Hypergraph Turán Problems (IPAM Tutorial)

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Introduction: From graphs to hypergraphs

Mantel's Theorem (1907)

The largest graph on a given vertex set with no triangle is bipartite.

- Generalisations led to Extremal Graph Theory.
- Hypergraph analogues?

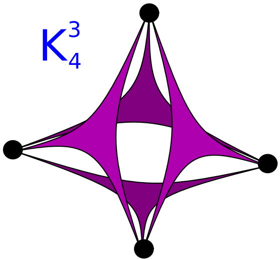
Definition

An *r-graph* G has a vertex set V and an edge set E.
Each edge $e \in E$ is a subset of V of size $|e| = r$.

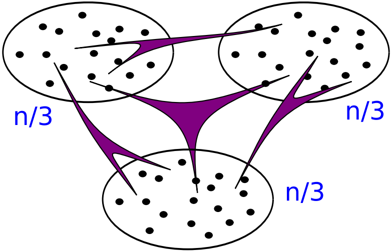
Question

What is the largest 3-graph on a given vertex set with no tetrahedron?

Turán's Conjecture



Tetrahedron



A conjectured optimum

Turán's Conjecture (Asymptotic Form)

Any tetrahedron-free 3-graph on $[n]$ has $< (5/9 + o(1))\binom{n}{3}$ edges?

- Chung-Lu (1999): density $\leq \frac{3+\sqrt{17}}{12} = 0.593592\cdots$
- Razborov (2009+): no tetrahedron & no 4 vertices inducing exactly 1 edge implies density $\leq 5/9$. Pikhurko (2009+): structure...
- Razborov's computations also *suggest* < 0.561666 for no tetrahedron.

The Turán problem for hypergraphs

Definitions

Suppose F is a fixed r-graph. The Turán number $\mathrm{ex}(n, F)$ is the maximum number of edges in an F-free r-graph on n vertices.
The Turán density is $\pi(F) = \lim_{n\to\infty} \mathrm{ex}(n, F)\binom{n}{r}^{-1}$.
Write K_s^r for the complete r-graph on s vertices.

Open problem (Turán 1941)

Determine $\pi(K_s^r)$ for some $s > r > 2$.

Bounds on $t(r, s) := 1 - \pi(K_s^r)$

de Caen 1983: $t(r, s) \geq \binom{r-1}{s-1}^{-1}$, Sidorenko 1981: $t(r, s) \leq \frac{s-1}{r-1}^{s-1}$.
Frankl-Rödl 1985: $t(r + a, r) \leq a(a + 4 + o_r(1))(\ln r)\binom{r}{a}^{-1}$.
Sidorenko 1997: $t(r, s) \leq (1 + o_s(1))(r - s + 1)\binom{r}{s}^{-1} \ln \binom{r}{s}$, $r \geq s + \frac{s}{\log_2 s}$.

$1/r \leq t(r + 1, r) \leq (1 + o(1))\frac{\ln r}{2r} \ldots$ & Lu-Zhao 2009+, Markström 2009+

Simple counting method

Proposition (de Caen)

$\pi(K_4^3) \leq 2/3$.

Proof.

Suppose G is a K_4^3-free 3-graph with n vertices and e edges.
Let f_i, $0 \leq i \leq 3$ be # 4-sets inducing i edges.
For $x, y \in V$ let $d_{xy} = \#\{z : xyz \in E\}$.

Double counting: $\sum_{i=0}^{3} f_i = \binom{n}{4}$; $\sum_{i=0}^{3} if_i = e(n-3)$; $\sum_{x,y} d_{xy} = 3e$; $\sum_{x,y} \binom{d_{xy}}{2} = f_2 + 3f_3$.

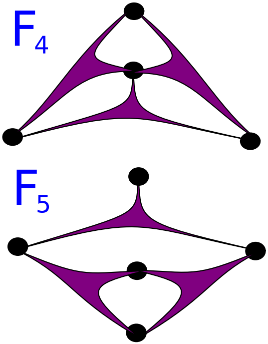
Inequality: $e(n-3) \geq f_2 + 3f_3 \geq \binom{n}{2}\binom{3e/\binom{n}{2}}{2}$ (Cauchy-Schwartz), so $e \leq (1/9 + o(1))n^3 = (2/3 + o(1))\binom{n}{3}$.

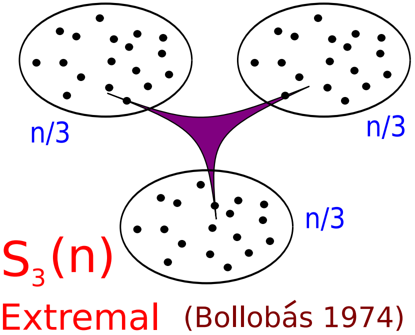
Cancellative hypergraphs

Question (Katona 1960's)

How many edges can a 3-graph G on $[n]$ have if it is cancellative?
$(A \cup C = B \cup C \rightarrow A = B, \forall A, B, C \in E(G))$

N.B. An ordinary graph is cancellative iff it is triangle-free.





Link graph method

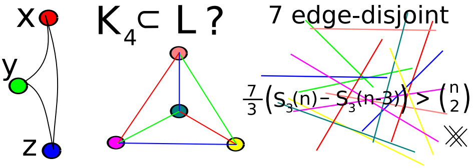
Definition

Suppose G is a 3-graph and $x \in V(G)$. The link (neighbourhood) graph is $G(x) = \{yz : xyz \in E(G)\}$.

Easy facts: Suppose G is cancellative & $xyz \in E(G)$. Then $G(x)$, $G(y)$, $G(z)$ are edge-disjoint. In the 3-coloured graph $L = G(x) \cup G(y) \cup G(z)$ all triangles are rainbow.

Sketch proof of Bollobás' Theorem (K.-Mubayi):

Suppose $e = xyz \in E(G)$. Induction $\rightarrow$ wma # edges meeting e $\geq s_3(n) - s_3(n-3) = t_3(n) - n + 1$, where $t_3(n) = \mathrm{ex}(n, K_4)$ (Turán).



Turán $\rightarrow$ # edges meeting e $\leq t_3(n-3) + (n-3) + 1 = t_3(n) - n + 1$. Therefore equality & structure holds. ∎

Structure and approximate structure

Andrásfai-Erdős-Sós Theorem

Any triangle-free graph G on n vertices with minimum degree $\delta(G) > 2n/5$ is bipartite.

Best possible: consider C_5-blowup.

Triangle Stability (Simonovits)

$\forall \epsilon \exists \delta$ s.t. if G is a triangle-free graph on n vertices with $> (1-\delta)n^2/4$ edges then $\exists$ bipartition $V(G) = A \cup B$ s.t. $e(A), e(B) < \epsilon n^2$.

Informally: if G triangle-free on $[n]$, $e(G) = (1+o(1))n^2/4$
then G is complete bipartite $\pm o(n^2)$ edges.

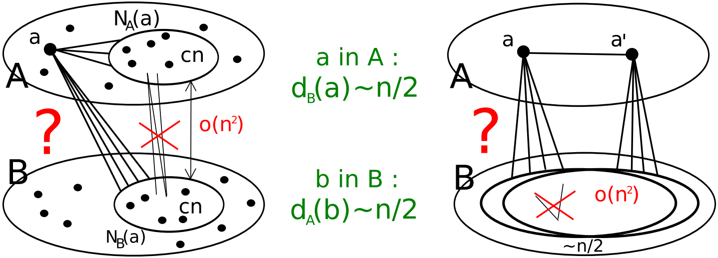
Both results generalise suitably, replacing 'triangle' with K_r.

The stability method

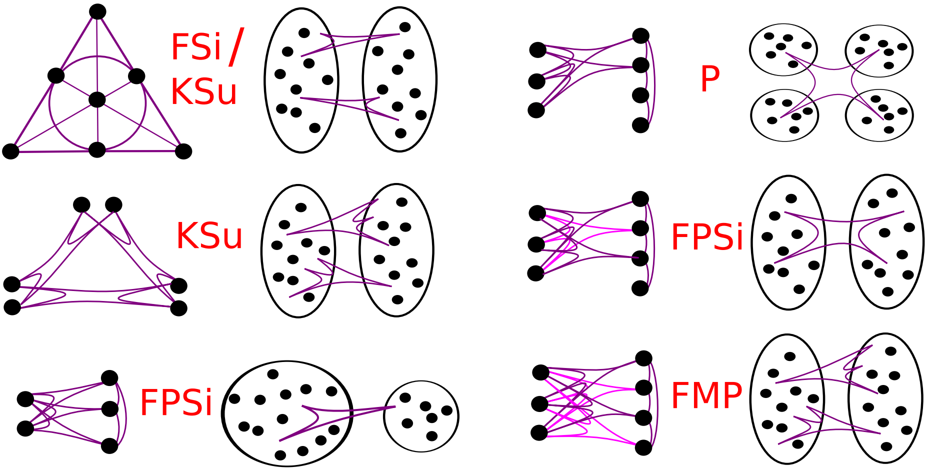
Step 1: Stability (approximate structure) version.
Step 2: Small imperfections → fewer edges than when perfect.

Example: $\mathrm{ex}(n, C_5) = \lfloor n^2/4 \rfloor$ for large n.
Stability: C_5-free & $e(G) \sim n^2/4 \to G \sim K_{n/2,n/2}$.

Sketch proof: Induction → can assume $\delta(G) \sim n/2$.
Optimal bipartition $V(G) = A \cup B$. Stability → $o(n^2)$ 'bad' edges.
Optimality → $d_B(a) \geq d_A(a)\ \forall a \in A$.



Some applications of stability



Füredi Keevash Mubayi Pikhurko Simonovits Sudakov

The Lagrangian method

Definitions

Consider an r-graph G on $[n]$. Introduce variables $x = (x_1, \cdots, x_n)$, let $p_G(x) = \sum_{e \in E(G)} \prod_{i \in E} x_i$ and $S = \{x : \sum x_i = 1, x_i \geq 0 \ \forall i\}$. Fix $x^* \in S$ with $p_G(x^*) = \max_{x \in S} p_G(x) := \lambda_G$ - the Lagrangian.

Properties of the Lagrangian:

- $\lambda_G \geq p_G(1/n, \ldots, 1/n) = e(G)/n^r$.
- Variation $\rightarrow$ wma $\text{supp}(x^*) = \{i : x_i^* \neq 0\}$ is a 2-cover: $i, j \in \text{supp}(x^*) \rightarrow \exists e \in E(G), \{i, j\} \subseteq e$.

  [Motzin-Strauss application: G triangle-free graph $\rightarrow$ $|\text{supp}(x^*)| \leq 2$, $\lambda_G = \max_S x_1 x_2 = 1/4$, $e(G) \leq n^2/4$.]
- Lagrange multipliers $\rightarrow \frac{\partial p_G}{\partial x_i}\Big|_{x^*} = \frac{\partial p_G}{\partial x_j}\Big|_{x^*} = C \ \forall i, j$, where $C = \sum x_i^* \frac{\partial p_G}{\partial x_i}\Big|_{x^*} = r p_G(x^*) = r\lambda_G$.

Cancellative 4-graphs

Theorem (Sidorenko)

Any cancellative 4-graph G on $[n]$ has $e(G) \leq (n/4)^4$.

Proof

ETS $\lambda_G = p_G(x^*) \leq (1/4)^4$. WMA $\text{supp}(x^*)$ 2-cover.

Cancellative $\rightarrow$ link 3-graphs edge-disjoint.

$n \cdot 4\lambda_G = \sum_i \frac{\partial p_G}{\partial x_i}\big|_{x^*} \leq \sum_{\{i,j,k\}} x_i^* x_j^* x_k^* \leq \binom{n}{3} n^{-3}$.

$\lambda_G \leq \frac{(n-1)(n-2)}{24n^3} \leq (1/4)^4$ for $n = 4$ or $n \geq 6$ ($n = 5$ not 2-cover).

Pikhurko: no $\{1234, 1235, 4567\}$ suffices (via stability).

Shearer: r-graph counterexample to $e(G) \leq (n/r)^r$ for $r \geq 10$.

Hypergraph limits and Flag algebras

The analytic viewpoint

r-graph $G \leftrightarrow$ (hom) density sequence $(d_F(G))_F$. (G_i) 'converges' $\leftrightarrow$ $d_F(G_i)$ converges $\forall F$. Regularity $\leftrightarrow$ compactness.

The algebraic viewpoint

$\mathbb{R}\mathcal{F}^0 = \{\sum_F a_F F\}$. r-graph $G \in \mathcal{F}^0 \leftrightarrow \sum_F p_F(G)F$. ($p_F$ = induced d_F.)
Chain rule: $p_F(G) = \sum_{L \in \mathcal{F}^0_\ell} p_F(L)p_L(G)$ if $v_F \leq \ell \leq v_G$.
$\mathcal{K}^0 = \langle\{G - \sum_{L \in \mathcal{F}^0_\ell} p_L(G)L\}\rangle$. Commutative algebra $\mathcal{A}^0 = \mathbb{R}\mathcal{F}^0/\mathcal{K}^0$.
$\text{Hom}^+(\mathcal{A}^0, \mathbb{R}) = \{\phi \in \text{Hom}(\mathcal{A}^0, \mathbb{R}) : \phi(F) \geq 0 \ \forall F \in \mathcal{F}^0\}$.

Flag algebras

Labelled r-graph σ. σ-flags $(G, \theta) \in \mathcal{F}^\sigma$, $\theta : \sigma \hookrightarrow G$. $\mathcal{A}^\sigma = \mathbb{R}\mathcal{F}^\sigma/\mathcal{K}^\sigma$.
Averaging $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}^0$. Theories... e.g. 'tetrahedron-free 3-graphs'.

Razborov's $(4,3)$-bound

Definitions

Let G_i be the 3-graph on 4 vertices with i edges. Say G is admissable if it has no induced G_0 or G_3. We work in the theory of admissable 3-graphs. Let $\rho \in \mathcal{F}_3^0$ be a single edge, and $e \in \mathcal{F}_3^1$ be a single edge with a distinguished vertex.

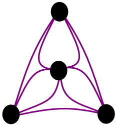
Theorem (Razborov)

$\rho \geq 4/9$, i.e. $\phi(\rho) \geq \phi(4/9) = 4/9 \ \forall \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$.

Sketch proof

- $[\![(e-4/9)(1-e)]\!]_1 = \frac{5}{9}[\![e-4/9]\!]_1 - [\![(e-4/9)^2]\!]_1 \leq \frac{5}{9}[\![e-4/9]\!]_1$.
- $[\![(e-4/9)(1-e)]\!]_1 \geq [\![Q_1]\!]_{G_1} + [\![Q_2]\!]_{G_2}$, where Q_1, Q_2 are explicit positive definite quadratic forms over $\mathcal{A}_5^{G_1}$, $\mathcal{A}_5^{G_2}$.

Open problems



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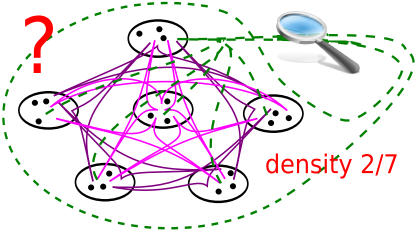

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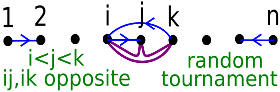
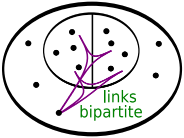
Every pair in cn edges,
no tetrahedron?

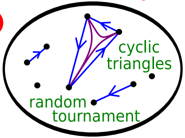


codegree density 1/2



density 1/4

?



?

Brown-Erdős-Sós problem

$f^r(n,v,e)$ = max #edges
in r-graph on n vertices s.t.
no v vertices span e edges

Ruzsa-Szemerédi: $f^3(n,6,3)=o(n^2)$

$f^3(n,7,4)$?