Extremal Set Theory (IPAM Tutorial)

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Sperner, LYM, Bollobás, Probability

Definitions

$\mathcal{P}[n]$: subsets of $[n] = \{1, \cdots, n\}$ (aka n-cube, Boolean lattice).
Antichain $\mathcal{A} \subseteq \mathcal{P}[n]$: $\nexists A, B \in \mathcal{A}$, $A \subsetneq B$.
Levels $\binom{[n]}{i} = \{A \subseteq [n] : |A| = i\}$ ('i-uniform')

Theorem (Sperner)

The largest antichain in $\mathcal{P}[n]$ is a level.

Theorem (LYM inequality)

$\mathcal{A} \subseteq \mathcal{P}[n]$ antichain, a_i sets of size $i \to \sum_{i=0}^{n} a_i \binom{n}{i}^{-1} \leq 1$.

Theorem (Bollobás)

$A_i \cap B_j = \emptyset$ iff $i = j$, $1 \leq i, j \leq m \to \sum_{i=1}^{m} \binom{|A_i|+|B_i|}{|A_i|}^{-1} \leq 1$.

Proof: Randomly order elements, events $E_i = \{a < b : \forall\ a \in A_i, b \in B_i\}$,
$\mathbb{P}(E_i) = \binom{|A_i|+|B_i|}{|A_i|}^{-1}$, $E_1, \cdots, E_m$ mutually exclusive $\to \sum_{i=1}^{m} \mathbb{P}(E_i) \leq 1$.

Intersecting families

Definition

t-intersecting $\mathcal{A} \subseteq \mathcal{P}[n]$: $|A \cap B| \geq t \ \forall \ A, B \in \mathcal{A}$.

Easy: $\mathcal{A} \subseteq \mathcal{P}[n]$ (1-)intersecting $\rightarrow |\mathcal{A}| \leq 2^{n-1}$.

Theorem (Erdős-Ko-Rado)

$\mathcal{A} \subseteq \binom{[n]}{k}$ intersecting, $n \geq 2k \rightarrow |\mathcal{A}| \leq \binom{n-1}{k-1}$.

Theorem (Katona)

$\mathcal{A} \subseteq \mathcal{P}[n]$ t-intersecting, $n+t$ even $\rightarrow |\mathcal{A}| \leq \sum_{i=(n+t)/2}^{n} \binom{n}{i}$.

Definition

t-majority $\mathcal{M}_{n,k,t,i} = \{A : A \subseteq [n], |A| = k, |A \cap [t+2i]| \geq t+i\}$.

Ahlswede-Khachatrian Complete Intersection Theorem

$\mathcal{A} \subseteq \binom{[n]}{k}$ t-intersecting $\rightarrow |\mathcal{A}| \leq \max_i |\mathcal{M}_{n,k,t,i}|$.

EKR via Katona circle method

Theorem (Erdős-Ko-Rado)

$\mathcal{A} \subseteq \binom{[n]}{k}$ intersecting, $n \geq 2k \to |\mathcal{A}| \leq \binom{n-1}{k-1}$.

Lemma

For any permutation $\sigma \in S_n$, at most k (cyclic) σ-intervals $B = \{\sigma(x), \cdots, \sigma(x+k-1)\}$ mod n belong to $\mathcal{A}$.

Proof of Lemma

Initial vertices of any two σ-intervals in $\mathcal{A}$ are at distance $\leq k-1$.

Proof of EKR

Random $\sigma \in S_n$, random σ-interval B, $|\mathcal{A}|/\binom{n}{k} = \mathbb{P}(B \in \mathcal{A}) \leq k/n$.

EKR via compressions

ij-compressions

Suppose $1 \leq i < j \leq n$. For $A \subseteq [n]$ let $C_{ij}(A) = A \setminus \{j\} \cup \{i\}$ if $j \in A$ and $i \notin A$, otherwise $C_{ij}(A) = A$. (Replace j by i where possible.) For $\mathcal{A} \subseteq \mathcal{P}[n]$ let $C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}, C_{ij}(A) \notin \mathcal{A}\} \cup \{A : A \in \mathcal{A}, C_{ij}(A) \in \mathcal{A}\}$.
(Replace each $A \in \mathcal{A}$ by $C_{ij}(A)$ unless already present.)
Compressed system $C(\mathcal{A})$: repeatedly apply compressions for all $i < j$.

Properties: $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$, $\mathcal{A}$ t-intersecting $\rightarrow$ $C_{ij}(\mathcal{A})$ t-intersecting.

Proof of EKR

Induction on n and k. Base cases: $k = 1$ trivial, $n = 2k$ pair $A \leftrightarrow \overline{A}$.

$\mathcal{B} = \{A \in C(\mathcal{A}) : n \notin A\}$ intersecting, $|\mathcal{B}| \leq \binom{n-2}{k-1}$.

$\mathcal{D} = \{A \setminus \{n\} : n \in A \in C(\mathcal{A})\}$ intersecting, $|\mathcal{D}| \leq \binom{n-2}{k-2}$. [Disjoint D_1, D_2 in $\mathcal{D} \rightarrow x \in [n-1] \setminus (D_1 \cup D_2)$, disjoint $D_1 \cup \{x\}$, $D_2 \cup \{n\}$ in $\mathcal{A}$.]

So $|\mathcal{A}| = |C(\mathcal{A})| = |\mathcal{B}| + |\mathcal{D}| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$.

Another compression proof

Theorem (Katona)

$\mathcal{A} \subseteq \mathcal{P}[n]$ t-intersecting, $n+t$ even $\rightarrow |\mathcal{A}| \leq \sum_{i=(n+t)/2}^{n} \binom{n}{i}$.

Proof (Ahlswede-Khachatrian)

Induction on n and t: $t=1$ easy ($|\mathcal{A}| \leq 2^{n-1}$), $t=n$ trivial.

$\mathcal{D} = \{A \setminus \{1\} : 1 \in A \in C(\mathcal{A})\}$ $(t-1)$-intersecting, $|\mathcal{D}| \leq \sum_{i=(n+t)/2-1}^{n-1} \binom{n-1}{i}$.

$\mathcal{B} = \{A \in C(\mathcal{A}) : 1 \notin A\}$ $(t+1)$-intersecting, $|\mathcal{B}| \leq \sum_{i=(n+t)/2}^{n-1} \binom{n-1}{i}$.

[If $|B_1 \cap B_2| = t$, B_1, B_2 in $\mathcal{B}$ take $x \in B_1 \cap B_2$,
$B_1' = C_{1x}(B_1) = B_1 \setminus \{x\} \cup \{1\} \in \mathcal{A}$, $|B_1' \cap B_2| < t$.]

So $|\mathcal{A}| = |C(\mathcal{A})| = |\mathcal{B}| + |\mathcal{D}| \leq \sum_{i=(n+t)/2-1}^{n-1} \binom{n-1}{i} + \sum_{i=(n+t)/2}^{n-1} \binom{n-1}{i}$
$= \sum_{i=(n+t)/2}^{n} \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right] = \sum_{i=(n+t)/2}^{n} \binom{n}{i}$.

Traces and down compression

Definitions

The trace of $\mathcal{A} \subseteq \mathcal{P}[n]$ on $S \subseteq [n]$ is $\mathcal{A}_S = \{A \cap S : A \in \mathcal{A}$. Write $Tr(n, \mathcal{F}) = \max |\mathcal{A}| : \mathcal{A}$ has no $\mathcal{F}$ trace. The VC-dimension $\dim_{VC}(\mathcal{A}) = \max k : \mathcal{A}$ has a $2^{[k]}$ trace ('shattered' k-set).

Sauer-Shelah / Shatter Function Lemma (Sa, Pe, Sh, VC)

$\dim_{VC}(\mathcal{A}) = k \to |\mathcal{A}| \leq Tr(n, 2^{[k]}) = \sum_{i=0}^{k-1} \binom{n}{i}$.

i-compressions

Suppose $1 \leq i \leq n$. For $A \subseteq [n]$ let $C_i(A) = A \setminus \{i\}$ if $i \in A$, otherwise $C_i(A) = A$. For $\mathcal{A} \subseteq \mathcal{P}[n]$ let $C_i(\mathcal{A}) = \{C_i(A) : A \in \mathcal{A}, C_i(A) \notin \mathcal{A}\}$ $\cup \{A : A \in \mathcal{A}, C_i(A) \in \mathcal{A}\}$. All i-compressions: $C(\mathcal{A})$.

Props: $|C_i(\mathcal{A})| = |\mathcal{A}|$, $|C_i(\mathcal{A})_S| \leq |\mathcal{A}_S|$, $\dim_{VC}(C_i(\mathcal{A})) \leq \dim_{VC}(\mathcal{A})$ [$\to$ SFL].

General trace problems? Anstee, Frankl-Pach, Balogh-K.-Sudakov, ...

Restricted intersections

Definitions

Suppose $L \subseteq \{0, 1, 2, \cdots\}$, $\mathcal{A} \subseteq \mathcal{P}[n]$. Say $\mathcal{A}$ is *L-intersecting* if $|A \cap B| \in L$ $\forall A, B \in \mathcal{A}$. Say $\mathcal{A}$ is *L-intersecting mod p* if $|A \cap B| \in L \bmod p$ $\forall A, B \in \mathcal{A}$. and $|A| \notin L \bmod p$ $\forall A \in \mathcal{A}$. *$L$-intersecting mod 0* $\equiv$ L-intersecting.

e.g. $\{1\}$-int $\rightarrow |\mathcal{A}| \leq n$ (proj plane); $\{0\}$-int mod 2 $\rightarrow \mathcal{A} \leq n$ (oddtown).

Biplane problem: Is there a $\{2\}$-intersecting $\mathcal{A}$ on $[n]$ of size n for large n?

Frankl–Ray-Chaudhuri–Wilson Theorems

Suppose p prime or 0; k, n positive integers; $|L| = s$.
(i) Any L-intersecting mod p family on $[n]$ has size at most $\sum_{i=0}^{s} \binom{n}{i}$.
(ii) Any k-uniform L-intersecting mod p family on $[n]$ has size at most $\binom{n}{s}$.

Definitions

$m(n, L)$: max L-int on $[n]$; $m(n, k, L)$: max k-uniform L-int on $[n]$;

Problems

Frankl: $m(n, k, L) = \Theta(n^{\alpha(k,L)})$? Deza-Erdős-Frankl: $m(n, 13, \{0, 1, 3\})$?
Füredi: $m(n, \{0, 2, 3\}) = \Theta(n^2)$, asymptotic?

The linear algebra method

Theorem

Suppose $\mathcal{A} = \{A_1, \cdots, A_m\}$ is L-intersecting mod p on $[n]$, where p is prime and $|L| = s$. Then $m \leq \sum_{i=0}^{s} \binom{n}{i}$.

Proof

Recall: $|A_i \cap A_j| \in L$ mod p for $i \neq j$, $|A_i| \notin L$ mod p for all i.

Let $f_i(x) = \prod_{\ell \in L}(x \cdot v_i - \ell) \in \mathbb{F}_p[x_1, \cdots, x_n]$. ($v_i$ incidence vectors.)

$v_i \cdot v_j = |A_i \cap A_j|$ mod p $\to$ $f_i(v_j) = 0$ iff $i \neq j$. Non-singular diagonal matrix.

Multilinearise $f_i \to \widetilde{f}_i$, $x_j^2 = x_j$ on 0/1-vectors, $\widetilde{f}_i(v_j) = f_i(v_j)$.

$\widetilde{f}_1, \cdots, \widetilde{f}_m$ linearly independent in vector space V of multilinear polys of degree $\leq s$ in $x = (x_1, \cdots, x_n)$; $m \leq \dim V = \sum_{i=0}^{s} \binom{n}{i}$.

Extra tricks: (i) Change $i \neq j$ to $i < j$: non-singular triangular matrix.
(ii) Add $\sum_{i=0}^{s-1} \binom{n}{i}$ linearly independent polys: $\leq \binom{n}{s}$ when k-uniform.

The sunflower (delta system) method

Definition

Sunflower $\mathcal{S}$ with t petals and centre C: $S_1, \cdots, S_t$ s.t. $S_i \cap S_j = C \; \forall \; i \neq j$.

Sunflower Lemma (Erdős-Rado)

$\mathcal{A}$ k-uniform, no t-sunflower $\rightarrow |\mathcal{A}| \leq k!(t-1)^k$.

Proof: max matching $\leq t-1$, $\exists |\mathcal{A}(x)| \geq |A|/(t-1)k$, induction.

Problem: $|A| \leq C(t)^k$? Open even for $t = 3$.

Key property: $|A| < t \rightarrow \exists i$ with $A \cap S_i = A \cap C$.

Definition

Non-trivial t-intersecting $\mathcal{A}$: t-intersecting & $|\cap \mathcal{A}| < t$.

$\mathcal{A}_1 = \{A \in \binom{[n]}{k} : [t] \subseteq A, A \cap [t+1, k+1] \neq \emptyset\} \cup \{[k+1] \setminus \{i\} : i \in [t]\}$.
$\mathcal{A}_2 = \{A \in \binom{[n]}{k} : |A \cap [t+2]| \geq t+1\}$.

Non-trivial t-intersecting families

Theorem (Frankl, Hilton-Milner $t = 1$)

k-uniform non-trivial t-intersecting $\mathcal{A}$ on $[n]$ large $\rightarrow |\mathcal{A}| \leq \max\{|\mathcal{A}_1|, |\mathcal{A}_2|\}$.
Equality only for $\mathcal{A} = \mathcal{A}_1$, $k \geq 2t + 2$ or $\mathcal{A} = \mathcal{A}_1$, $k \leq 2t + 1$.

Proof

Generating family $\mathcal{B}$: repeatedly replace A by $A' \subsetneq A$ while still t-int.

$\mathcal{B}$ no k-sunflower (or replace $\mathcal{S}$ by centre C), so $|\mathcal{B}| \leq k!k^k$.

$\mathcal{B}_i = \{B \in \mathcal{B} : |B| = i$, $\mathcal{B}_i = \emptyset$ for $i \leq t$ (non-trivial).

wma $\max\{|\mathcal{A}_1|, |\mathcal{A}_2|\} \leq |\mathcal{A}| \leq \sum_i |\mathcal{B}_i| \binom{n-i}{k-i} = |\mathcal{B}_i| \binom{n-t+1}{k-t+1} + O(n^{k-t-2})$, so
$|\mathcal{B}_{t+1}| \geq \max\{k - t + 1, t + 2\}$. Choose $B_1, B_2, B_3 \in \mathcal{B}_{t+1}$.

Suppose $B_1 \cap B_2 = B_1 \cap B_3 = C$. Then $\mathcal{B}_{t+1}$ sunflower center C, non-triv
$\rightarrow \exists A \in \mathcal{A} : |A \cap C| < t$, so $|\mathcal{B}_{t+1}| \leq k - t + 1$, $\mathcal{A} = \mathcal{A}_1$.

Otherwise, $D = B_1 \cup B_2$ size $t + 2$, $|A \cap D| \geq t + 1$ $\forall A \in \mathcal{A}$, $\mathcal{A} = \mathcal{A}_2$.

Spectral methods

Theorem (Hoffman bound)

d-regular graph G, adjacency matrix A, eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$
$\rightarrow$ independence number $\alpha(G) \leq \frac{-\lambda_n n}{d-\lambda_n}$.

Proof

Suppose I independent with characteristic function f. Take uniform measure $\mu(i) = 1/n \; \forall i$. Write $\alpha := |I|/n = \|f\|_1 = \|f\|_2^2$ (0/1 function).

Take $v_1, \cdots, v_n$ orthonormal eigenvectors of A, where $v_1 = (1, \cdots, 1)$.
Write $f = \sum a_i v_i$. Then $a_1 = \|f\|_1 = \alpha$, and Parseval $\rightarrow \|f\|_2^2 = \sum a_i^2 = \alpha$.

Now I indep $\rightarrow fAf^T = 0$, so $\sum \lambda_i a_i^2 = 0$, so $0 \geq \lambda_1 \alpha^2 + (\alpha - \alpha^2)\lambda_n$.

Definition (expansion)

A graph G is an α-expander if $|N(W)| \geq \alpha |W| \; \forall \; W \subset V$, $|W| \leq n/2$,
where $N(W) = \{v \in V \setminus W : \exists w \in W, vw \in E\}$.

Theorem

G is a $(d - \lambda_2)/2d$-expander.

Intersection stability: Friedgut's proof

Disjointness graph D_n on $\{0,1\}^n \leftrightarrow \mathcal{P}[n]$: $A \sim B$ iff $A \cap B = \emptyset$.
Independent set = intersecting family.

$\mu_p(1) = p$, $\mu_p(0) = q = 1 - p$, $A_1 = \begin{pmatrix} 1 - p/q & p/q \\ 1 & 0 \end{pmatrix}$, eigenvectors $(1,1)$, $(\sqrt{p/q}, -\sqrt{q/p})$ evals 1, $-p/q$, orthonormal wrt μ_p.

$A = A_1^{\otimes n}$ pseudo-adjacency matrix, eigenvectors $\{\chi_S\}_{S \subseteq [n]}$: p-biased Walsh functions $\chi_S(x) = \prod_{i \in S} \chi_{\{i\}}(x)$, where $\chi_{\{i\}}(x)$ is $\sqrt{p/q}$ if $x_i = 0$, $-\sqrt{q/p}$ if $x_i = 1$, eigenvalues $\lambda_S = (-p/q)^{|S|}$.

I independent=intersecting, f char function, $f = \sum_S \hat{f}(S)\chi_S$.

Parseval $\rightarrow \sum_S \hat{f}(S)^2 = \|f\|_2^2 = \|f\|_1 = \hat{f}(\emptyset) = \mu_p(\mathcal{A}) := \alpha$, so
$0 = fAf^T = \sum (-p/q)^{|S|} \hat{f}(S)^2 \geq \alpha^2 + (\alpha - \alpha^2)(-p/q)$, $\alpha \leq p$.

μ_p stability: $\alpha \sim p \rightarrow \sum_{|S|>1} \hat{f}(S)^2 = o(1) \rightarrow_{FKN} \exists i \ \|f - x_i\| = o(1)$.

EKR stability: k-uniform int $|\mathcal{A}| \sim \binom{n-1}{k-1}$, $p \approx k/n$, $\mathcal{A}^\uparrow$ upset of $\mathcal{A}$, $\mu_p(\mathcal{A}^\uparrow) \sim p$, $\exists \mathcal{B} = \{B : i \in B\}$, $\mu_p(\mathcal{A}^\uparrow \setminus \mathcal{B}) = o(1)$, $|\mathcal{A} \setminus \mathcal{B}| = o\binom{n-1}{k-1}$.

Another spectral proof of intersection stability (K.)

Recall circle method: $\mathcal{A} \subseteq \binom{[n]}{k}$ intersecting ($n \geq 2k$), contains $\leq k$ σ-intervals $\forall \sigma \in S_n$, averaging $\rightarrow |\mathcal{A}| \leq \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$.

Definition

σ is v-complete if $\mathcal{A}$ contains all k σ-intervals containing v.

N.B. $|\mathcal{A}| \sim \binom{n-1}{k-1} \rightarrow o(n!)$ incomplete permutations.

Lemma [just case checking]

If σ is v-complete and $\tau = \sigma \circ (i\ i+1)$ is complete then τ is v-complete.

Represent circular orders by $S_{n-1} = \{\sigma : \sigma(n) = n\}$. Cayley graph C generated by $T = \{(12), (23), \cdots, (n-2\ n-1)\}$. Regular $d = n-2$, Bacher $\rightarrow d - \lambda_2 = 2 - 2\cos(\pi/(n-1)) > 2/n^2$, so $1/n^3$-expander.

If $|W| > n^3 \cdot$#incomplete then $\exists$ complete $\sigma \in N(W)$, so $C' = C$[complete] has component of size $\sim n!$. Lemma $\rightarrow \exists v$ s.t. σ is v-complete $\forall \sigma \in C'$. Then EKR $\rightarrow$ all but $o\binom{n-1}{k-1}$ sets of $\mathcal{A}$ contain v.

More open problems

Frankl's union-closed set problem

Suppose $\mathcal{A}$ satisfies $A \cup B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
Is there some element in at least half of the sets in $\mathcal{A}$?

Chvátal's intersecting subfamily problem

Suppose $\mathcal{A}$ is a downset = simplicial complex, i.e. $B \subseteq A \in \mathcal{A} \to B \in \mathcal{A}$.
Is there a maximum size intersecting subfamily $\mathcal{A}'$ of $\mathcal{A}$ that is a star, i.e. $\mathcal{A}' = \{A : x \in A \in \mathcal{A}\}$ for some x?

Chvátal's simplex problem

A *d-simplex* is a collection of $d+1$ sets with empty intersection, every d of which have non-empty intersection. Suppose $k \geq d+1 \geq 2$, $n > k(d+1)/d$ and $\mathcal{A}$ is k-uniform on $[n]$ with no d-simplex.
How large can $\mathcal{A}$ be? At most $\binom{n-1}{k-1}$, equality only for a star?