Quantifying Uncertainty in Climate Change Science: Empirical Information Theory, Fluctuation Dissipation Theorems, and Physics Based Statistics

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- A. Majda and B. Gershgorin:
  Quantifying Uncertainty in Climate Change Science through Empirical Information Theory, 2010 PNAS in press
- A. Majda, R. Abramov, B. Gershgorin:
  High skill in low frequency climate response through fluctuation dissipation theorems despite structural instability, 2010 PNAS, Vol. 107, no. 2, pp 581-586
- B. Gershgorin, A. Majda:
  A Test Model for Fluctuation-Dissipation Theorems with Time Periodic Statistics
  Physica D, submitted October 2009
- A. Majda, B. Gershgorin, Y. Yuan:
  Low Frequency Climate Response and Fluctuation-Dissipation Theorems: Theory and Practice,
  2010 JAS, Vol. 67, pp. 1186-1201

Practical questions in climate change science

- How will the mean temperature change if the heating from the sun increases?

- How will the variance of the temperature respond to the changes of CO_2 concentration?

- How will the mean velocity profile in the ocean behave if the salinity starts changing?

- How will the mean temperature in April change if the heating in January decreases?

Quantifying Uncertainty in Climate Change Science through Empirical Information Theory

Quantifying the uncertainty for the present climate and the predictions of climate change in the suite of imperfect Atmosphere Ocean Science (AOS) computer models is a central issue in climate change science.
Basic questions:

- A How to measure the skill of a given model in reproducing the present climate and predicting the future climate in an unbiased fashion?

- B How to make the best possible estimate of climate sensitivity to changes in external or internal parameters by utilizing the imperfect knowledge available of the present climate? What are the most dangerous parameters for climate change given uncertain knowledge of the present climate?

- C How do coarse-grained measurements of different functionals of the present climate affect the assessments in A), B)? What are the weights which should be assigned to different functionals of the present climate as targets to improve the performance of the imperfect AOS models? Which new functionals of the present climate should be observed in order to improve the assessments in A), B)?

Difficulty: Don't know dynamics for actual climate!

Empirical Information Theory

Jaynes 1957

Majda, Abramov, Grote 2005 AMS

Majda, Wang 2006, Cambridge Press

Empirical information theory and climate science

With a subset of variables $\vec{u} \in \mathbb{R}^N$ and a family of measurement functionals $\vec{E}_L(\vec{u}) = (E_j(\vec{u}))$, $1 \leq j \leq L$, for the present climate, <u>empirical information theory builds the least biased probability measure $\pi_L(\vec{u})$ consistent with the L measurements</u> of the present climate, $\overline{\vec{E}}_L$.

There is a unique functional on probability densities to measure this given by the entropy

$$\mathcal{S} = -\int \pi \ln \pi,$$

and $\pi_L(\vec{u})$ is the unique probability so that $\mathcal{S}(\pi_L(\vec{u}))$ has the largest value among those probability densities consistent with the measured information, $\overline{\vec{E}}_L$.

$$\pi_L(\vec{u}) = e^{-\alpha_0 - \vec{\alpha}_L \cdot \vec{E}_L(\vec{u})},$$

L Lagrange multipliers $\vec{\alpha}_L = (\alpha_1, \ldots, \alpha_L)$ are chosen

$$\overline{\vec{E}}_L = \int \vec{E}(\vec{u}) e^{-\alpha_0 - \vec{\alpha}_L \cdot \vec{E}_L(\vec{u})},$$

α_0 is determined by the normalization

$$e^{\alpha_0} = \int e^{-\vec{\alpha}_L \cdot \vec{E}_L(\vec{u})},$$

The natural way to measure the lack of information in one probability density, $q(\vec{u})$, compared with the true probability density, $p(\vec{u})$, is through the relative entropy, $\mathcal{P}(p,q)$, given by

$$\mathcal{P}(p,q) = \int p \ln\left(\frac{p}{q}\right).$$

This functional on probability densities has two attractive features as a metric for climate change science:

1) $\mathcal{P}(p,q) \geq 0$ with equality if and only if $p = q$,
2) $\mathcal{P}(p,q)$ is invariant under general nonlinear changes of variables.

$\mathcal{P}(\pi, \pi_L)$ precisely quantifies the intrinsic error in using the L measurements of the present climate, $\overline{\vec{E}}_L$.

An AOS model for the present climate is described by $\pi^M(\vec{u})$, intrinsic model error in the climate statistics is given by

$$\mathcal{P}(\pi, \pi^M).$$

Consider a class of imperfect models, $\mathcal{M}$, for the climate, the best climate model for the coarse-grained variable $\vec{u}$ is the $M_* \in \mathcal{M}$ so that the true climate has the smallest additional information beyond the modelled climate distribution $\pi^{M_*}(\vec{u})$, i.e.,

$$\mathcal{P}(\pi, \pi^{M_*}) = \min_{M \in \mathcal{M}} \mathcal{P}(\pi, \pi^M).$$

Also, actual improvements in a given climate model with distribution $\pi^M(\vec{u})$ either through higher resolution or improved parameterization resulting in a new $\pi^M_{post}(\vec{u})$ should result in improved information for the actual climate, so that

$$\mathcal{P}(\pi, \pi^M_{post}) \leq \mathcal{P}(\pi, \pi^M),$$

otherwise, objectively, the model has not been improved compared with the original climate model.

Fact1 : $$\mathcal{P}(\pi, \pi_{L'}^{M}) = \mathcal{P}(\pi, \pi_L) + \mathcal{P}(\pi_L, \pi_{L'}^{M})$$
$$= (\mathcal{S}(\pi_L) - \mathcal{S}(\pi)) + \mathcal{P}(\pi_L, \pi_{L'}^{M}) \text{ for } L' \leq L.$$

The unbiased intrinsic error in the finite number of climate measurements in of the actual climate is exactly the entropy difference. With **Fact1** and a fixed family of L measurements of the actual climate, the optimization principles can be computed explicitly by replacing the unknown π by the hypothetically known π_L in these formulas so that for example, π^{M_*} is calculated by

$$\mathcal{P}(\pi_L, \pi_{L'}^{M_*}) = \min_{M \in \mathcal{M}} \mathcal{P}(\pi_L, \pi_{L'}^{M}).$$

Suppose new observations of the climate became available so that $L > L'$, then according to **Fact1**,

$$\mathcal{S}(\pi_L) + \mathcal{P}(\pi_L, \pi_{L'}^M) = \mathcal{S}(\pi_{L'})) + \mathcal{P}(\pi_{L'}, \pi_L^M).$$

Since π_L involves more climate measurements than $\pi_{L'}$, it follows immediately that the absolute uncertainty satisfies $\mathcal{S}(\pi_L)) < \mathcal{S}(\pi_{L'}))$ so

$$\mathcal{P}(\pi_L, \pi_{L'}^M) > \mathcal{P}(\pi_{L'}, \pi_{L'}^M).$$

There is an increase of uncertainty of all the current AOS models, $\pi_{L'}^M$, due to the additional new measurements of the present climate; thus the current AOS models need to be recalibrated for their skill.

An appealing way to recalibrate the current climate models and improve them simultaneously is through active "on the fly" filtering or data assimilation.

Research/expository article: Majda, Harlim, Gershgorin (2010)

Algorithms for effective calculation of the empirical metrics for climate uncertainty

Practical setup for calibration of contemporary AOS models: climate measurements and model measurements involve only mean and covariance of $\vec{u}$ so that π_L is Gaussian with climate mean $\overline{\vec{u}}$ and covariance R while π^M is Gaussian with model mean $\overline{\vec{u}}_M$ and covariance R_M.
$\mathcal{P}(\pi_L, \pi^M)$ has the explicit formula:

$$\mathcal{P}(\pi_L, \pi^M) = \left[\frac{1}{2}(\overline{\vec{u}} - \overline{\vec{u}}_M)^* (R_M)^{-1} (\overline{\vec{u}} - \overline{\vec{u}}_M)\right] + \left[-\frac{1}{2}\log\det(RR_M^{-1}) + \frac{1}{2}(\mathrm{tr}(RR_M^{-1}) - N)\right].$$

First term is the signal, reflecting the model error in the mean but weighted by the inverse of the model covariance, R_M^{-1}, while the second term, the dispersion, involves only the model error covariance ratio, RR_M^{-1}.
This intrinsic metric is invariant under any (linear) change of variables which maps Gaussian distributions to Gaussians and the signal and dispersion terms are individually invariant under these transformations.

Non-Gaussian statistics: Kleeman (2002), Majda Kleeman Cai (2002), Haven Majda Abramov (2005), Abramov (2006-7-9)

Empirical theory for finding the most dangerous climate change directions from the present climate

Consider a family of parameters $\vec{\lambda} \in \mathbb{R}^p$ with $\pi_{\vec{\lambda}}$ the true climate that occurs; $\vec{\lambda}$ — external parameters, changes in forcing, internal variability, change in dissipation.

The most dangerous perturbed climate is the one with largest uncertainty of the present climate

$$\mathcal{P}(\pi_{\vec{\lambda}_*}, \pi) = \max_{\vec{\lambda} \in \mathbb{R}^p} \mathcal{P}(\pi_{\vec{\lambda}}, \pi).$$

$$\pi_{\vec{\lambda}}\Big|_{\vec{\lambda}=0} = \pi, \text{ then for small values of } \vec{\lambda}:$$

$$\mathcal{P}(\pi_{\vec{\lambda}}, \pi) = \vec{\lambda} \cdot I(\pi)\vec{\lambda} + O(|\vec{\lambda}|^3).$$

Fisher information:

$$\vec{\lambda} \cdot I(\pi)\vec{\lambda} = \int \frac{(\vec{\lambda} \cdot \nabla_{\vec{\lambda}}\pi)^2}{\pi}.$$

<u>Fact 2</u>: The most dangerous climate change direction occurs along the unit direction $\vec{e}_\pi^{\,*} \in \mathbb{R}^p$
which is associated with the largest eigenvalue, λ_π^*, of the above quadratic form.

Theoretical solution to the climate sensitivity of the present climate is hampered by both our lack of information in both the present climate and for the gradients, $\vec{\lambda} \cdot \nabla_{\vec{\lambda}} \pi$.
Platonic ideal for climate change science simplifies when the observed climate distribution $\pi_L(\vec{u})$ is utilized

<u>Fact3</u> $: \mathcal{P}(\pi_{\vec{\lambda},L}, \pi_L) = \vec{\lambda} \cdot I(\pi_L)\vec{\lambda} + O(|\vec{\lambda}|^3),$

with $\quad \vec{\lambda} \cdot I(\pi_L)\vec{\lambda} = (\vec{\lambda} \cdot \nabla_{\vec{\lambda}} \overline{\vec{E}}_L)^T \mathcal{C}_L^{-1} \vec{\lambda} \cdot \nabla_{\vec{\lambda}} \overline{\vec{E}}_L,$

and $\mathcal{C}_L$ is the $L \times L$ climate correlation matrix

$$\mathcal{C}_L = \overline{(\vec{E}_L(\vec{u}) - \overline{\vec{E}}_L)(\vec{E}_L(\vec{u}) - \overline{\vec{E}}_L)^T},$$

For fewer measurements $L' \leq L$, the compressed quadratic form is

$$\vec{\lambda} \cdot I(\pi_L)\vec{\lambda} = (\vec{\lambda} \cdot \nabla_{\vec{\lambda}} \overline{\vec{E}}_{L'})^T \mathcal{C}_L^{-1} \vec{\lambda} \cdot \nabla_{\vec{\lambda}} \overline{\vec{E}}_{L'}, \quad \vec{E}_{L'} = (E_1, \ldots, E_{L'}, 0, \ldots, 0)^T,$$

<u>Link to FDT</u>: for such a climate model, one can calculate the unknown information $\nabla_{\vec{\lambda}} \overline{\vec{E}}_L(\vec{u})$ through statistics of the present modelled climate from a suitable version of algorithms based on the fluctuation-dissipation theorem (FDT)

Exactly solvable test models for climate change science

$U(t) = \bar{U}(t) + U'(t),$ zonal jet, seasonal cycle

$v(x,t),$ turbulent Rossby waves

$T(x,t) + \alpha y = \text{"}T\text{"},$ passive tracer with mean gradient (CO_2, CO, etc)

$$\frac{dU}{dt} = -\gamma(U - \bar{U}(t)) + \sigma \dot{W},$$ similar equation for each Fourier mode of v

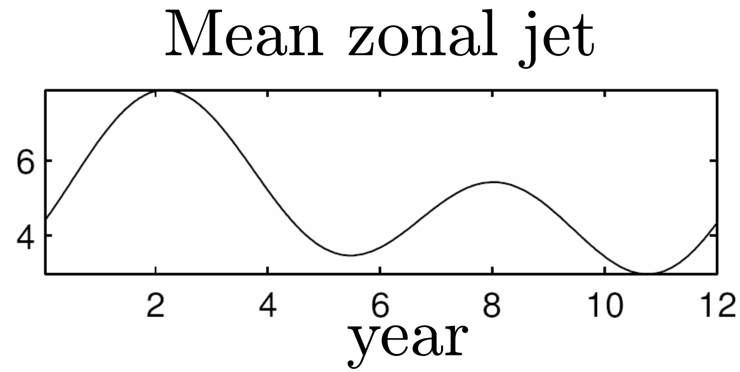
$$\frac{\partial T}{\partial t} + U(t)\frac{\partial T}{\partial x} = -\alpha v(x,t) + \kappa \frac{\partial^2 T}{\partial x^2},$$

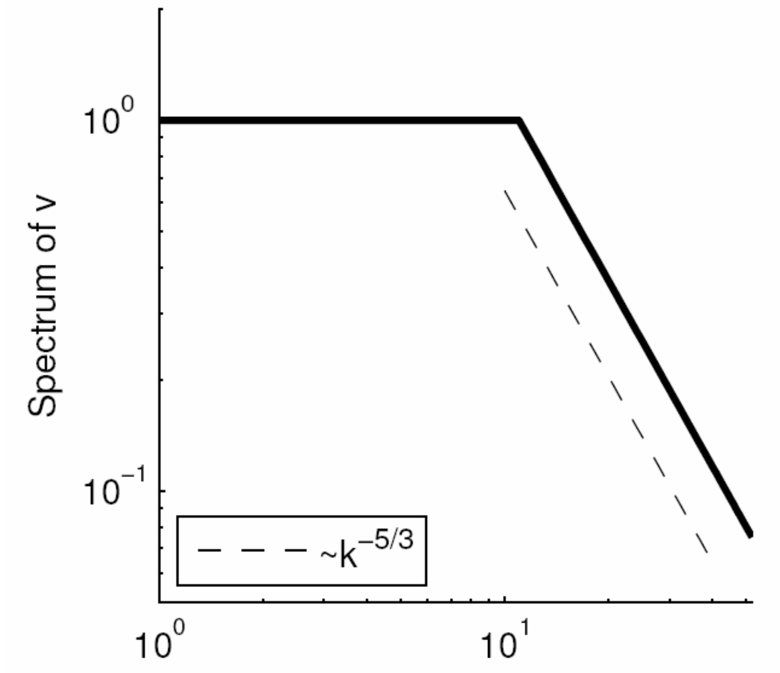
Statistically exactly solvable

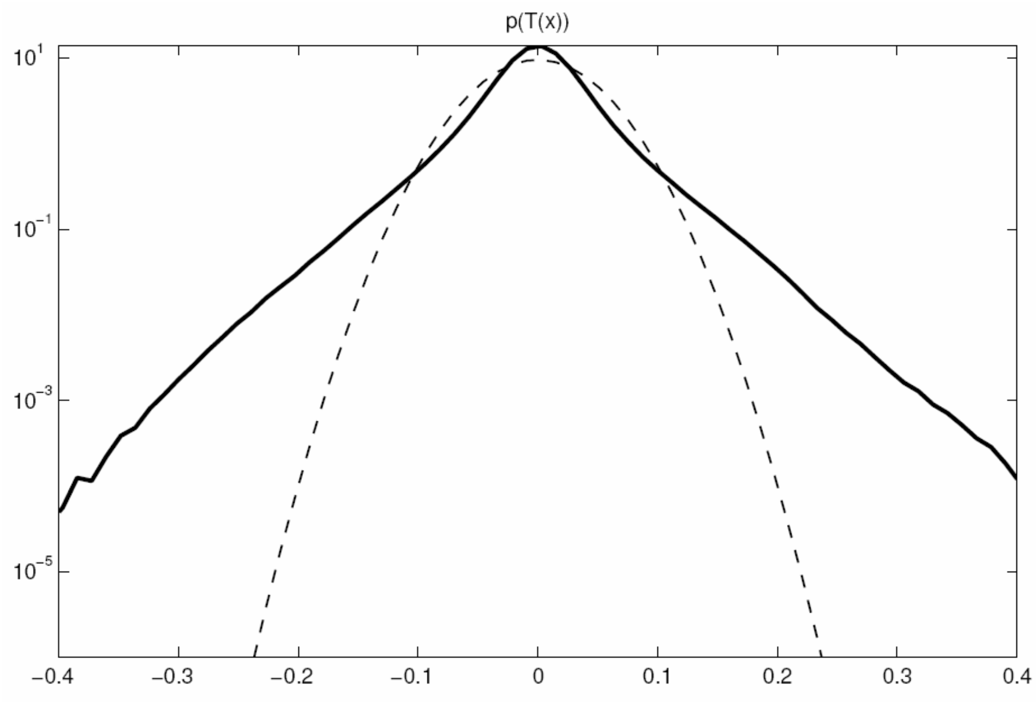
Gershgorin, Majda (2010), Bourlioux, Majda (2002)

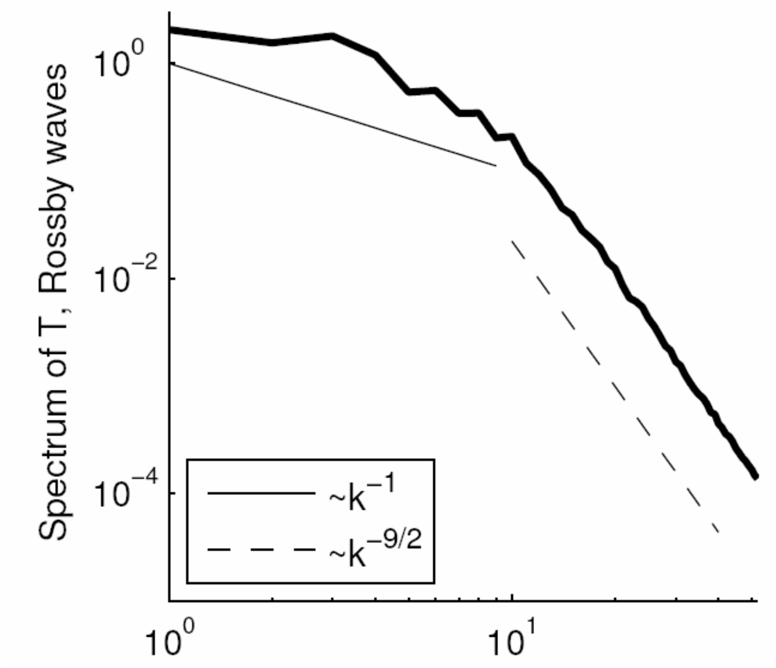
$U^M, \; v^M, \; T^M,$ solutions with Model Error

Mimic GCM: increase damping, γ_M, eddy duffusivity for T^M



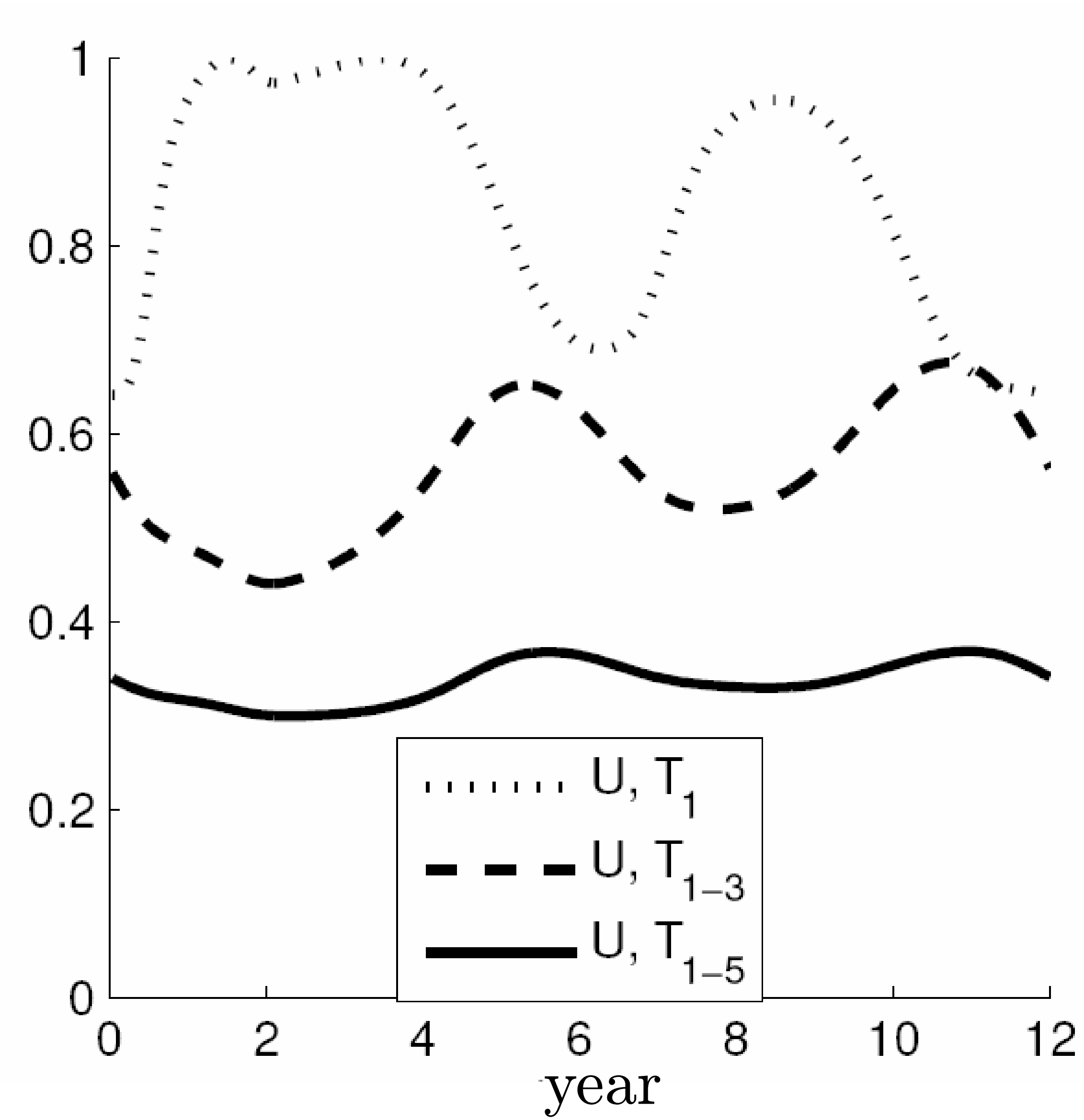






Pdf for T like atmospheric tracers in observations, Neelin et al (2010)

Fraction of the signal part in the total lack of information $\mathcal{P}$



Stochastic models for low-frequency climate dynamics

Atmosphere-ocean system:

$$\vec{u}_t = \underbrace{\vec{B}(\vec{u},\vec{u})}_{\text{quadratic}} + \overbrace{L\vec{u}}^{\text{skew-symmetric}} - \alpha(t)\vec{u} + \vec{F}(t)$$

$$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{matrix} \text{slow} \\ \text{fast} \end{matrix}$$

Energy conservation:

$$\vec{u}\cdot\vec{B}(\vec{u},\vec{u}) = 0$$

$$\mathrm{div}_{\vec{u}}\vec{B}(\vec{u},\vec{u}) = 0$$

Effective low-frequency dynamics:

$$dx = [F + ax + bx^2 - cx^3]dt + (A - Bx)dW + \sigma dW_A$$

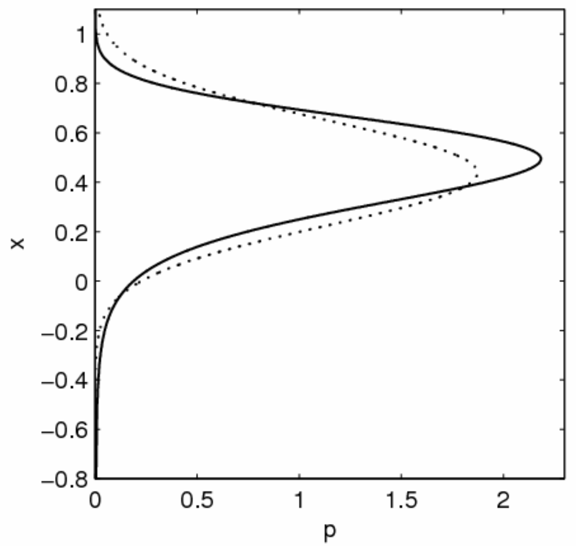
Come from the same physical phenomena: dyad interactions

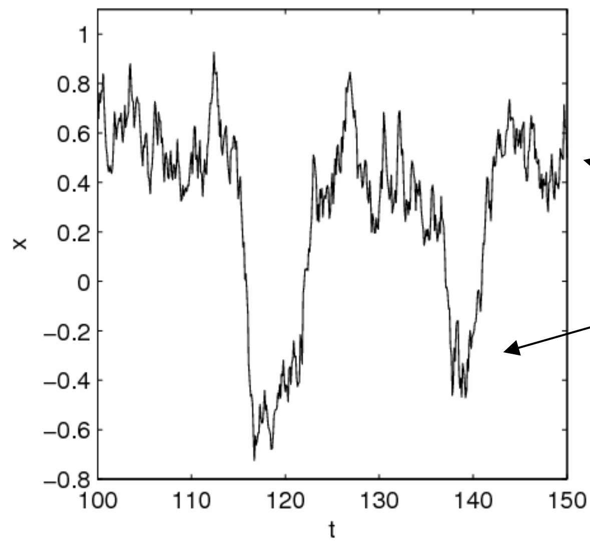
What is the statistical and dynamical description of the model?

Majda, Franzke, Cromelin PNAS (2009)

Invariant measure and ideal response

Invariant pdf: $p(x) = \frac{N_0}{\left((Bx-A)^2+\sigma^2\right)^{a_1}} e^{d\,\arctan\left(\frac{Bx-A}{\sigma}\right)} e^{\frac{-c_1 x^2 + b_1 x}{B^4}}$





Distinct regimes of behavior

Ideal mean response to the changes of forcing: $\frac{\partial\langle x\rangle}{\partial F} = \frac{\partial}{\partial F}\int x p(x)\,dx$

How do we choose interesting test cases?

Deterministic structural instability

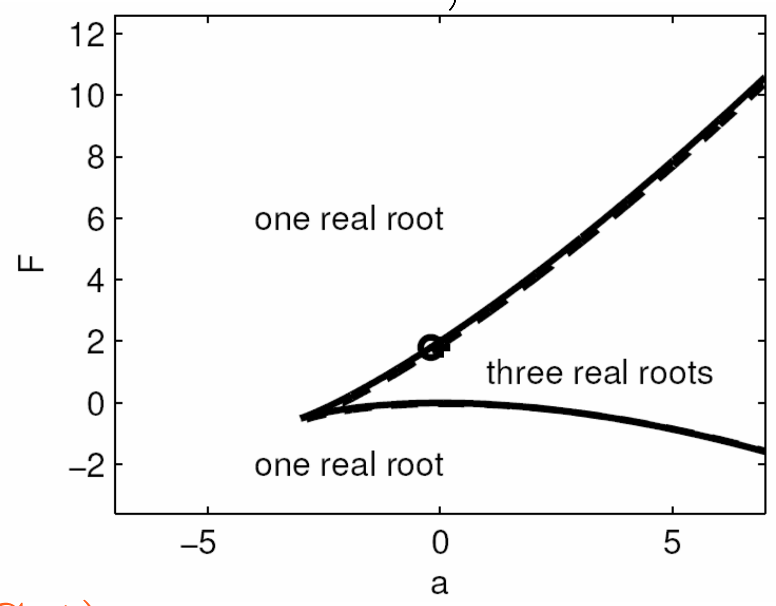
$$\dot{x} = F + ax + bx^2 - cx^3$$

Deterministic structural instability:
equilibrium points: either 1 stable
or 2 stable and one unstable
or the boundary

North-Atlantic Oscillation (NAO)

Leading Principal Component (PC-1)
has features of Arctic Oscillation

$b = -6,\ c = 4$



The effective model is clearly nonlinear and non-Gaussian
but we still can apply FDT in its original form!

The most dangerous climate change directions in a stochastic model for low frequency variability

The advantage: we know the perturbed true climate explicitly, so we also know the true climate change behavior explicitly. The two dimensional parameters for external forcing and dissipation are the natural parameters which are varied.

The mean and variance for the climate equilibrium are the measured climate functionals identifying the most dangerous perturbation direction, $\vec{e}$.
For PC-1, the exact most dangerous direction $\vec{e}^*_\pi = (0.969, 0.249)^T$, the projection on changes in external forcing is roughly 80%.
The functional with model with a Gaussian approximate climate:
$\vec{e}^*_G = (0.937, 0.349)^T$, error of 6.0° with $\vec{e}^*_\pi$.
The model error functional with the mean alone but the non-Gaussian climate:
$\vec{e}^*_1 = (0.989, 0.150)^T$, error of -5.8° with $\vec{e}^*_\pi$.
PC-1 is most sensitive to changes in external forcing.
Most dangerous perturbation direction for the NAO is $\vec{e}^*_\pi = (-0.076, 0.997)$ and is overwhelmingly dominated by changes in dissipation.
Remarkably, all three approximations reproduce $\vec{e}^*_\pi$ here exactly within three significant figures.

FDT for climate systems

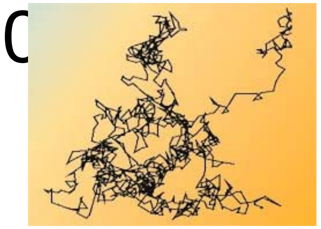
The linear response of a climate system to small perturbations of the external forcing can be predicted by observing appropriate statistics of the system in equilibrium without the need of applying any perturbations.

Valid for Statistical Physics of identical particles in equilibrium

Leap to use for forced-dissipative systems in climate(Leith,1975)

Historical note: general FDT

- A. Einstein (1905), M. Smoluchowski (190
Brownian motion



$$\underset{\text{diffusion}}{D} = \underset{\text{mobility of particles}}{\mu} k_B T$$

- Johnson–Nyquist noise (1928)

$$\underset{\text{voltage variance}}{\sigma_V^2} = 4 k_B T \Delta f \underset{\text{resistance}}{R}$$

- Callen and Green (1952) formulation and proof of FDT for the systems modeled by a Langevin equation

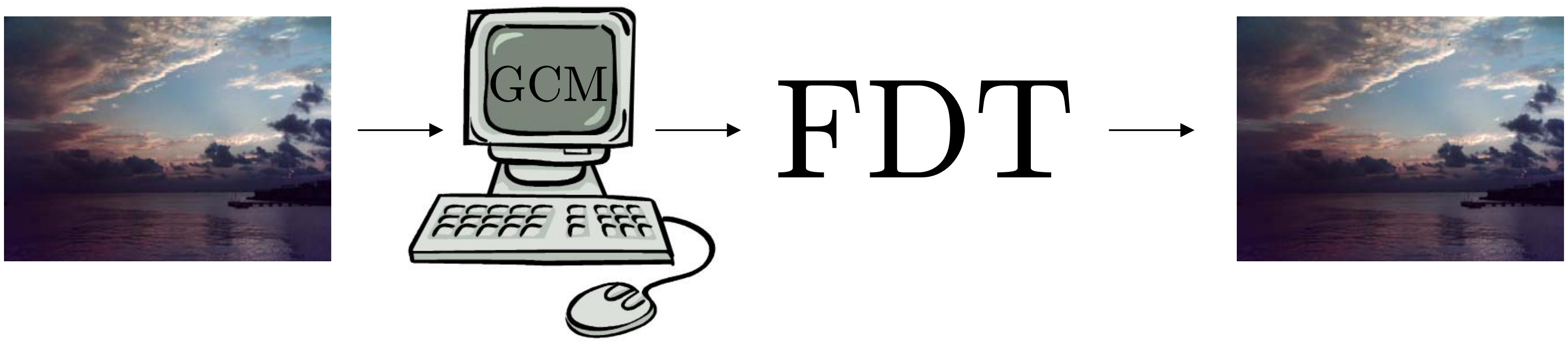
- Kubo (1966) FDT for general nonlinear dynamical systems in thermal equilibrium

Historical note: FDT for the atmosphere

Leith (1975) proposed that in order to determine climatic sensitivity (response) to external perturbations one needs to observe natural variability of the atmosphere (Gaussian assumption).

Bell, Gritsun, Branstator, Majda, Abramov…

GCMs, New Algorithms, Math Theory



Advantages of FDT

- only need data in the unperturbed state
- one response operator can be used with multiple perturbation vectors
- analyse the response operator for the most dangerous perturbations
- inverse modelling: perturbation can be recovered from the known response

General properties of FDT

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) + \sigma(\mathbf{u})\dot{\mathbf{W}}$$

nonlinear dynamical system (e.g., GCM)

$$p_t = -\mathrm{div}_{\mathbf{u}}(\mathbf{F}(\mathbf{u})p) + \frac{1}{2}\mathrm{div}_{\mathbf{u}}\nabla_{\mathbf{u}}(Qp) \equiv L_{FP}p$$

Fokker-Planck equation

$$L_{FP}p_{eq} = 0$$

invariant measure

$$A(\mathbf{u})$$

a nonlinear functional of $\mathbf{u}$
(e.g., mean variance, skewness)

$$\langle A(\mathbf{u})\rangle = \int A(\mathbf{u})p_{eq}(\mathbf{u})d\mathbf{u}$$

How does $\langle A(\mathbf{u})\rangle$ change if $\mathbf{F}(\mathbf{u})$ is perturbed?

General properties of FDT

$\mathbf{F} \to \mathbf{F} + \delta\mathbf{F}$ $\quad\quad$ $\delta\mathbf{F} = \delta\mathbf{h}(\mathbf{u})f(t)$ forcing perturbation

$$\delta\langle A(\mathbf{u})\rangle(t) = \langle A(\mathbf{u}^\delta)\rangle - \langle A(\mathbf{u})\rangle = \int_0^t R(t-s)\delta f(s)ds$$

$$R(\tau) = \langle A(\mathbf{u}(\tau))B(\mathbf{u}(0))\rangle$$

$$B(\mathbf{u}) = -\frac{\mathrm{div}_{\mathbf{u}}(\mathbf{h}p_{eq})}{p_{eq}}$$

FDT

$R(\tau)$ is computed through a correlation function in the unperturbed climate

How do we generalize FDT for a time-periodic case?

Majda and Wang 2010; Gershgorin and Majda 2009

Quasi-Gaussian approximation to FDT

p_{eq} is usually unknown exactly.

One way to apply FDT is to assume that p_{eq} is Gaussian (qG-FDT)

$$p^G_{eq} = C_N \exp\left(-\frac{1}{2}(\mathbf{u}-\bar{\mathbf{u}})^T \mathcal{C}^{-1}(\mathbf{u}-\bar{\mathbf{u}})\right)$$

$$\begin{aligned} R^G(\tau) &= \langle A(\mathbf{u}(\tau))B^G(\mathbf{u}(0))\rangle \\ B^G(\mathbf{u}) &= -\frac{\mathrm{div}_{\mathbf{u}}(\mathbf{w}p^G_{eq})}{p^G_{eq}} \end{aligned}$$

for perturbations of external forcing $\mathbf{w}(\mathbf{u})_\mathbf{i} = \mathbf{e}_\mathbf{i}$

$$R^G(t) = \langle A(\mathbf{u})(t)\mathcal{C}^{-1}(\mathbf{u}-\bar{\mathbf{u}})(0)\rangle \quad \text{computable}$$

How do we use test models for FDT?

Test models for FDT

Test model

$$\frac{du}{dt} = F(u,t)$$

Ideal response R^{ideal}

$$\delta\langle A(u)\rangle = R^{ideal}\delta F$$

- analytically
- numerically

Approximation in

- model
- FDT algorithm
- both

$$R^{ideal} \overset{?}{\approx} R^{FDT}$$

Standard approach to FDT in climate community: Linear Regression Models (LRM)

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) + \sigma(\mathbf{u})\dot{\mathbf{W}} \longrightarrow \frac{d\mathbf{u}^R}{dt} = L\mathbf{u}^R + F + \sigma\dot{\mathbf{W}}$$

linear

L, σ, F can be found by matching
(i) mean, (ii) variance, (iii) functional of a lag covariance

Variance response for the LRM is zero

$$A(\mathbf{u}) = (\mathbf{u} - \bar{\mathbf{u}}, Q(\mathbf{u} - \bar{\mathbf{u}}))$$

Odd centered moment vanishes for Gaussians

$$R^R(t) = \langle(\mathbf{u}^R(t) - \bar{\mathbf{u}}^R) \cdot Q(\mathbf{u}^R(t) - \bar{\mathbf{u}}^R)(\mathcal{C}^R)^{-1}(\mathbf{u}^R(0) - \bar{\mathbf{u}}^R)\rangle = 0$$

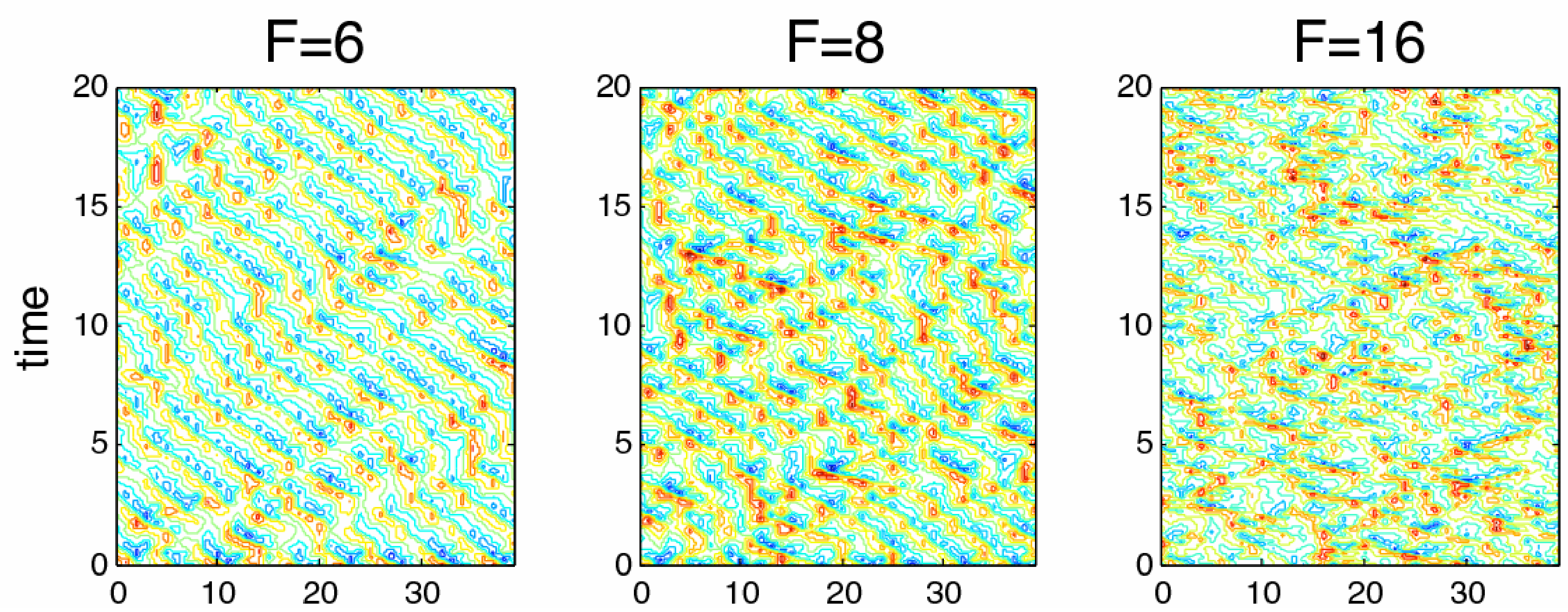
Variance response to the changes in forcing cannot be predicted by FDT through LRM! (No skill for variance)

However, mean response for LRM can be skillful

Lorenz 40 mode model as a test model for FDT

$$\frac{du_j}{dt} = \underbrace{(u_{j+1} - u_{j-2})u_{j-1}}_{\text{advection}} - \underbrace{u_j}_{\text{dissipation}} + \underbrace{F}_{\text{forcing}}$$

Model for baroclinic turbulence in the mid'latitude atmosphere

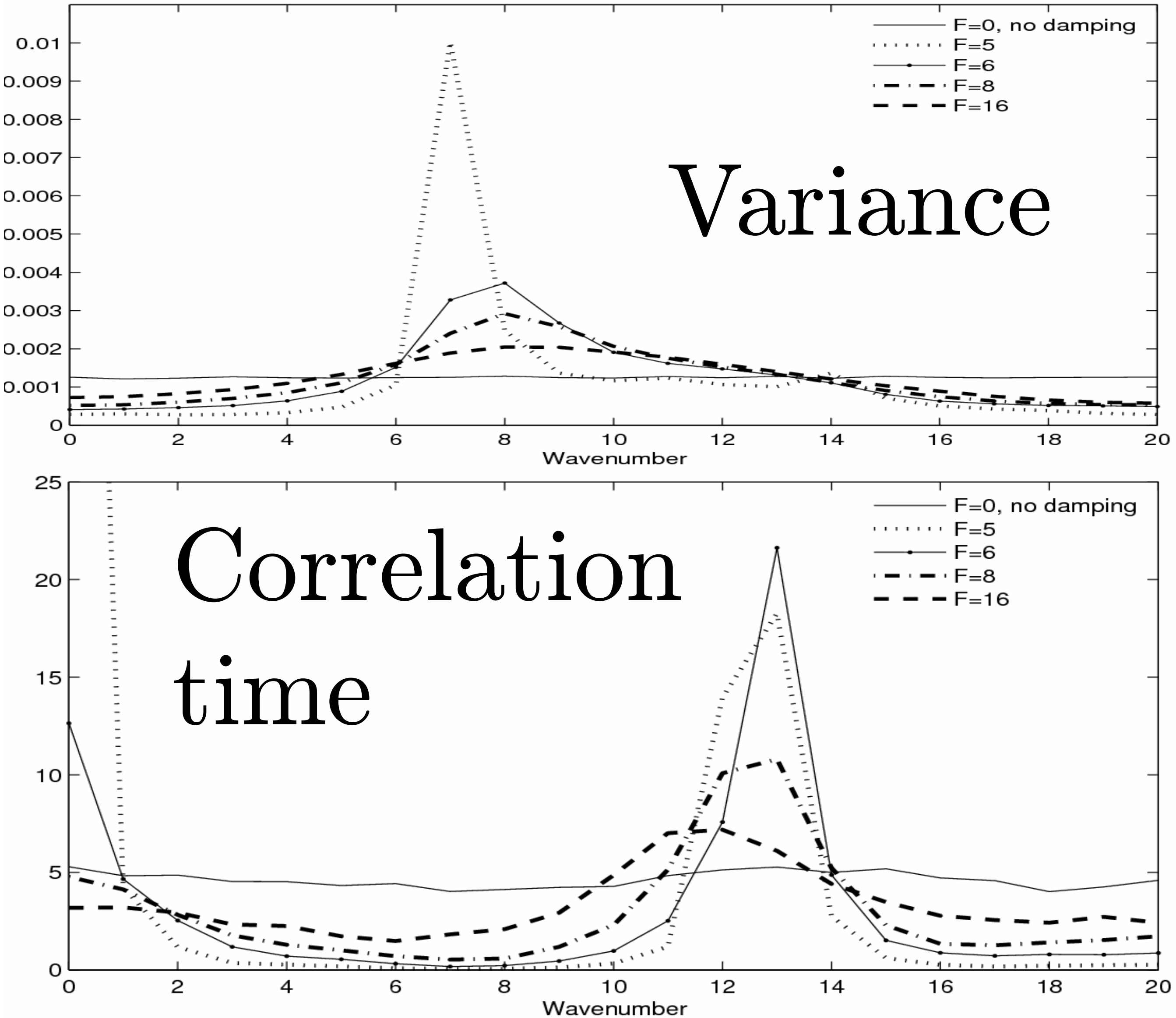


weakly chaotic $\lambda_1 = 1.02$

strongly chaotic $\lambda_1 = 1.74$

turbulent $\lambda_1 = 3.95$

Statistics of Lorenz 40 mode model



Modes with the largest variance do not always contribute the largest climate response

However, it is not always the case in GCMs

Approximation of the nonlinear term

$$u_j \xrightarrow{\text{rescale}} \tilde{u}_j \xrightarrow{\text{Fourier}} \hat{u}_k \xrightarrow[\text{regression}]{\text{linear}} \hat{v}_k$$

$$E_p = 1/2 \sum_{j=0}^{J-1} \overline{(u_j - \bar{u})^2}$$

$$(\tilde{u}_{j+1} - \tilde{u}_{j-2})\tilde{u}_{j-1}$$

$$(-d_k + i\omega_k)\hat{v}_k + \sigma_k \dot{W}_k$$

perfect regression

$$-d_k \hat{v}_k + \sigma_k \dot{W}_k$$

standard regression

$$\frac{d\hat{v}_k}{dt} = \big(\underline{-\,d_k} + \omega_1(k) + i(\omega_2(k) + \underline{\omega_k})\big)\hat{v}_k + \frac{1}{E_p}(F - \bar{u})\delta_k + \underline{\sigma_k \dot{W}_k}$$

$$\frac{d\hat{v}_k}{dt} = \big(\underline{-\,d_k} + \omega_1(k) + i\omega_2(k)\big)\hat{v}_k + \frac{1}{E_p}(F - \bar{u})\delta_k + \underline{\sigma_k \dot{W}_k}$$

Find d_k, ω_k, σ_k by matching Var_k and $\int_0^\infty Corr_k(\tau)d\tau$ in the original and linear models.

Regression coefficients

$$Var(\hat{u}_k) = Var_k, \quad \int_0^\infty Corr_{\hat{u}_k}(\tau)d\tau = T_k - i\theta_k$$

perfect regression

$$\boxed{\begin{aligned} d_k &= \omega_1(k) + \frac{T_k}{T_k^2 + \theta_k^2}, \\ \omega_k &= -\omega_2(k) + \frac{\theta_k}{T_k^2 + \theta_k^2}, \\ \sigma_k^2 &= 2Var_k \frac{T_k}{T_k^2 + \theta_k^2}. \end{aligned}}$$

always realizable

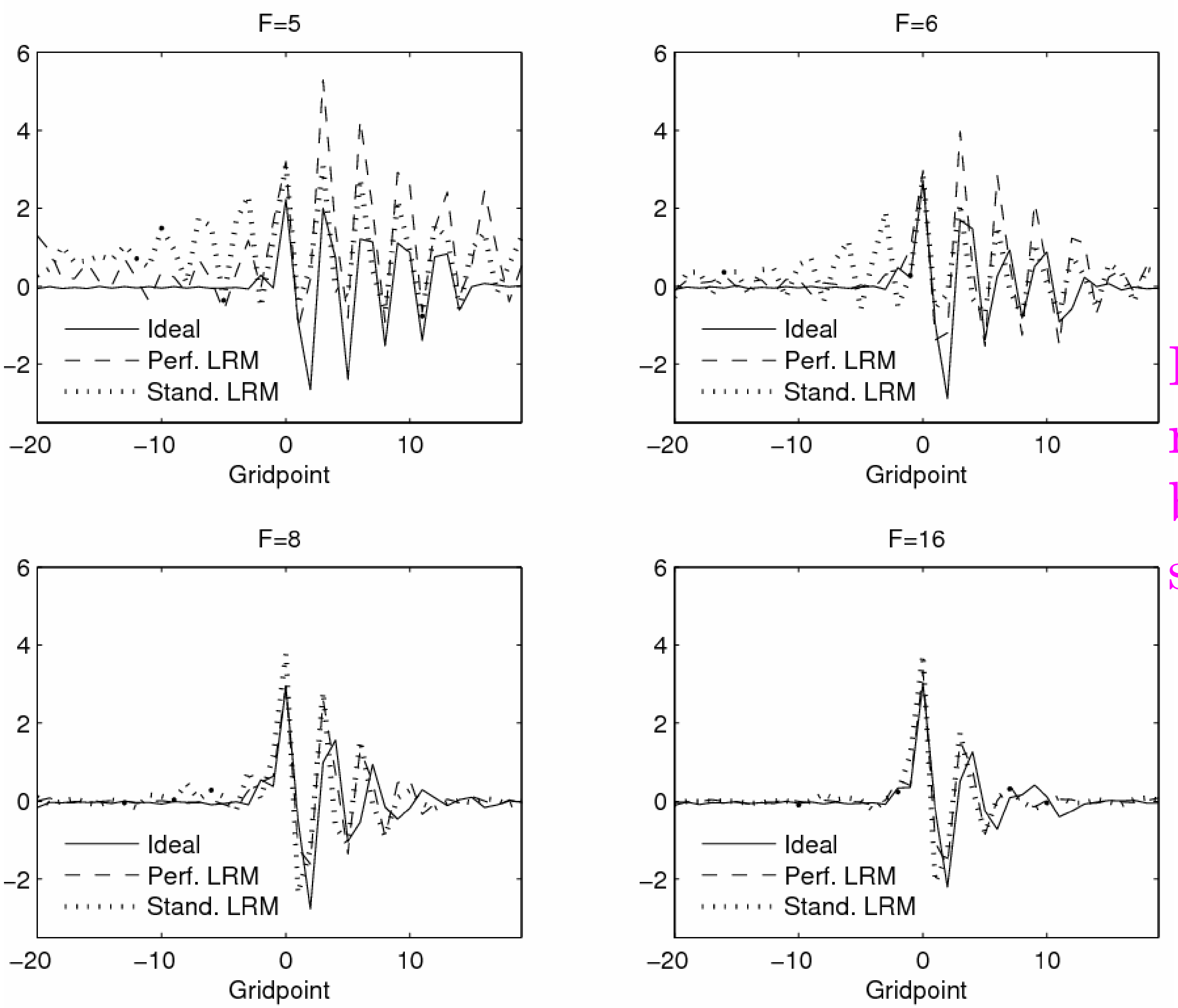
standard regression

$$\boxed{\begin{aligned} d_k &= \omega_1(k) + \frac{1 \pm \sqrt{1 - 4T_k^2\omega_2(k)^2}}{2T_k} \\ \sigma_k^2 &= 2(d_k - \omega_1(k))Var_k. \end{aligned}}$$

not always realizable

How well does FDT for LRM predict
the actual response of the Lorenz model?

Results: response operator



Perfect regression recovers the pattern better than standard regression

Results: perfect vs standard regression

F	T	Perfect LRM	Standard LRM	qG-FDT	Blended Response
5	5	0.8268	0.5823	0.9148	0.9998
	20	0.7154	0.5073	0.7392	0.9996
	∞	0.6641	0.5043	0.7322	0.9346
6	5	0.8733	0.6625	0.9718	0.9999
	20	0.6938	0.5091	0.7866	0.9994
	∞	0.7150	0.5064	0.7765	0.9827
8	5	0.9429	0.9668	0.9707	0.9999
	20	0.7757	0.7304	0.8076	0.9976
	∞	0.7637	0.6140	0.7618	0.9741
16	5	0.9640	0.9752	0.9924	0.9999
	20	0.9136	0.8638	0.9279	0.9982
	∞	0.8862	0.7875	0.9082	0.9892

New Blended Response FDT Algorithms; Abramov and Majda 2007 Nonlinearity 2009 J. Atmos. Sci.

FDT for LRM with perfect regression works reasonably well for the mean response to the changes in forcing but completely misses the variance response

Major shortcomings of Standard LRM in computing mean response especially for $F = 5, 6$

Stochastic models for low-frequency climate dynamics

Atmosphere-ocean system:

$$\vec{u}_t = \underbrace{\vec{B}(\vec{u},\vec{u})}_{\text{quadratic}} + \overset{\text{skew-symmetric}}{L}\vec{u} - \alpha(t)\vec{u} + \vec{F}(t)$$

$$\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{matrix} \text{slow} \\ \text{fast} \end{matrix}$$

Energy conservation:

$$\vec{u}\cdot\vec{B}(\vec{u},\vec{u}) = 0$$

$$\mathrm{div}_{\vec{u}}\vec{B}(\vec{u},\vec{u}) = 0$$

Effective low-frequency dynamics:

$$dx = [F + ax + bx^2 - cx^3]dt + (A - Bx)dW + \sigma dW_A$$

Come from the same physical phenomena: dyad interactions

What is the statistical and dynamical description of the model?

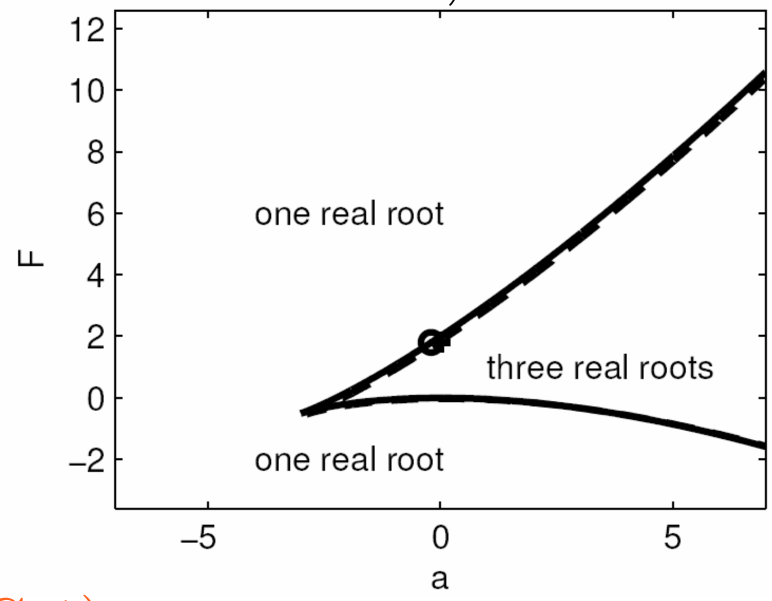
Majda, Franzke, Cromelin PNAS (2009)

Deterministic structural instability

$$\dot{x} = F + ax + bx^2 - cx^3$$

Deterministic structural instability: equilibrium points: either 1 stable or 2 stable and one unstable or the boundary

$b = -6,\ c = 4$



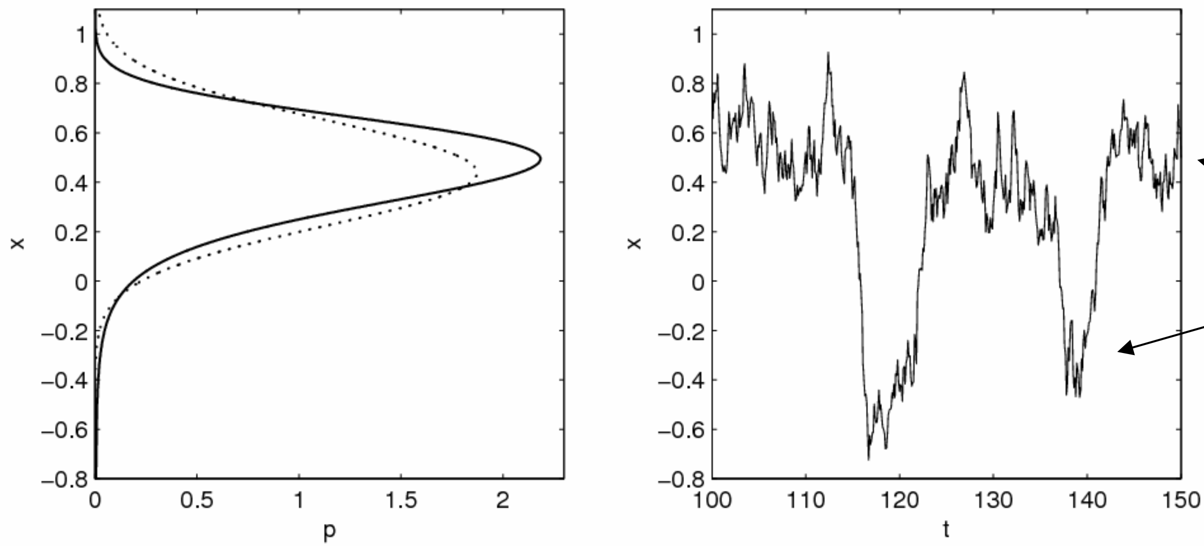
North-Atlantic Oscillation (NAO)

Leading Principal Component (PC-1) has features of Arctic Oscillation

The effective model is clearly nonlinear and non-Gaussian but we still can apply FDT in its original form!

Invariant measure and ideal response

Invariant pdf: $p(x) = \frac{N_0}{\left((Bx-A)^2+\sigma^2\right)^{a_1}} e^{d\arctan\left(\frac{Bx-A}{\sigma}\right)} e^{\frac{-c_1x^2+b_1x}{B^4}}$



Distinct regimes of behavior

Ideal mean response to the changes of forcing: $\frac{\partial\langle x\rangle}{\partial F} = \frac{\partial}{\partial F}\int xp(x)dx$

How do we choose interesting test cases?

FDT set up

$$dx = [F + ax + bx^2 - cx^3]dt + (A - Bx)dW + \sigma dW_A$$

Mean and variance response

$A_M(x) = x$ $\quad$ $A_{Var}(x) = (x - \langle x \rangle)^2$

to the perturbations of forcing or dissipation

$h_F(x)p = p$ $\quad$ $h_a(x)p = -xp$

$$\begin{aligned} R(\tau) &= \langle A(x(\tau))B(x(0))\rangle \\ B(x) &= -\frac{\partial/\partial x(hp)}{p} \end{aligned}$$

$$B^{(F)}(x) = -2\frac{(AB+F)+(a-B^2)x+bx^2-cx^3}{\sigma^2+(A-Bx)^2}$$

$$B^{(F)}_{qG}(x) = \frac{x-\mu}{Var}$$

What are the results?

Results

$\mathscr{R}$	Case	Ideal	FDT	qG-FDT	FDT % error	qG-FDT % error	
	S	0.1814	0.1827	0.3197	0.70	76	
	B	0.1621	0.1633	0.3425	0.72	111	
$\mathscr{R}_M^{(F)}$	U	0.1555	0.1591	0.8655	2.29	457	
	PC-1	40.54	40.52	41.84	0.05	3.21	+
	NAO	5.41	5.46	5.51	0.93	1.84	+
	S	0.0533	0.0536	−0.0937	0.57	276	
	B	0.0486	0.0499	−0.1639	2.55	437	
$\mathscr{R}_M^{(a)}$	U	0.0401	0.0334	−0.8988	16.71	2339	
	PC-1	6.13	6.66	16.91	8.69	176	−
	NAO	−71.00	−71.16	−71.37	0.22	0.51	+
	S	−0.0522	−0.0540	−0.2331	3.4	346	
	B	−0.0490	−0.0463	−0.3218	5.5	557	
$\mathscr{R}_{\mathrm{Var}}^{(F)}$	U	−0.0630	−0.0702	−1.2797	11.5	1932	
	PC-1	11.23	11.24	16.02	0.08	42	−
	NAO	0.470	0.473	0.715	0.72	52	−
	S	0.0083	0.0071	0.157	15	1798	
	B	0.0104	0.0107	0.291	3.1	2714	
$\mathscr{R}_{\mathrm{Var}}^{(a)}$	U	0.0207	0.0304	1.56	47	7431	
	PC-1	36.25	36.16	34.30	0.23	5.4	+
	NAO	−5.185	−5.186	−8.1	0.03	56	−

+ High Skill qG-FDT

− Low Skill qG-FDT

FDT response has high skill always!

Case	Skew	Flat
S	−1.35	8.56
B	−1.50	10.8
U	−2.05	17.5
PC-1	0.27	2.72
NAO	0.21	3.11