Improved linear response for stochastically driven systems

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Preliminaries

- Stochastic (Itō) differential equation

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x})\mathrm{d}t + \boldsymbol{\sigma}(\mathbf{x})\mathrm{d}\mathbf{W}_t$$

- Observable function $A(\mathbf{x})$
- Given initial condition $\mathbf{x}$, the expectation of A at time t is $\mathbb{E}_{\mathbf{x}}^{t}[A]$, with $\mathbb{E}_{\mathbf{x}}^{0}[A] = A(\mathbf{x})$

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- Given initial condition $\mathbf{x}$, the expectation of A at time t is $\mathbb{E}^t_{\mathbf{x}}[A]$, with $\mathbb{E}^0_{\mathbf{x}}[A] = A(\mathbf{x})$
- Assume that an ensemble of initial conditions is distributed according to a probability measure ρ, then, the average of A at time t is given by

$$\langle A \rangle_t = \int \mathbb{E}^t_{\mathbf{x}}[A]\mathrm{d}\rho(\mathbf{x})$$

Response to external perturbation

- Add small perturbation

$$\mathrm{d}\mathbf{x} = [\mathbf{f}(\mathbf{x}) + \mathbf{b}(\mathbf{x})\eta(t)]\mathrm{d}t + \boldsymbol{\sigma}(\mathbf{x})\mathrm{d}\mathbf{W}_t$$

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- Then, for the same initial condition $\mathbf{x}$, the expectation of A at time t will be different: $\hat{\mathbb{E}}_{\mathbf{x}}^{t}[A]$, with the average with respect to ρ given by

$$\langle \hat{A} \rangle_t = \int \hat{\mathbb{E}}_{\mathbf{x}}^{t}[A]\mathrm{d}\rho(\mathbf{x})$$

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- We denote the *response* of A as the difference between the perturbed and unperturbed averages:

$$\delta_t \langle A \rangle = \langle \hat{A} \rangle_t - \langle A \rangle_t = \int \left(\hat{\mathbb{E}}_{\mathbf{x}}^t[A] - \mathbb{E}_{\mathbf{x}}^t[A] \right) \mathrm{d}\rho(\mathbf{x})$$

Classical linear response

- If ρ is the invariant measure, and the forcing $\boldsymbol{\eta}(t)$ is small, then the response can be linearized with respect to $\boldsymbol{\eta}(t)$ as

$$\delta_t \langle A \rangle = \int_0^t \mathbf{R}(t-\tau)\boldsymbol{\eta}(\tau)\mathrm{d}\tau,$$
$$\mathbf{R}(t) = \int A(\mathbf{x}(t))\nabla \cdot (\mathbf{b}(\mathbf{x})p(\mathbf{x}))\,\mathrm{d}\mathbf{x},$$

where $p(\mathbf{x})$ is the probability density of ρ

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- One can also replace measure average by time average:

$$\mathbf{R}(t) = -\lim_{r\to\infty}\frac{1}{r}\int_0^r A(\mathbf{x}(t+s))[\nabla \cdot \mathbf{b} + \mathbf{b}\nabla \log p](\mathbf{x}(s))\mathrm{d}s$$

Quasi-Gaussian linear response

- Explicit approximation is needed for p
- When the Gaussian density with appropriate mean state and covariance matrix is used, it is called the quasi-Gaussian response
- Not too precise when p is not Gaussian

Stochastic short-time linear response

- A different formula is available:

$$\mathbf{R}_{SST}(t) = \int \mathbb{E}\left[\nabla A(\mathbf{x}(t))\mathbf{T}^t_{\mathbf{x}}\mathbf{b}(\mathbf{x})\right] \mathrm{d}\rho(\mathbf{x}),$$

  where $\mathbf{T}$ is the tangent map along a given realization of the Wiener path

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- Does not require a density approximation, however, can be numerically unstable for longer response times

Stochastic Lorenz 96 model

Here we “stochasticize” the well-known Lorenz 96 model by adding a stochastically driven term to the right-hand side:

$$\mathrm{d}x_k = [x_{k-1}(x_{k+1} - x_{k-2}) - x_k + F]\mathrm{d}t + \boldsymbol{\sigma}_k(\mathbf{x}) \cdot \mathrm{d}\mathbf{W}_t$$

- Constant forcing $F = 6$

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- Constant forcing $F = 6$
- Diagonal noise matrix: $\sigma_{kl}(\mathbf{x}) = \sigma_k(\mathbf{x})\delta_{kl}$
    - Additive noise: $\sigma_k(\mathbf{x})$ is set to 0 or 1
    - Multiplicative noise: $\sigma_k(\mathbf{x}) = 0.2x_k$ or $0.5x_k$

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- Forcing and response: $b(\mathbf{x}) = I$, $\boldsymbol{\eta} = const$, $A(\mathbf{x}) = \mathbf{x}$, $\delta_t\langle\mathbf{x}\rangle = \boldsymbol{\mathcal{R}}(t)\boldsymbol{\eta}$

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- Forcing and response: $b(\mathbf{x}) = I$, $\boldsymbol{\eta} = const$, $A(\mathbf{x}) = \mathbf{x}$, $\delta_t\langle\mathbf{x}\rangle = \boldsymbol{\mathcal{R}}(t)\boldsymbol{\eta}$
- SST-FDT, classical (with Gaussian p) FDT (qG-FDT), and blended SST/qG (at 3 Lyapunov characteristic times) are compared with the “ideal” directly measured response

Diagnostics

- Relative errors between the predicted and "ideal" response,

$$Error = \frac{\|\mathbf{R} - \mathbf{R}_{ideal}\|}{\|\mathbf{R}_{ideal}\|}$$

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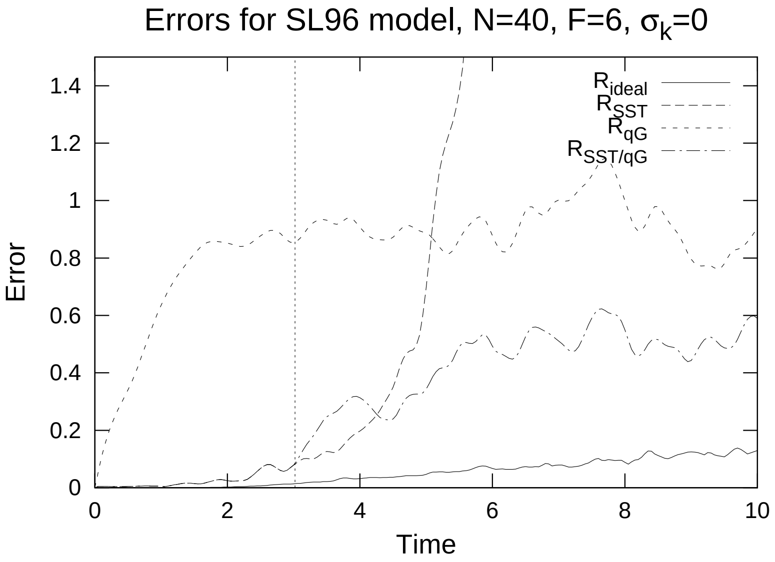
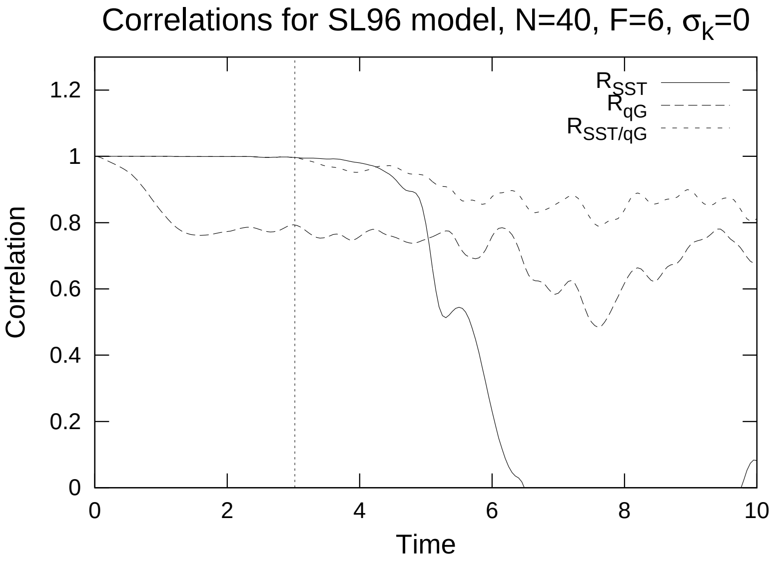
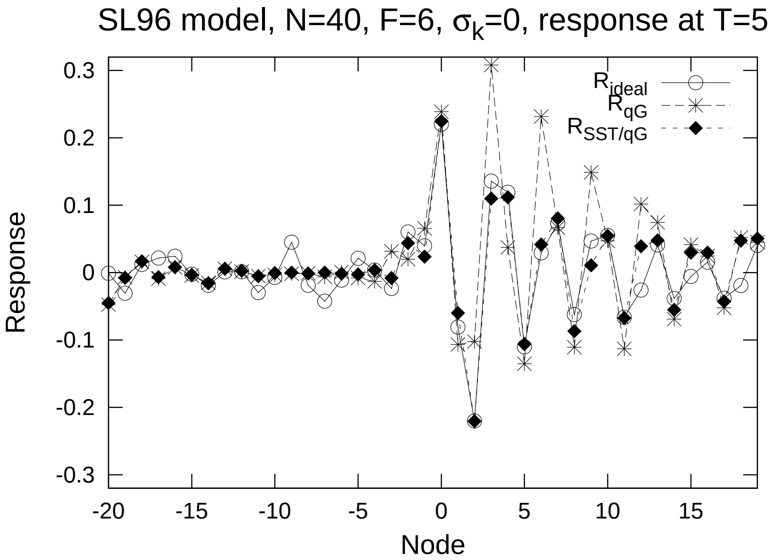
- Direct comparison between the response operators

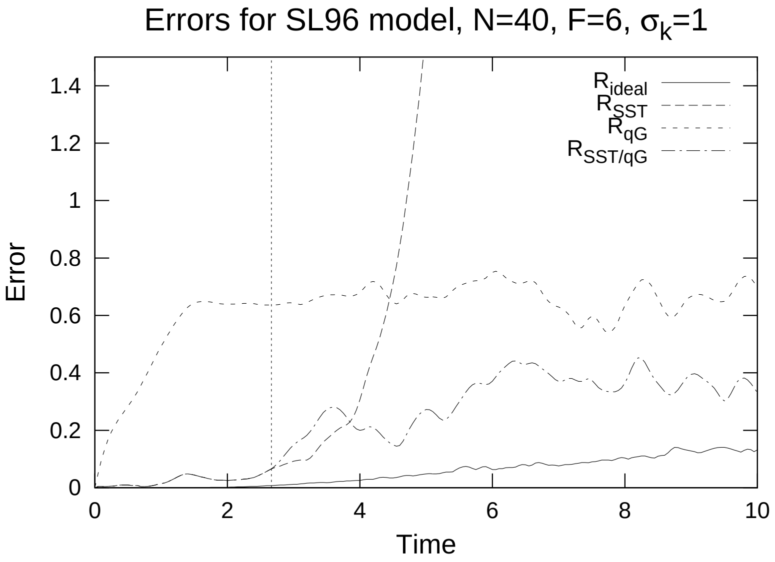


Errors for SL96 model, N=40, F=6, σ_k=0
Error
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10
R_{ideal}
R_{SST}
R_{qG}
$R_{SST/qG}$

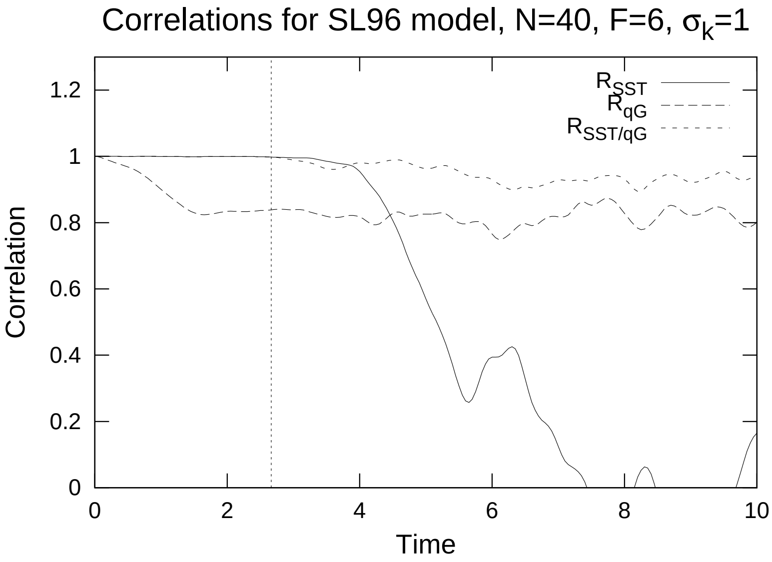

Correlations for SL96 model, N=40, F=6, σk=0
Correlation
Time
RSST
RqG
RSST/qG
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10



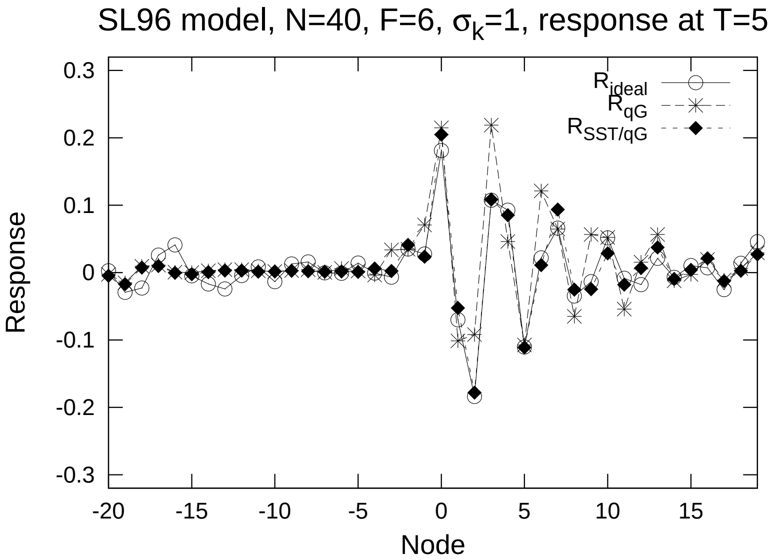
SL96 model, N=40, F=6, σ_k=0, response at T=5
Response
Node
R_{ideal}
R_{qG}
$R_{SST/qG}$
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-20
-15
-10
-5
0
5
10
15



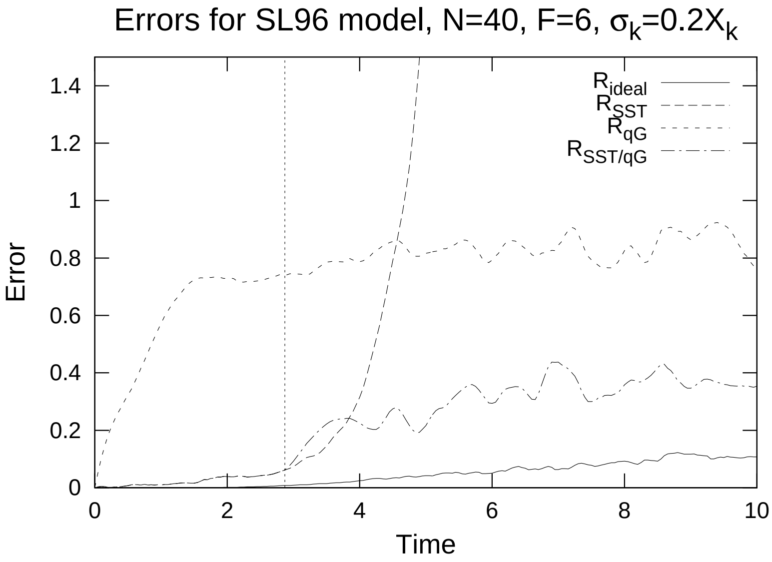
Errors for SL96 model, N=40, F=6, σ_k=1
Error
Time
R_{ideal}
R_{SST}
R_{qG}
$R_{SST/qG}$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10



Correlations for SL96 model, N=40, F=6, σk=1
RSST
RqG
RSST/qG
Correlation
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
Time

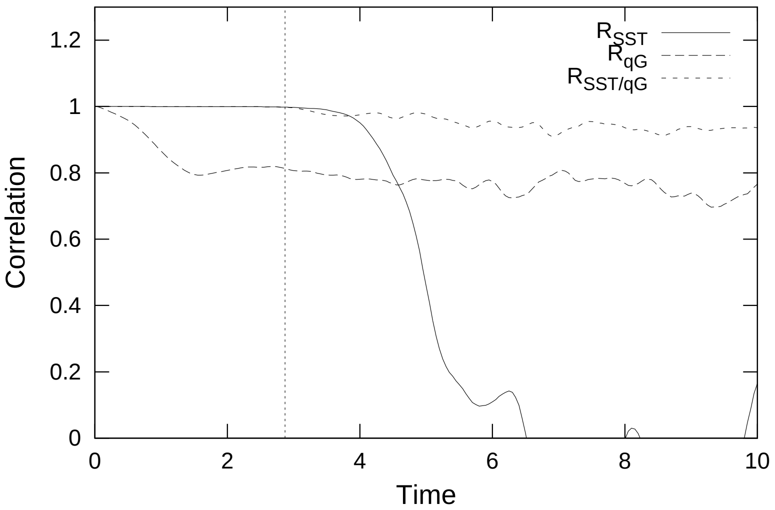


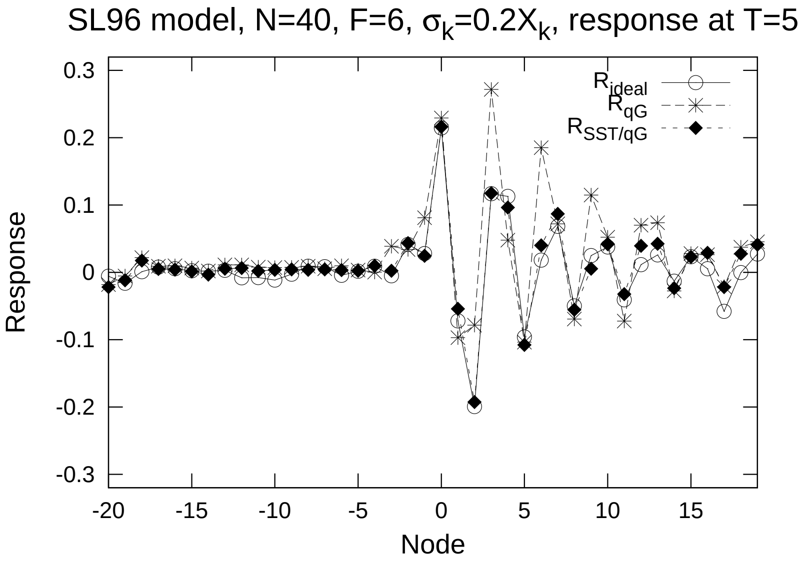
SL96 model, N=40, F=6, σk=1, response at T=5
Response
Node
R ideal
R qG
R SST/qG
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-20
-15
-10
-5
0
5
10
15



Errors for SL96 model, N=40, F=6, σ_k=0.2X_k
R_{ideal}
R_{SST}
R_{qG}
$R_{SST/qG}$
Error
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10

Correlations for SL96 model, N=40, F=6, σ_k=0.2X_k





SL96 model, N=40, F=6, σ_k=0.2X_k, response at T=5
Response
Node
R_{ideal}
R_{qG}
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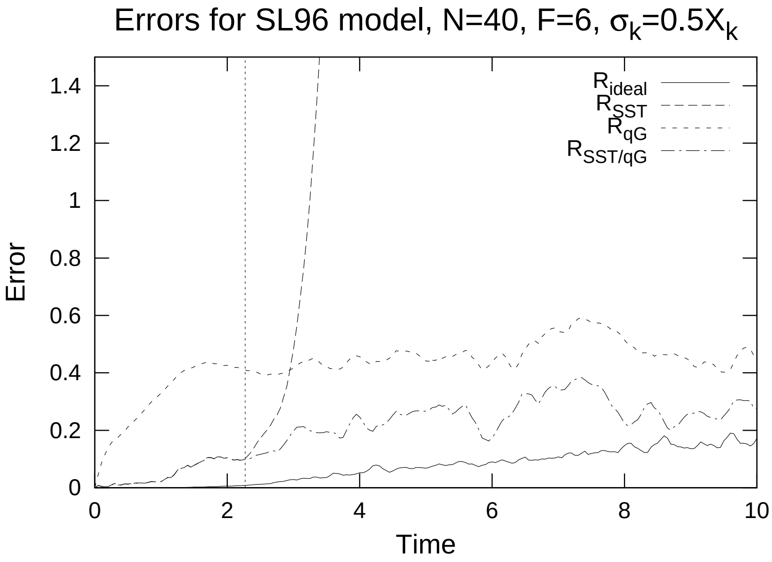
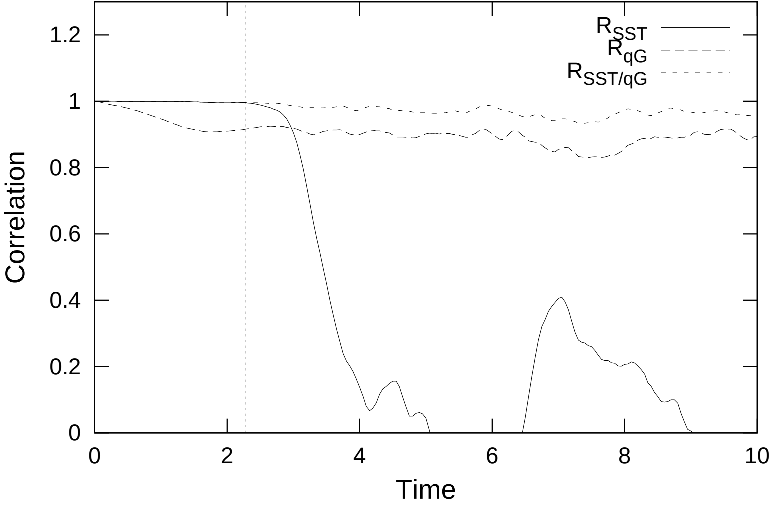
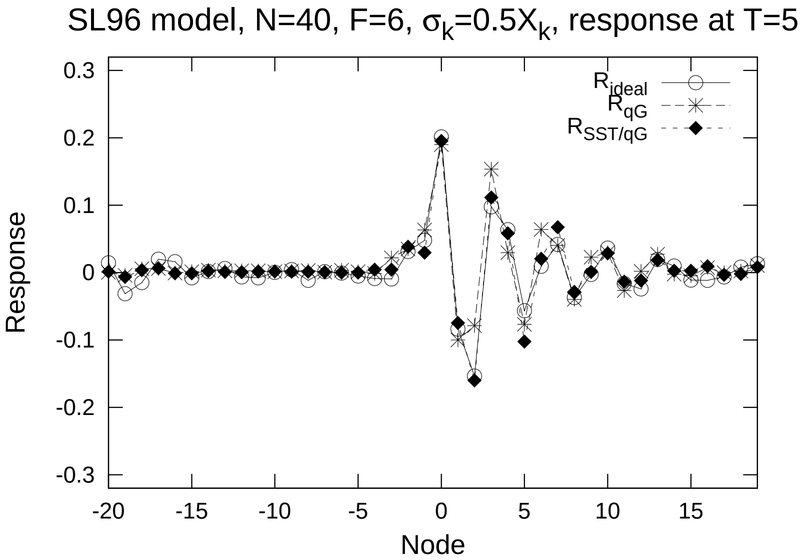


Errors for SL96 model, N=40, F=6, σk=0.5Xk
Error
Time
R_ideal
R_SST
R_qG
R_SST/qG
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10

Correlations for SL96 model, N=40, F=6, $\sigma_k=0.5X_k$




SL96 model, N=40, F=6, σk=0.5Xk, response at T=5
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R_qG
R_SST/qG
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- Reference: R. Abramov, *Improved linear response for stochastically driven systems*, submitted to Journal of Nonlinear Science