorem:

*Number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.*

$$\prod_{i=1}^{\infty}(1+t^i) = \prod_{i=1}^{\infty}\frac{1}{1-t^{2i-1}}$$

**Proof:**

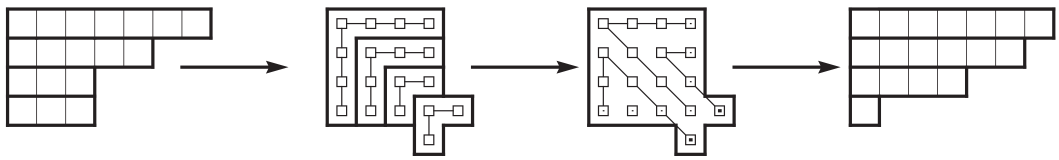
$$\prod_{i=1}^{\infty}(1+t^i) = \prod_{i=1}^{\infty}\frac{(1-t^i)(1+t^i)}{(1-t^i)} = \prod_{i=1}^{\infty}\frac{1-t^{2i}}{(1-t^{2i-1})(1-t^{2i})} = \prod_{i=1}^{\infty}\frac{1}{1-t^{2i-1}}$$

**Glaisher's bijection** (1870's) :

$$(7,6,4,1) \to (7,4,3,3,1) \to (7,3,3,2,2,1) \to (7,3,3,2,1,1,1)$$
$$\to (7,3,3,1,1,1,1,1)$$

**Extension** [Franklin, 1883; Wilf, 2000]: *The number of partitions of $n$ with $k$ even part sizes is equal to the number of partitions of $n$ with $k$ repeated part sizes.*

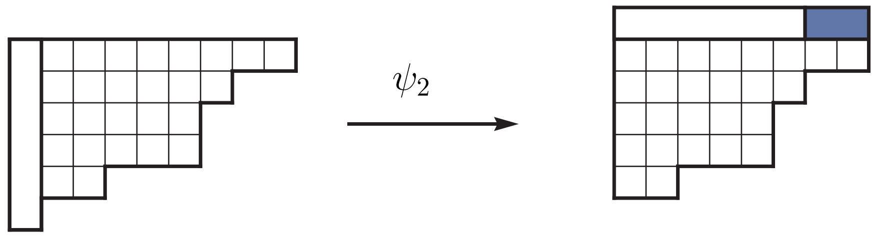
**Sylvester's bijection** (1882) :



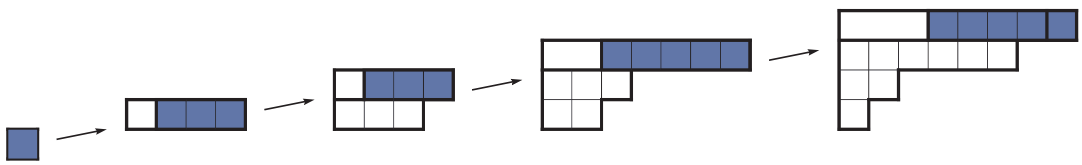
**Extension** [Sylvester, 1882]: *The number partitions of $n$ into odd parts with $k$ distinct part sizes is equal to the number of partitions of $n$ into distinct parts with $k$ contiguous sequences of parts.*

**Extension** [Fine, 1948; P., 2003]: *The number of partitions of $n$ into distinct parts with the largest part $k$ is equal to the number of partitions $\lambda$ of $n$ into odd parts with $\lambda_1 + 2\ell(\lambda) = 2k + 1$.*

# Iterated Dyson's map:




Iterated Dyson's map $\Psi : (5,5,3,3,1) \to (8,6,2,1)$.



**Theorem** [P., 2003] *This works.*

**Extension** [Fine, 1948; Andrews, 1983; P., 2003]:
*The number of partitions of* $n$ *into distinct parts with Dyson's rank* $2k$ *or* $2k+1$ *is equal to the number of partitions* $\lambda$ *of* $n$ *into odd parts with* $\lambda_1 = 2k+1$.

# The elusive R-R bijection

**George Andrews** (70's): *Will give* \$100 *for a R-R bijection!*

**Garsia–Milne** (1980): *Here is one!*
(based on the *involution principle*)

**The idea:**

(1) Schur's 1915 proof of R-R identities by an explicit involution proving Schur's identity combined with Jacobi triple product identity.

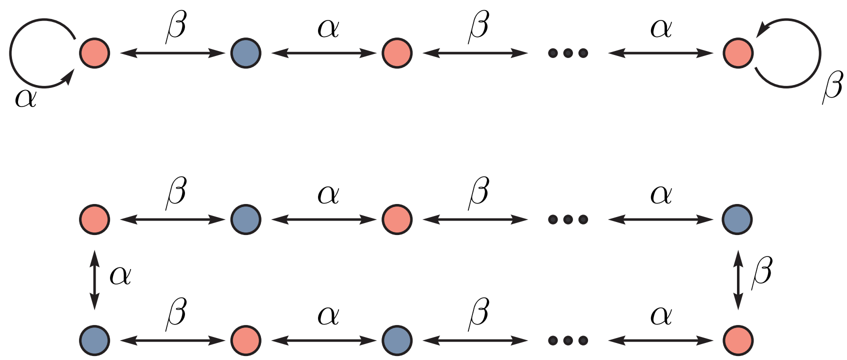
(2) Sylvester's 1882 proof of the Jacobi triple product identity by an explicit involution.

(3) Garsia–Milne method of combining involutions.

# The involution principle

Let $\alpha, \beta$ be two involutions on a finite set $X$: $\alpha^2 = \beta^2 = 1$.

Dihedral group $D_\infty = \langle \alpha, \beta \rangle$ acts on $X$ with the following orbits:



Suppose now $X = X_+ \cup X_-$ and both $\alpha$ and $\beta$ are sign reversing:
$\alpha(X_-) \subset X_+, \quad \beta(X_-) \subset X_+.$
Denote by $A, B \subset X_+$ the fixed points of $\alpha, \beta$. Then:

$$|A| \ =_\alpha \ |X_+| - |X_-| \ =_\beta \ |B|$$

Then the action of $D_\infty$ gives a bijection $\Phi : A \to B$.

**Remark.** From TCS point of view, there is an easy polynomial time algorithm to produce a *R-R bijection*, based on lexicographic ordering of partitions. On the other hand, the bijection of Garsia and Milne is so complicated, it may in fact require $\exp(n^{\Omega(1)})$ steps.

## Recent work:

**1.** The only known "good" involution principle bijection (*O'Hara's algorithm*) is actually pretty bad.
[Konvalinka–P., 2009]

**2.** When restricted to *natural* geometric partition bijections, there is no R-R bijection. [P., 2006, 2009]

# Andrews's identities

Let $\overline{a} = (a_1, a_2, \ldots)$ with $a_i \in \{1, 2, \ldots, \infty\}$
$\mathcal{A}$ to be the set of partitions $\lambda$ with the number of parts $i$

$$m_i(\lambda) < a_i, \quad \text{for all } i.$$

$\mathcal{A}_n$ is the subset of $\mathcal{A}$ of partitions of size $n$.
$\operatorname{supp}(\overline{a}) = \{i \colon a_i < \infty\}$ the *support* of the sequence $a$.

Let $\overline{a} = (a_1, a_2, \ldots)$ and $\overline{b} = (b_1, b_2, \ldots)$. Define $\overline{a} \sim_\varphi \overline{b}$, if:
$\varphi : \operatorname{supp}(\overline{a}) \to \operatorname{supp}(\overline{b})$ is a bijection such that $ia_i = \varphi(i) b_{\varphi(i)}$ for all $i$.

**Theorem** [Andrews, 1978] If $\overline{a} \sim \overline{b}$, then $|\mathcal{A}_n| = |\mathcal{B}_n|$ for all $n$.

**Note:** This generalizes Euler's theorem, Schur's theorem, and several Sylvester's results.

## Examples

**(1)** Let $\overline{a} = (2, 2, \ldots)$ and $\overline{b} = (\infty, 1, \infty, 1, \ldots)$
$\mathcal{A}_n$ is the set of all partitions of $n$ into distinct parts
$\mathcal{B}_n$ is the set of partitions of $n$ into odd parts
The bijection $\varphi : i \mapsto 2i$ between $\operatorname{supp}(\overline{a}) \to \operatorname{supp}(\overline{b})$ satisfies $ia_i = \varphi(i) b_{\varphi(i)}$, i.e. $\overline{a} \sim_\varphi \overline{b}$.
The theorem then implies that $|\mathcal{A}_n| = |\mathcal{B}_n|$.

**(2)** Let $\overline{a} = (1, 1, 4, 5, 3, 1, 1, \ldots)$ and $\overline{b} = (1, 1, 5, 3, 4, 1, 1, \ldots)$
$\varphi(3) = 4$, $\varphi(4) = 5$, $\varphi(5) = 3$, and $\varphi(i) = i$ for $i \neq 3, 4, 5$;
observe that $\overline{a} \sim_\varphi \overline{b}$.
The theorem then implies that $|\mathcal{A}_n| = |\mathcal{B}_n|$.

# O'Hara's algorithm

**Algorithm**
*Fix:* sequences $\overline{a} \sim_{\varphi} \overline{b}$
*Input:* $\lambda \in \mathcal{A}$
`Set:` $\mu \leftarrow \lambda$
*While:* $\mu$ contains more than $b_j$ copies of $j$ for some $j$
    *Do:* remove $b_j$ copies of $j$ from $\mu$,
    add $a_i$ copies of $i$ to $\mu$, where $\varphi(i) = j$
*Output:* $\psi(\lambda) \leftarrow \mu$

In Example **(1)**, we get Glaisher's bijection.

**Example (2):**

$$3^3 4^4 5^2 \to 3^7 4^1 5^2 \to 3^2 4^1 5^5 \to 3^2 4^6 5^1 \to 3^6 4^3 5^1 \to$$

$$\to 3^{10} 4^0 5^1 \to 3^5 4^0 5^4 \to 3^0 4^0 5^7 \to 3^0 4^5 5^3 \to 3^4 4^2 5^3$$

**Theorem** [Konvalinka-P., 2009]
*For general $\varphi$, the O'Hara algorithm is mildly exponential:* $\exp \Omega(\sqrt[3]{n})$.
*For general $\varphi$ with support of size $m$, the O'Hara algorithm takes* $n^{\Omega(m)}$
*For general $\varphi$ with support of size $m$, the* O'Hara algorithm function *can be computed in polynomial time.*

# Random partitions

**Algorithm** [Friestedt, 1993]
*Input* $n$. Choose $q > 0$ wisely.
For each $i = 1, \ldots, n$, let $m_i \leftarrow \mathrm{Geo}\big(1 - q^i\big)$.
Let $\lambda \leftarrow 1^{m_1} 2^{m_2} \ldots n^{m_n}$.
If $|\lambda| = n$, *Output* $\lambda$.

Here in $X \leftarrow \mathrm{Geo}(p)$ is a geometric random variable:
$P(X = k) = (1-p)^k p$, $k = 0, 1, 2, \ldots$

We choose $q$ in to maximize $f_n(q) = q^n (1-q) \cdots (1-q^n)$.

**Proof:** The probability to obtain any given partition:

$$\begin{aligned} P(\lambda) &= (1-q) q^{1 m_1} \cdot (1-q^2) q^{2 m_2} \cdots (1-q^n) q^{n m_n} \\ &= (1-q)(1-q^2) \cdots (1-q^n) \cdot q^{|\lambda|} \end{aligned}$$

Thus

$$P(\lambda \mid \lambda_1 + \lambda_2 + \ldots = n) = \frac{1}{p(n)}\,.$$

**Note:** Often called *Boltzmann sampling*. Generalized by Flajolet et al. to other unlabeled structures (2007). This approach was extensively used by Dembo–Vershik–Zeitouni (2001), Pittel (2007) and others, to obtain delicate results on the shape of random partitions.