Scale-Free Distributions In Nature:

An Overview of Self-Organized Criticality

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OUTLINE

Power-law distributions in nature

The challenge for statistical physics

Self-Organized Criticality: scaling without tuning

Sandpile models: activity threshold + slow drive

Connection with absorbing-state phase transitions

Alternatives to SOC

Power-law distributions in nature

Distributions of event sizes for many natural phenomena appear to follow power laws, in some cases over many orders of magnitude

Examples: earthquakes $N \sim m^{-b}$

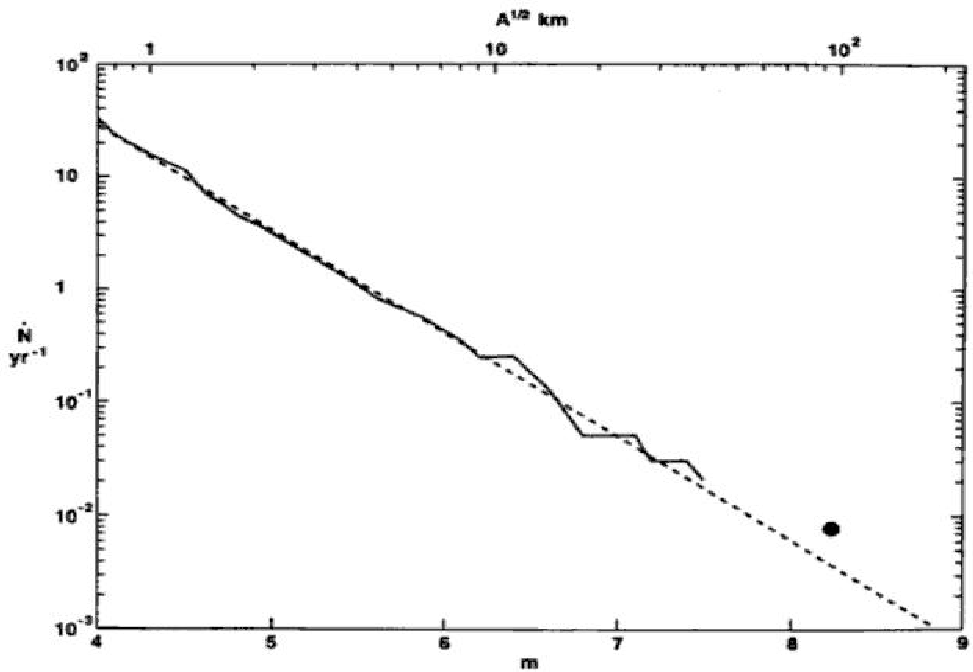


Figure 4.2. Number of earthquakes per year $\dot{N}$ occurring in southern California with magnitudes greater than m as a function of m. The solid line is the data from the southern California earthquake network for the period 1932–1994. The straight dashed line is the correlation with (4.1) taking $b = 0.923$ ($D = 1.846$) and $\dot{a} = 1.4 \times 10^5$. The solid circle is the observed rate of occurrence of great earthquakes in southern California (Sieh *et al.*, 1989).

D Turcotte, ***Fractals and Chaos in Geology and Geophysics***, Cambridge, 1997

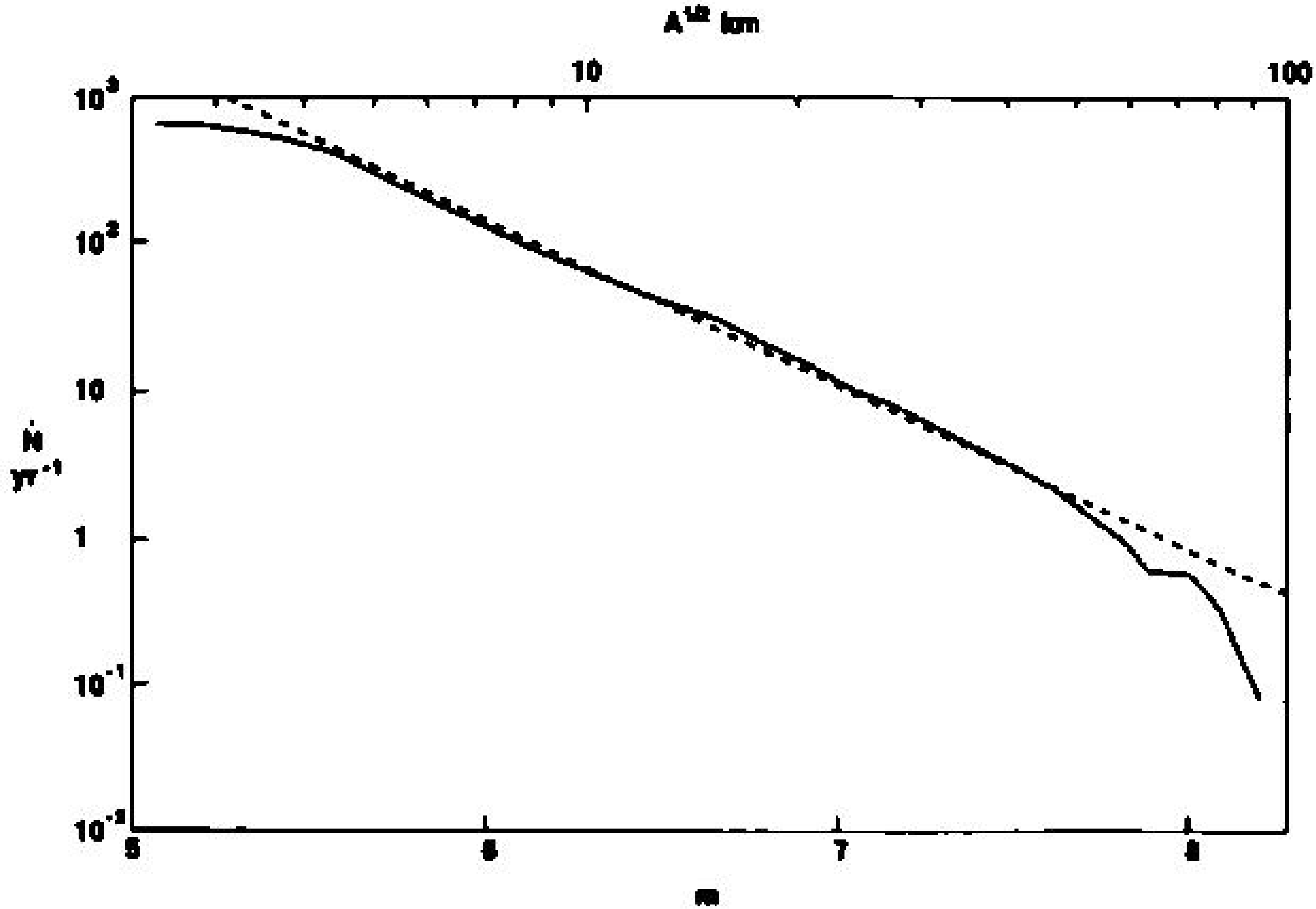


Figure 4.1. Worldwide number of earthquakes per year, $\dot{N}$, with magnitudes greater than m as a function of m. The square root of the rupture area A is also given. The solid line is the cumulative distribution of moment magnitudes from the Harvard Centroid Moment Tensor Catalog for the period January 1977 to June 1989 (Frohlich and Davis, 1993). The dashed line represents (4.1) with $b = 1.11$ ($D = 2.22$) and $\dot{a} = 6 \times 10^8$ yr^{-1}.

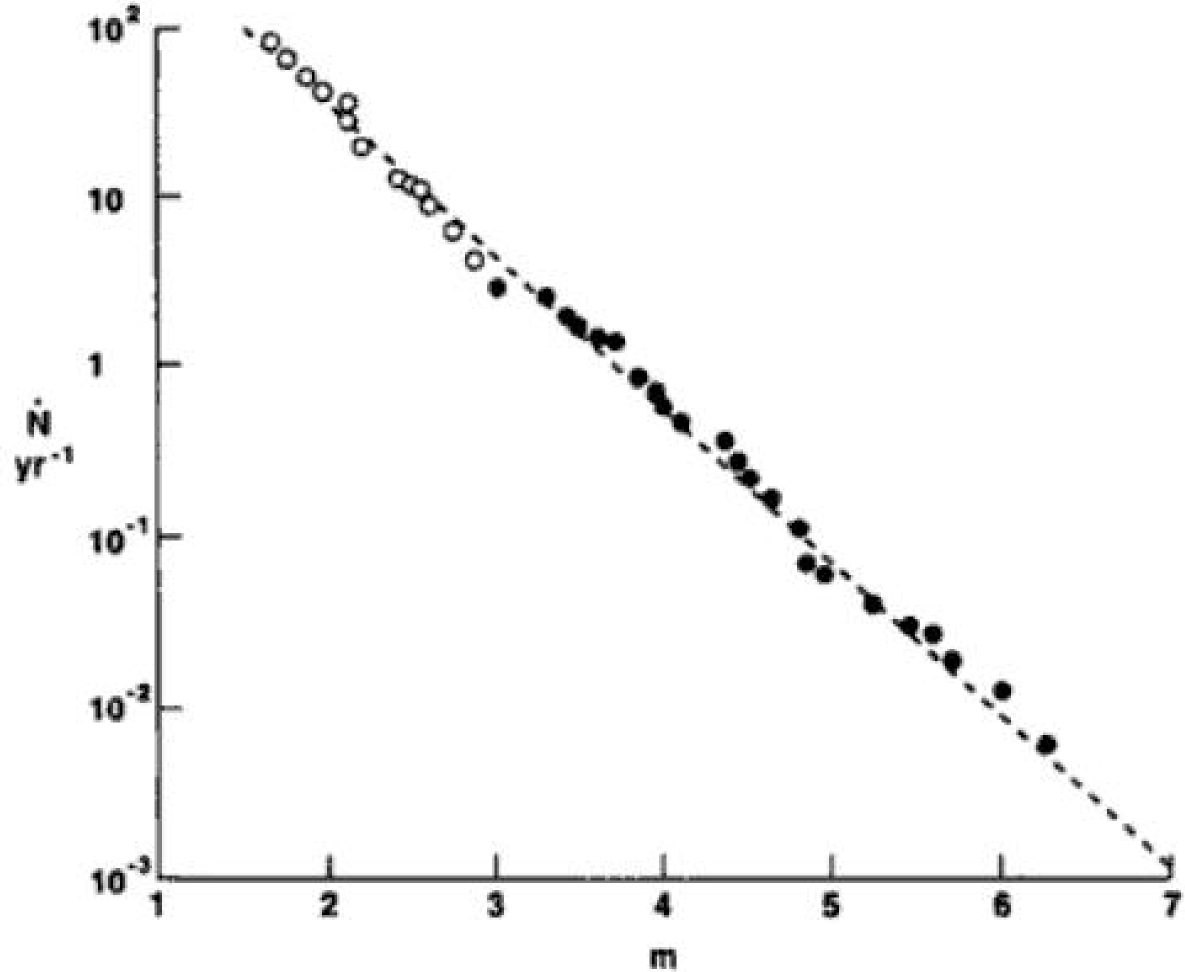


Figure 4.4. The cumulative number of earthquakes per year $\dot{N}$ occurring in the Memphis–St. Louis (New Madrid, Missouri) seismic zone with magnitudes greater than m as a function of m (Johnston and Nava, 1985). The data are for the period 1816–1983. The open circles represent instrumental data and the solid circles historical data. The dashed line represents (4.1) with $b = 0.90$ ($D = 1.80$) and $\dot{a} = 2.24 \times 10^3$ yr^{-1}.

Volcanic eruptions

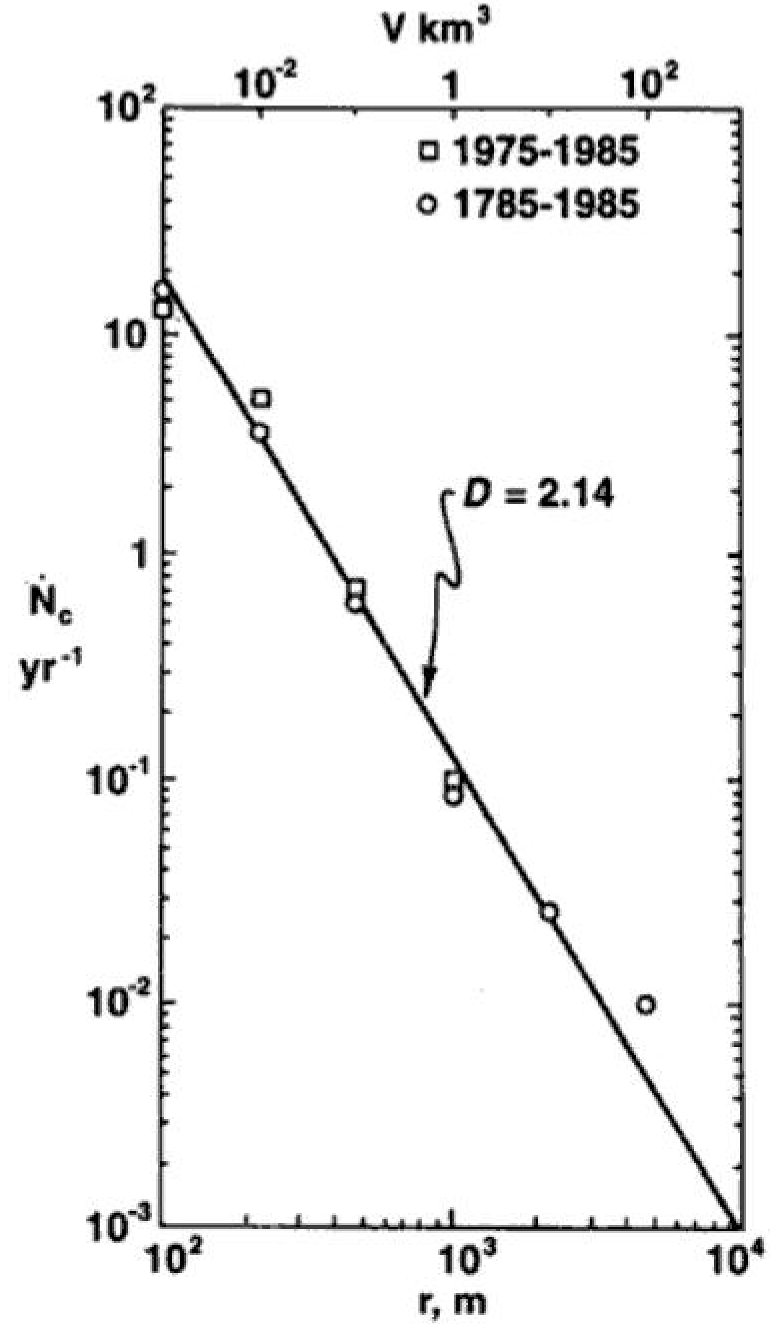
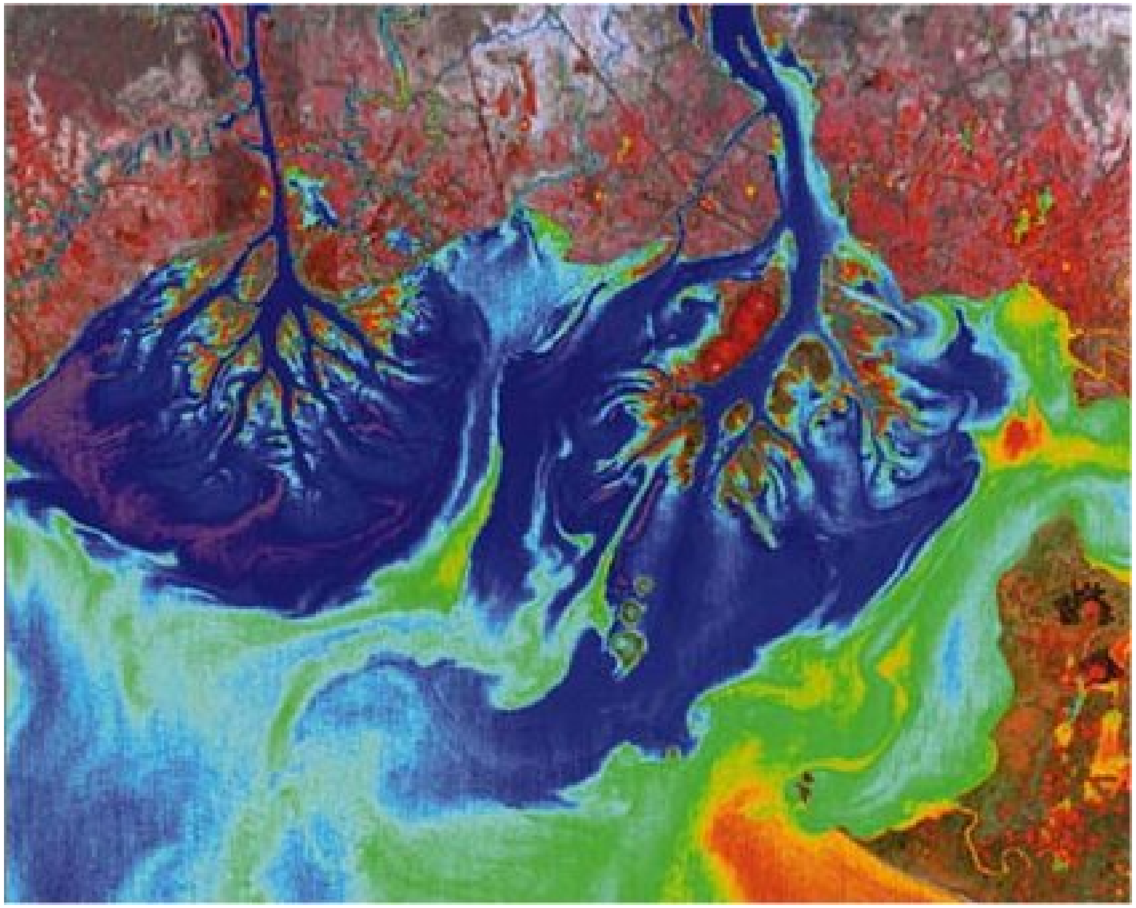


Figure 4.13. Number of volcanic eruptions per year $\dot{N}_c$ with a tephra volume greater than V as a function of V for the period 1975–1985 (squares) and for the last 200 years (circles) (McClelland *et al.*, 1989). The line represents the correlation with (2.6) taking $D = 2.14$.

River basins



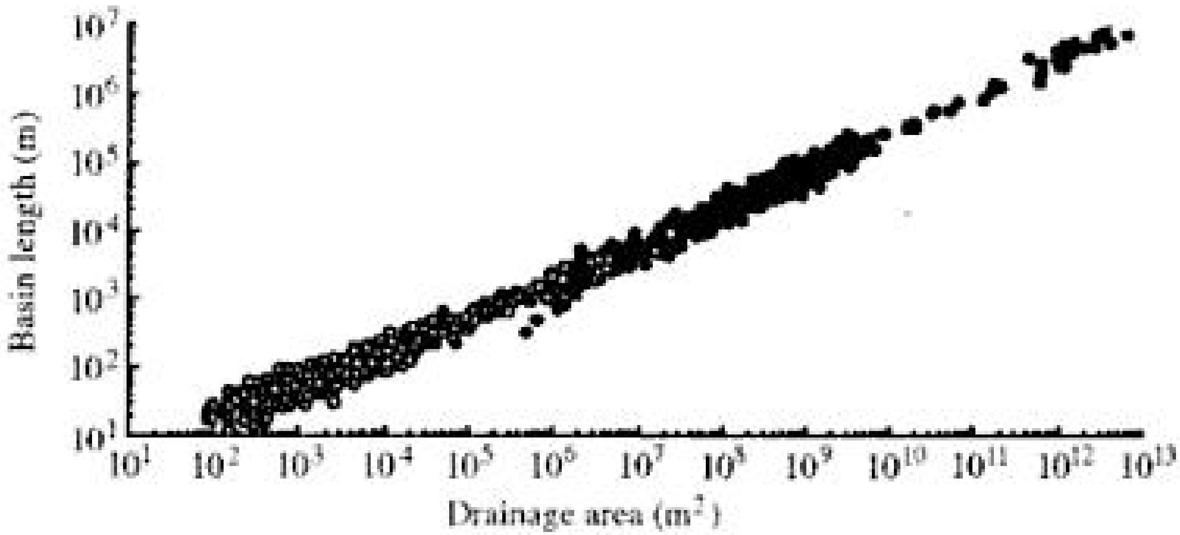


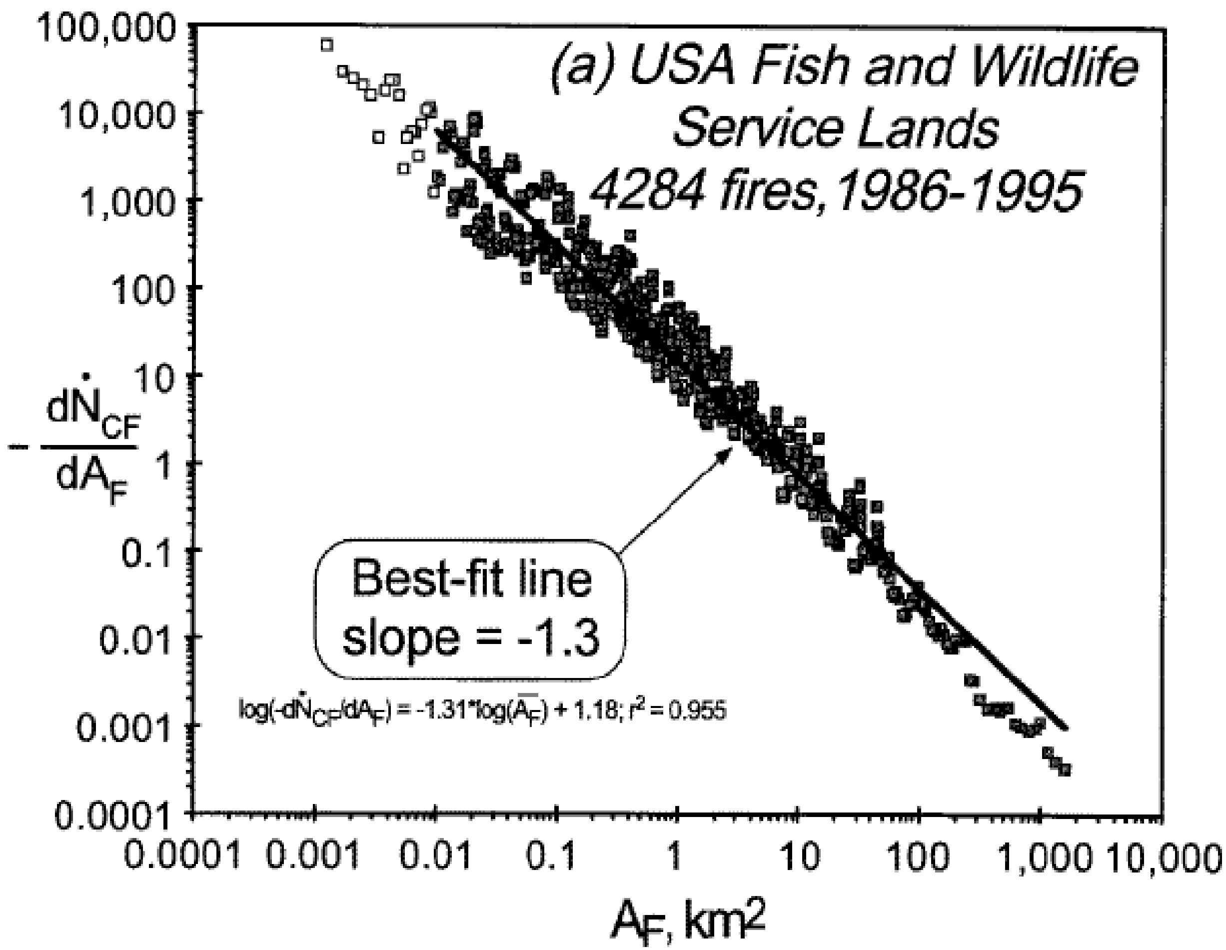
Figure 1.6. Basin length versus area for unchanneled valleys, source areas and low-order channels (empty circles). Solid circles are reported data for large channel networks [after Montgomery and Dietrich, 1992].

Hack's equation is

$$\frac{A}{L^2} \approx \frac{1}{2} A^{-0.2}$$

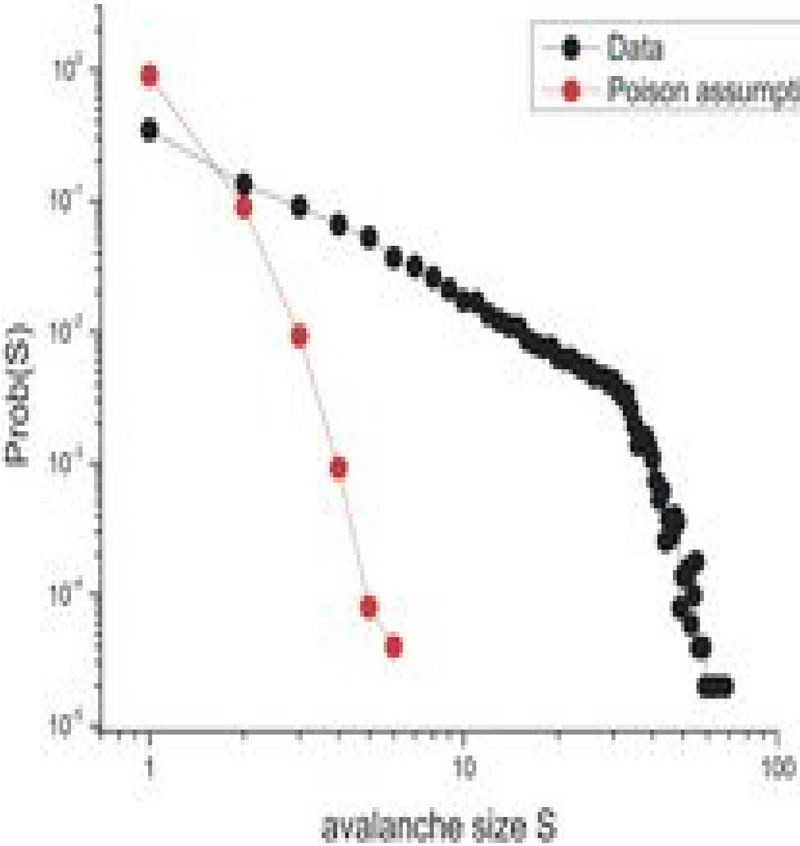
I Rodrigues-Iturbe, ***Fractal river basins : chance and self-organization***
Cambridge University Press, 1997

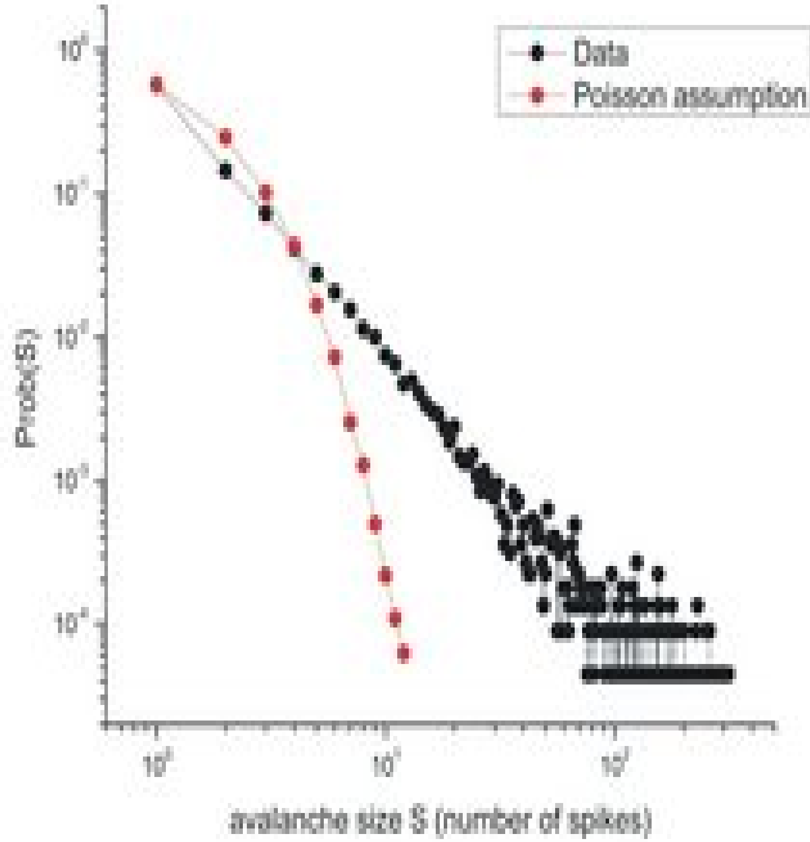
Forest Fires



D Turcotte, Rep. Prog. Phys., 1999

Neuronal Avalanches





Beggs and Plenz, J. Neurosci, 2003

Other examples of scale-invariant (power-law) distributions in nature:

- Solar flares
- Earth's magnetosphere
- Rain and drought intensity/duration
- Stock prices
- Wars

In the laboratory:

- Magnetic domain wall motion (Barkhausen noise)
- Ricepile experiment

Why is the observation of scale-invariant behavior without parameter tuning a problem?

The systems in which power-law distibutions and 1/f noise are documented are composed of many interacting units, and as such are described by ***statistical mechanics***

Statistical mechanics shows that most systems generically exhibit ***non***-power-law distributions (i.e., Poisson or exponential), with scale-invariant behavior ***only at a critical point*** (a continuous phase transition)

To reach criticality, one or more parameters (temperature, pressure...) must be adjusted precisely

Who is adjusting the parameters of Earth's tectonic plates or of forest-fire propagation?

Self-Organized Criticality

Bak, Tang and Wiesenfeld (Phys Rev Lett,1987) argued that scale-invariant distributions and 1/f noise could arise without tuning in systems far from equilibrium, with a threshold for activity and spatial coupling between elements, when subject to a slow external drive.

Threshold dynamics: The response of each element to perturbations below a certain value is minimal; strong response or activity above this threshold

Spatial coupling: When an active element relaxes, it perturbs its neighbors, which may themselves become active → ***avalanches***

Slow loss mechanism: When activity reaches the edge of the system, some is lost

Slow external drive: In the absence of activity, the system is excited at a rate << rate of relaxation/propagation

Sandpile Models: The BTW Sandpile

Square lattice of LxL sites

Each site (i,j) harbors z(i,j) particles or “sand grains” (z is called “height”)

$z(i,j) = 0, 1, 2, 3,$ or 4

If $z(i,j) = 4$, the site “topples”, transferring one grain to each neighbor:

$z(i,j) \longrightarrow 0$ and $z(i+1,j) \longrightarrow z(i+1,j) + 1$ and similarly for the other three neighbors of site (i,j)

This may cause other neighbors to topple, and so on (avalanche)

When a site at the edge of the system topples, one (or more) grains are lost

When all sites have $z(i,j) < 4$, a new grain is added at a randomly chosen site – infinitely slow drive

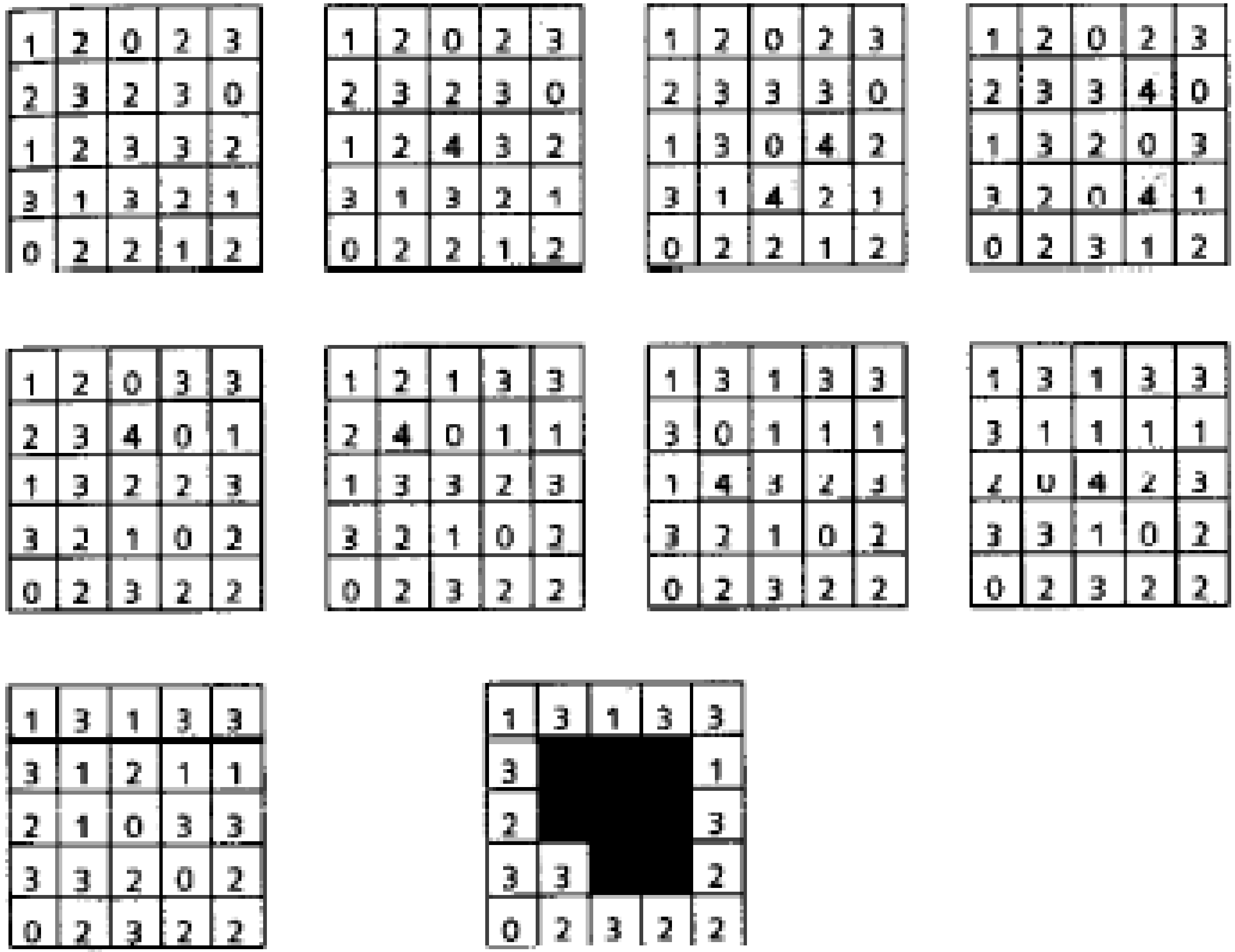


Figure 12. Illustration of toppling avalanche in a small sandpile. A grain falling at the site with height 3 at the center of the grid leads to an avalanche composed of nine toppling events, with a duration of seven update steps. The avalanche has a size $s = 9$. The black squares indicate the eight sites that toppled. One site toppled twice.

P Bak, ***How Nature Works*** (Springer-Verlag, 1996)

The simple rules of the BTW sandpile give rise to a scale-invariant avalanche-size distribution in the stationary state, apparently without adjusting any parameters

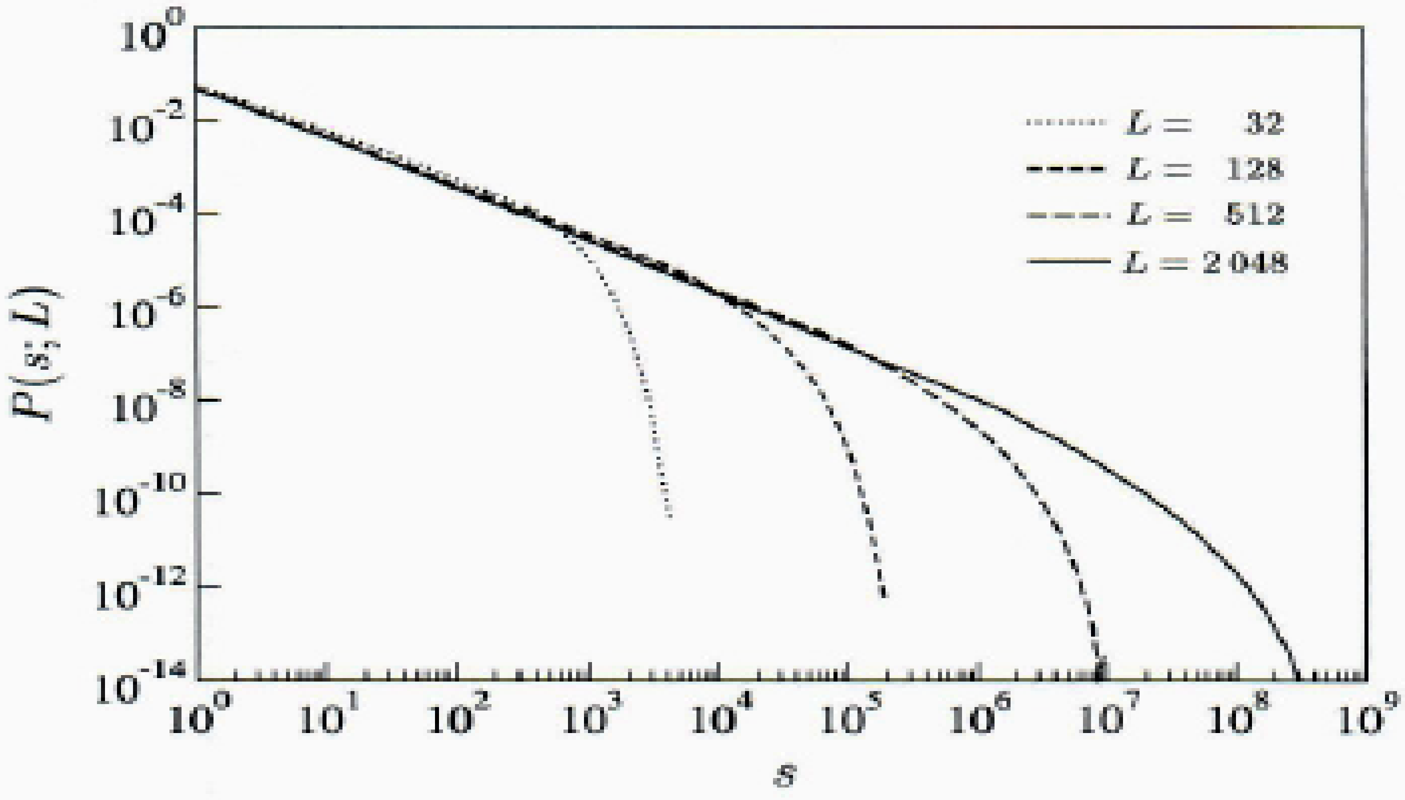


Fig. 3.13 Numerical results of the avalanche-size probability, $P(s; L)$, versus the avalanche size, s, for the two-dimensional BTW model on square lattices of size $L = 32, 128, 512, 2\,048$ marked with lines of increasing dash length. The frequency of avalanches decays with size. There is no typical size of an avalanche except for a cutoff avalanche size which increases with system size.

K Christensen and N R Moloney, ***Complexity and Criticality***, Imperial Coll. Press, 2005

BTW Sandpile

Avalanche-size distribution:

$$P(s;L) \sim s^{-\tau_s} G(s/s_c)$$

with avalanche size exponent $\tau_s \approx 1.33$

and cutoff size $s_c \sim L^{D}$, with $D \approx 2.86$

But: dissipative and nondissipative avalanches ***scale differently***

BTW sandpile is a deterministic dynamical system
Toppling invariants lead to many conserved quantities, lack of ergodicity

Stochastic Sandpile Models

Scaling behavior appears to be simpler in the ***stochastic*** sandpile (Manna 1991)

Here z(i,j) = 0, 1, or 2. Sites with z = 2 topple, sending two grains to ***randomly chosen*** neighbors

The stochastic sandpile again features loss of grains at the edges and addition when there are no toppling sites

This model also produces scale-invariant avalanche distributions, with somewhat different exponents than the BTW model

Why do sandpile models exhibit scale-free avalanche distributions without tuning of parameters?

Connection with absorbing-state phase transitions:

The protocol of grain addition (in absence of activity) and loss of grains at boundaries pins the system at a critical point

(RD, M. A. Muñoz, A. Vespignani and S. Zapperi, Braz. J. Phys., 2000)

Phase Transitions

Examples: liquid-vapor, magnetic, binary mixtures...

Formal definition: singular dependence of macroscopic properties (e.g., density) on control parameters (temperature, pressure) in a system with a very large number of degrees of freedom

Example: magnetic systems (ferromagnetic/paramegnetic transition)
Control parameters are temperature (T) and external magnetic field (H)
Order parameter: magnetization

At the ***critical point*** (zero field, T=Tc) there are long-range correlations:

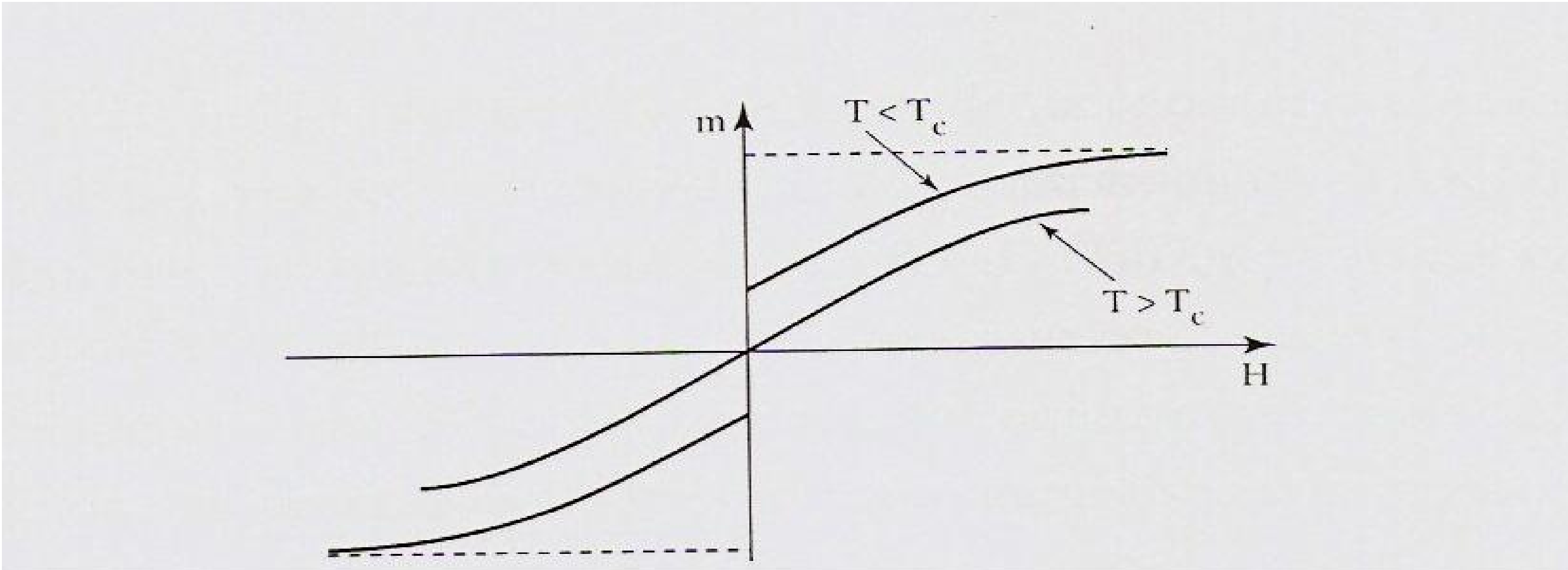
The correlation function

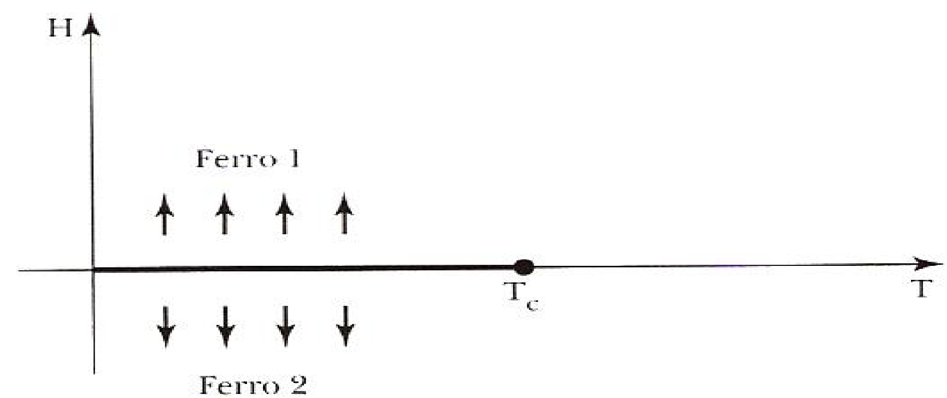
$$C(r) = \mathrm{cov}[m(r), m(0)]$$

decays as a power law [m(r) is the local magnetization]

The distribution of cluster sizes also follows a power law

Magnetic Phase Transition

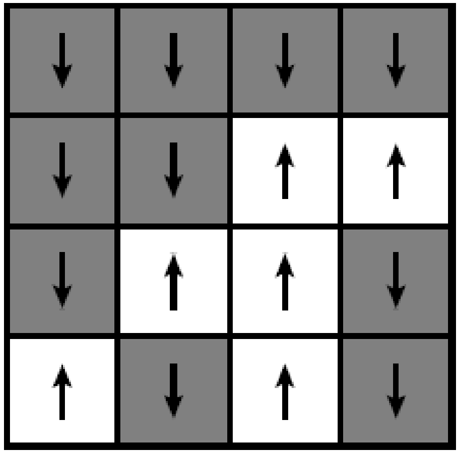




Minimal description of a magnetic phase transition: the Ising model

“Spins” localized at sites on a lattice interact with their neighbors and with the external magnetic field

The spin s_i at site i can take values of +1 or -1



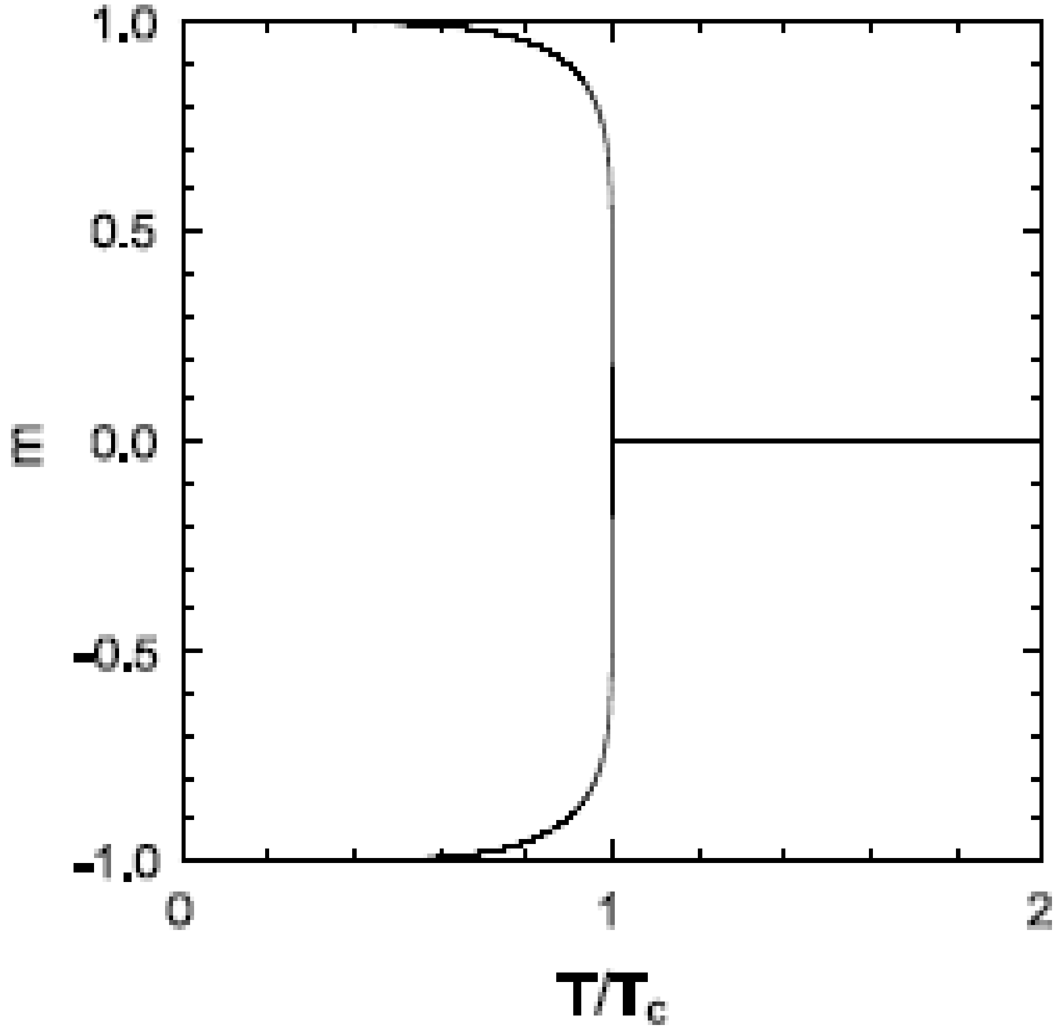
The energy of the system is

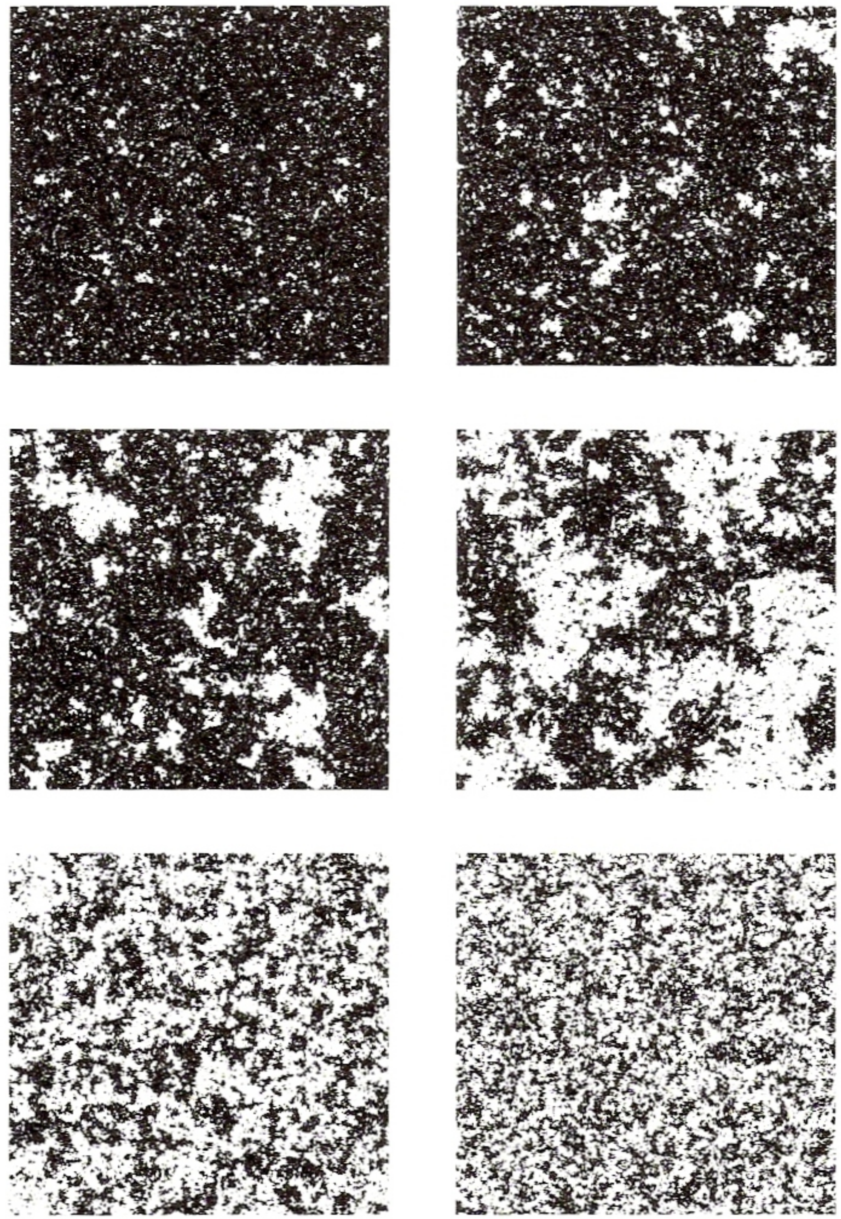
$$\mathcal{H} = \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i$$

For H=0 the minimum-energy configurations have all spins equal. Using statistical mechanics, one can calculate the magnetization and other properties as functions of T and H

There is a transition between a disordered phase (m=0) and an ordered one in dimensions d>1

Ising model: magnetization for H=0





Ising model: typical configurations at various temperatures

At the critical point (and only there), the mean cluster size in the Ising model diverges and the cluster-size distribution follows a power law

Away from the critical point the distribution decays exponentially

This raises two problems:

1. The natural and social systems exhibiting scale-free behavior are far from equilibrium, as are the sandpile models. Can we observe critical phenomena in far from equilibrium systems?

2. How can such critical phenomena appear without our having to tune parameters?

We shall see that (1) it is not difficult to find critical phenomena in systems out of equilibrium and (2) in the case of an ***absorbing-state phase transition***, parameter tuning can be hidden.

Absorbing state of a Markov process:

Consider a population of organisms, population size $N(t)$

N evolves via a stochastic dynamics with transitions from N to $N+1$ (reproduction), and to $N-1$ (death)

$N=0$ is an ***absorbing state***: if $N=0$ at some time t, then $N(t') = 0$ for all times $t' > t$

Systems with spatial structure: **phase transitions** between active and absorbing states are possible in infinite-size limit

Of interest in population dynamics, epidemiology, self-organized criticality, condensed-matter physics, social system modelling...

Processes with an absorbing state are intrinsically far from equilibrium, as the dynamics is ***irreversible***

Examples of absorbing-state phase transitions:

Directed percolation* (DP) (contact process)

Parity-conserving (branching-annihilating random walks)

Conserved DP (conserved stochastic sandpile)**

*Experiment: Takeuchi et al, Phys Rev Lett **99** 234503 (2007)

**Experiment: L Corté, P M Chaikin, J P Gollub and D J Pine, Nature Phys 2008
Transition between reversible and irreversible deformation in sheared colloidal suspension

General references on absoring-state phase transitions:
J Marro and R Dickman, *Nonequilibrium Phase Transitions in Lattice Models*, (Cambridge Press, 1999).
H Hinrichsen, Adv. Phys. **49** 815 (2000).
G Odór, Rev. Mod. Phys. **76**, 663 (2004)

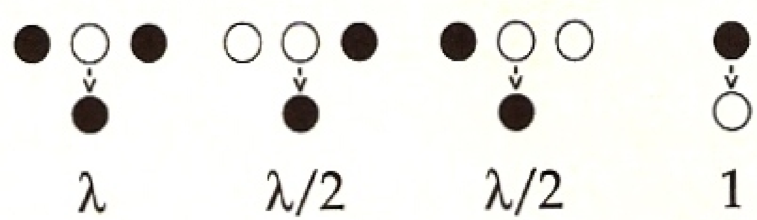
Contact Process (Harris 1972): a birth-and-death process with spatial structure

Lattice of L^d sites in d dimensions

Each site can be either active ($\sigma_i = 1$) or inactive ($\sigma_i = 0$)
An active site represents an organism

Active sites become inactive at a rate of unity, indep. of neighbors
An inactive site becomes active at a rate of λ times the fraction of active neighbors

The state with all sites inactive is absorbing



Rates for the one-dimensional CP.

Contact Process: order parameter $\boldsymbol{\rho}$ is fraction of active sites

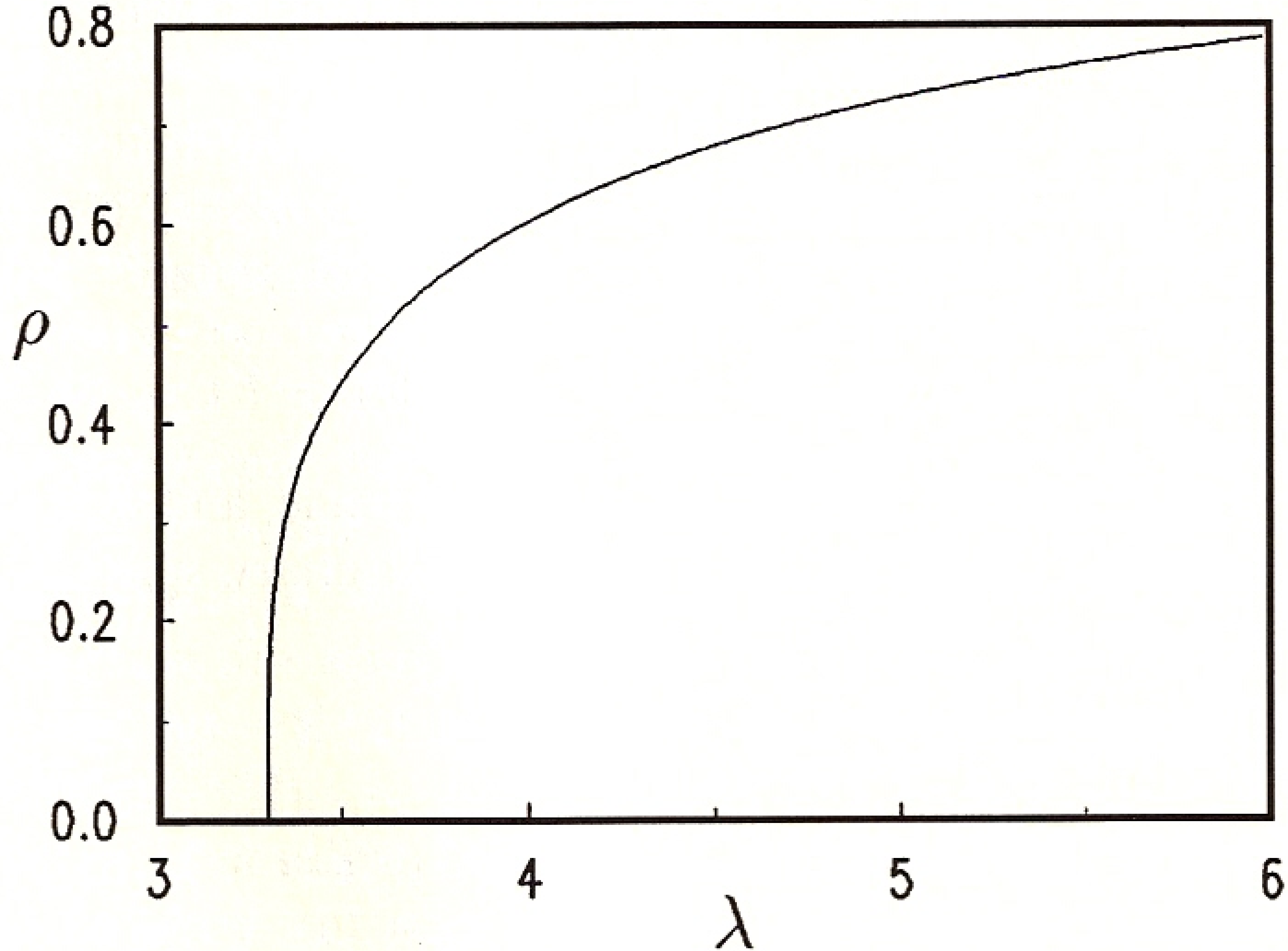
Rigorous results: continuous phase transition between active and absorbing state for d ≥ 1, at some λ_c (Harris, Grimmet...)

Order parameter: $\rho \sim (\lambda - \lambda_c)^{\beta}$

(Mean-field theory: λ_c = 1, β = 1)

Results for λ_c, critical exponents: series expansion, simulation, analysis of the master equation, ε-expansion

Types of critical behavior: static, dynamic, spread of activity



Order parameter in the one-dimensional contact process:
series expansion analysis

subcritical critical supercritical

Spread of activity in contact process (avalanches)
At critical point avalanche-size distribution is power-law

The contact process is a good example of a critical point in a far from equilibrium system, but to observe power-law scaling we must adjust the creation rate to its critical value

Let's consider another simple model, ***activated random walkers***

A Markov process defined on a lattice of $\mathbf{L}^{\mathbf{d}}$ sites with ***periodic boundaries***

Particles perform random walks on the lattice

Let n_i denote the number of particles at site i (n_i = 0, 1, 2,...)

Initially **N** particles are distributed randomly over the lattice

Dynamics: any site with $n_i \geq 2$ is ***active***

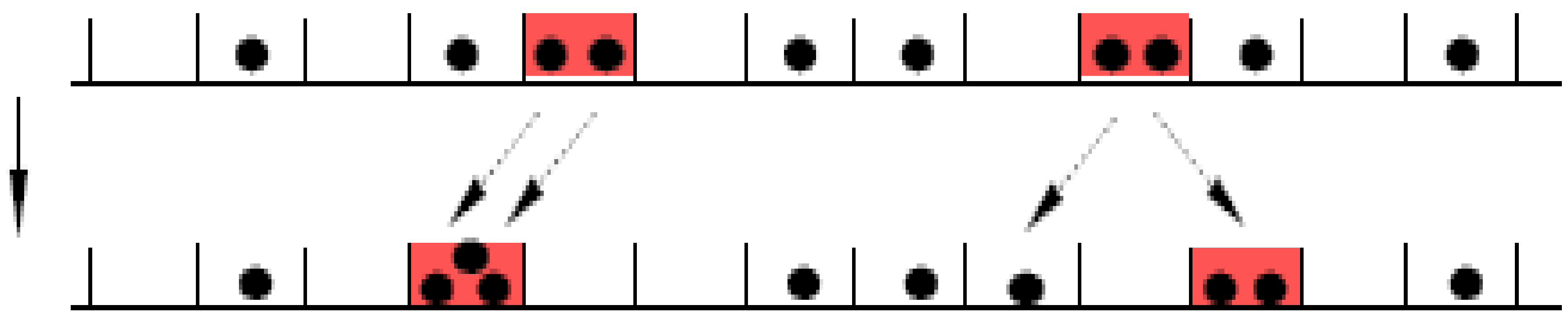
Active sites ***topple*** at a rate of unity, sending two particles to randomly chosen neighbors

The number of particles remains constant throughout the evolution

$\zeta = N/L^{\mathbf{d}}$ is a control parameter

Activated random walkers (ARW): when site i topples two particles jump from i to a nearest neighbor, independently

Examples of topplings in one dimension

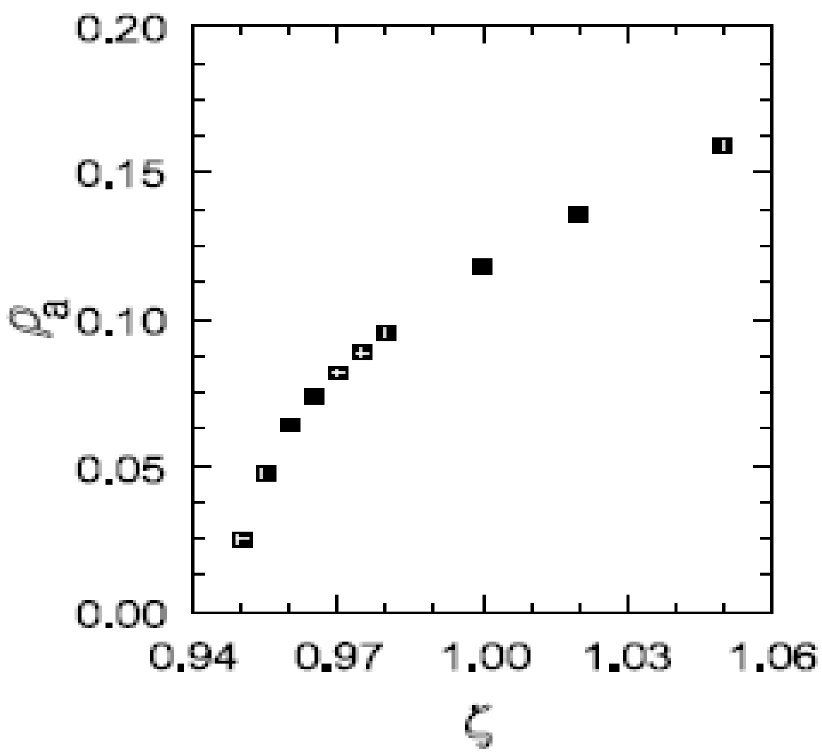


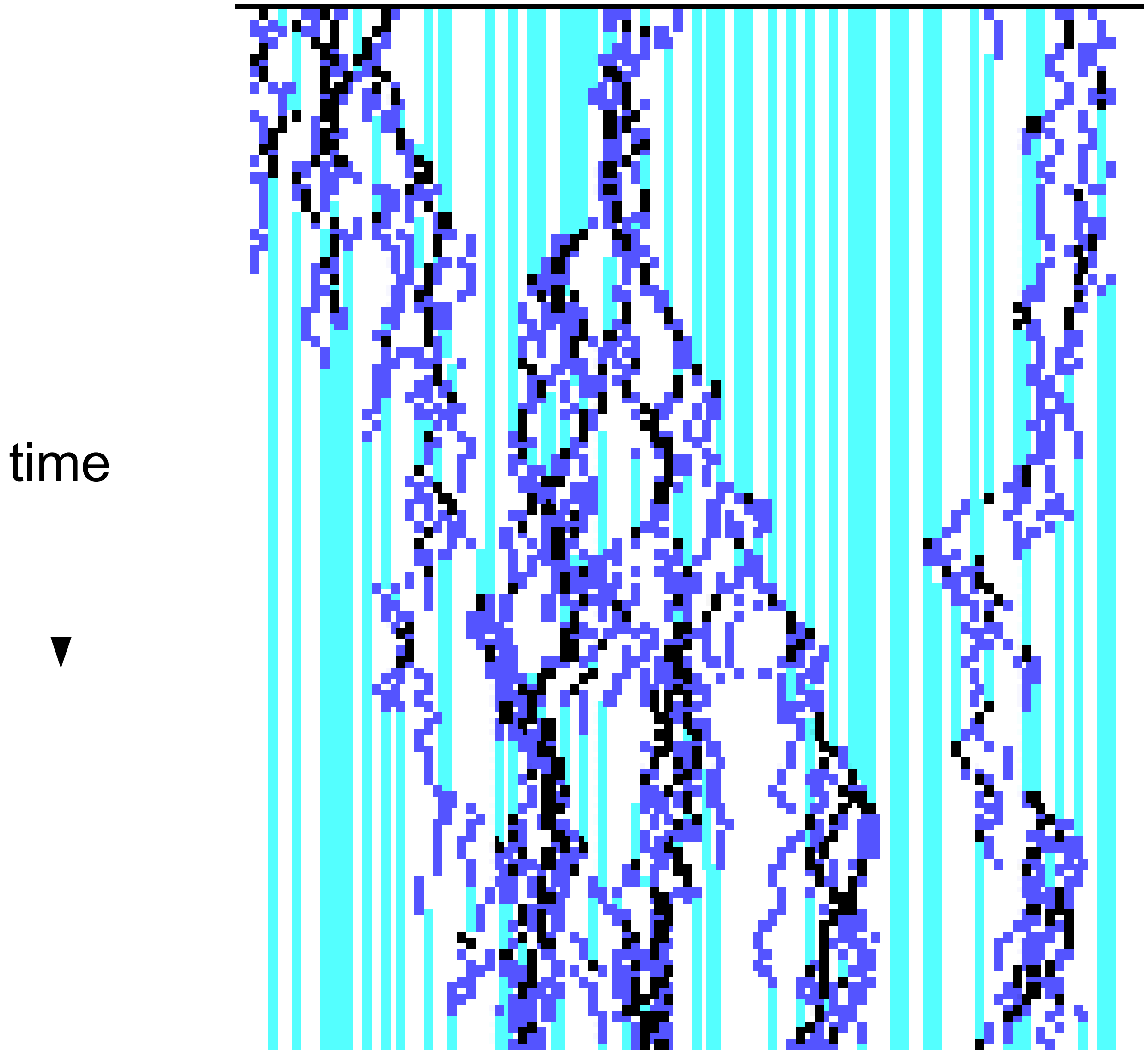
Activated random walkers: any configuration with without active sites is absorbing

Such configurations exist for $\zeta < 1$

There is an absorbing-state phase transition at $\zeta = \zeta_c$ (= 0.94885 in one dimension)

Order parameter: ρ, the fraction of active sites





Typical evolution of ARW process

As in the contact process, the activated random walkers process exhibits scale-invariance at the critical point, but o reach this point we must tune ζ to its critical value

Now we make two simple changes in the process:

1. Replace the periodic boundary condition with open boundaries
   When a site at the edge topples, particles may be lost

2. Eventually the system reaches an absorbing configuration
   When this happens a new particle is added at a randomly chosen site

Evidently this converts the ARW process into the Manna sandpile

These changes ***force*** the ARW process to its critical point:

If $\zeta > \zeta_c$ there is activity and ζ can only decrease

If $\zeta < \zeta_c$ activity will stop and ζ will then increase

Absorbing-state mechanism for SOC: self-organized criticality in a slowly driven system corresponds to an absorbing-state phase transition in the model with the same local dynamics, but with strict conservation

Simulations confirm that the critical exponents in SOC and in the absorbing phase transition are related

As the system size increases, the fluctuations of ζ in the ***driven*** sandpile are restricted to an ever smaller region centered on the critical density of the ***conserved*** model

The SOC and absorbing "ensembles" are however distinct (Pruessner and Peters, Phys. Rev. E, 2006, arXiv:0912.2305)

In deterministic sandpiles, the critical density in the conserved version is a ***tiny*** bit higher than in the SOC version! (Fey et al., Phys Rev Lett, 2010)

In sandpile and related models, an infinite timescale separation between activity (toppling) and driving is realized by prohibiting addition while activity is in progress

In natural systems, we can't expect the driving mechanism to “wait” for all activity to cease before perturbing the system

If the perturbation rate **h** is very small, scale-free distributions can be generated over a finite range: the avalanche duration distribution is cut off at a time **~1/h**

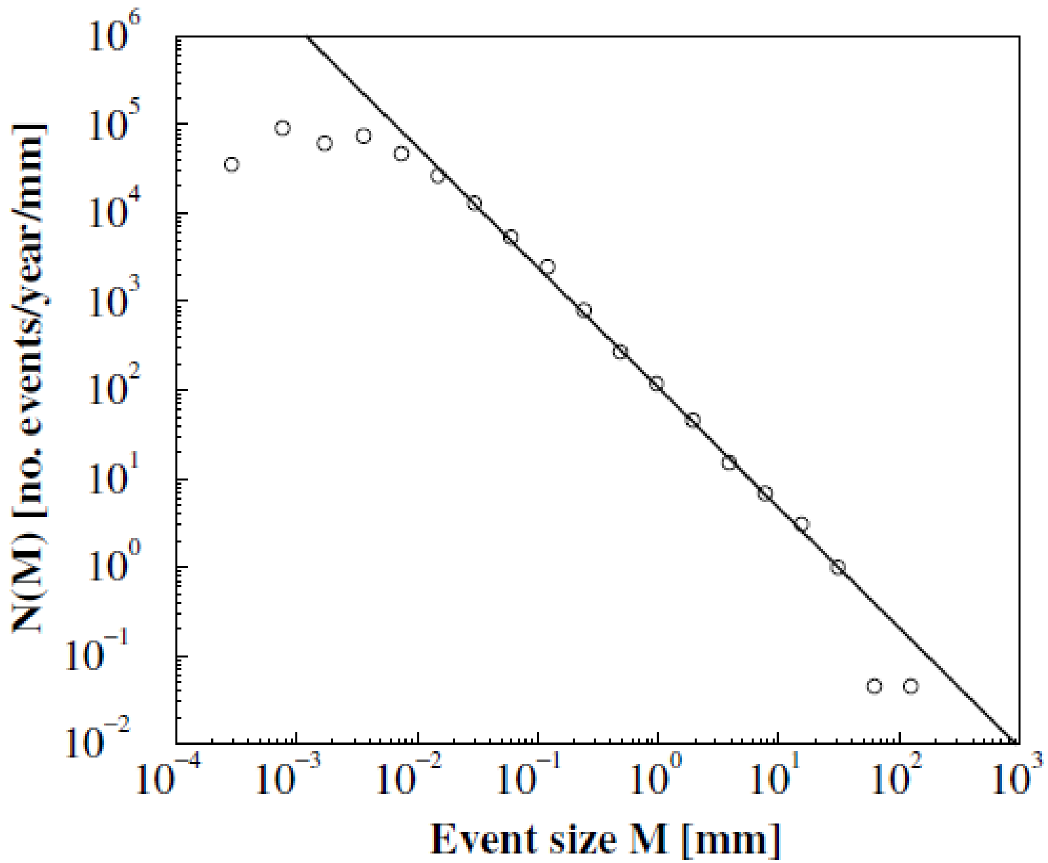
This should be fine from an empirical viewpoint!

Alternatives to SOC

In some instances, the validity of power-law distributions have been questioned; in others, alternative explanations have been proposed

Example: Scale-invariant rain and drought distributions
Rain event intensity: integrated precipitation over a rainy period
The intensity distribution follows a power law

Peters, Hertlein, and Christensen, Phys. Rev. Lett., 2002



The distribution of time intervals between passage of consecutive raindrops also involves power laws

$$f(\tau) = p\frac{\alpha a^{\alpha}}{(\tau + a)^{\alpha+1}} + q\frac{\beta b^{\beta}}{(\tau + b)^{\beta+1}},$$

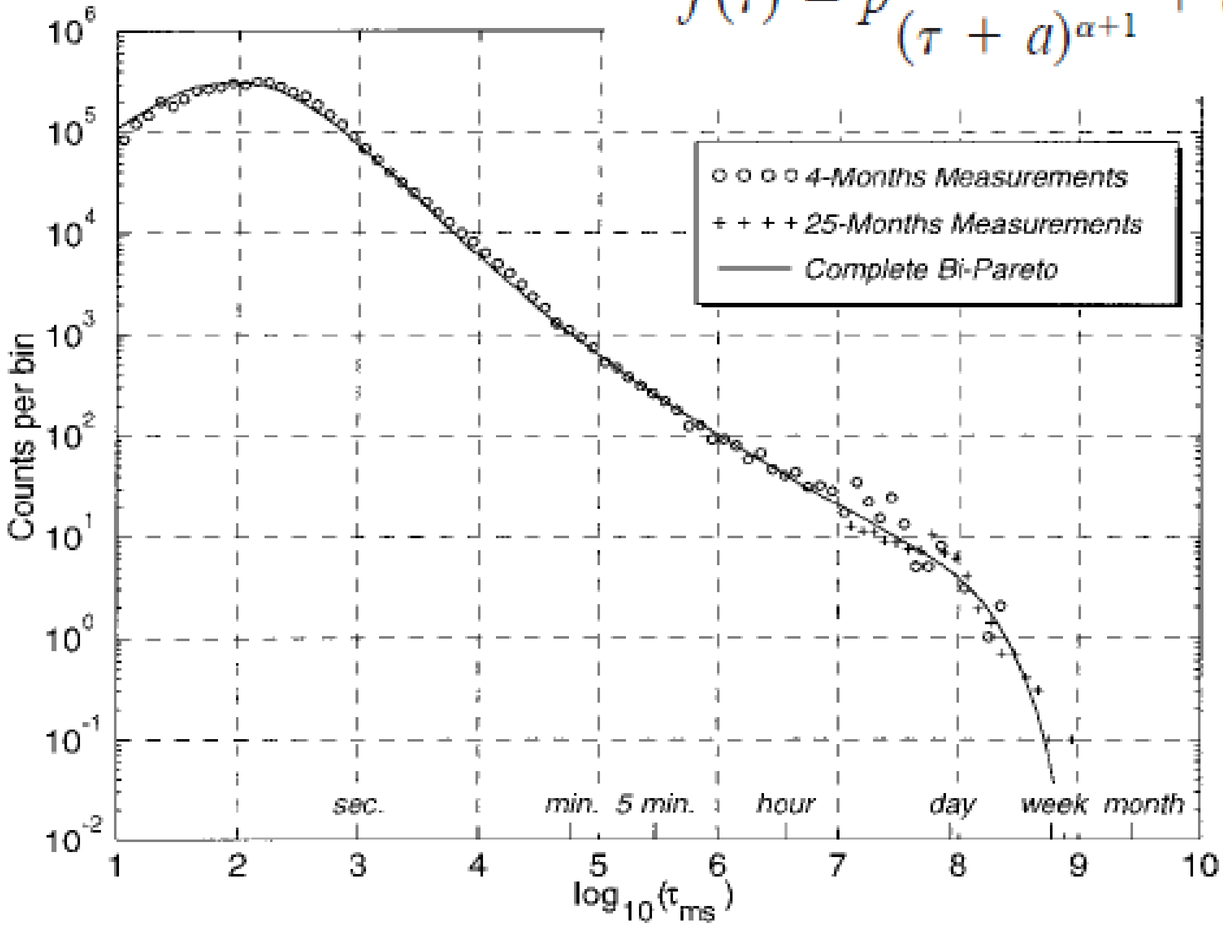


FIG. 4. Distributions of interdrop time intervals obtained from 4 months of data (○) and from 25 months of data (+).

J Lavergnat and P Golé, Journal of Applied Meteorology, 1998

Peters et al suggest the observed power laws are evidence of SOC in the dynamics of evaporation and condensation in Earth's atmosphere

Activity: Rapid condensation above a threshold value of humidity

Slow drive: Energy influx from Sun, causing evaporation

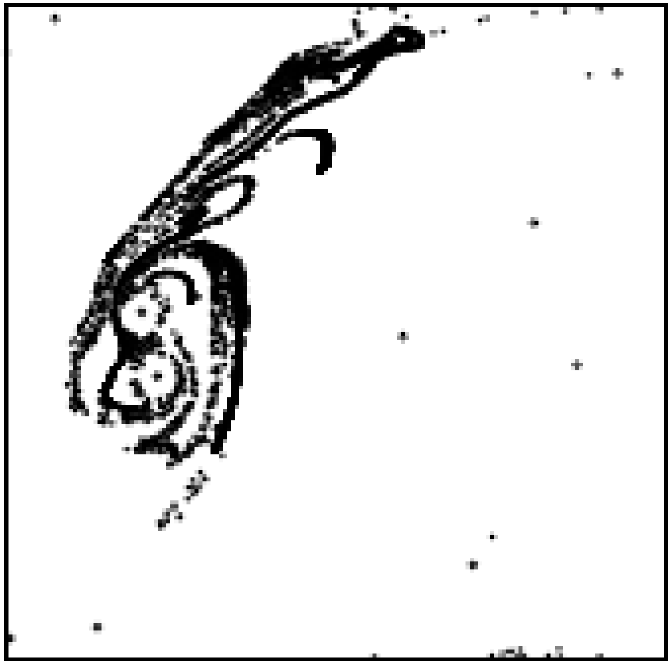
Loss: Rain falling to Earth

Coupling mechanism between nearby regions of atmosphere not clear - Winds associated with rainfall? Cold pools?

Alternative model (RD Phys Rev Lett, 2003)

If condensation occurs in localized regions, chaotic advection can generate power-law distributions of rain intensities at fixed observation sites

Simple model: two-dimensional fluid with rain treated as passive tracers, fluid motion generated by a systen of ideal vortices



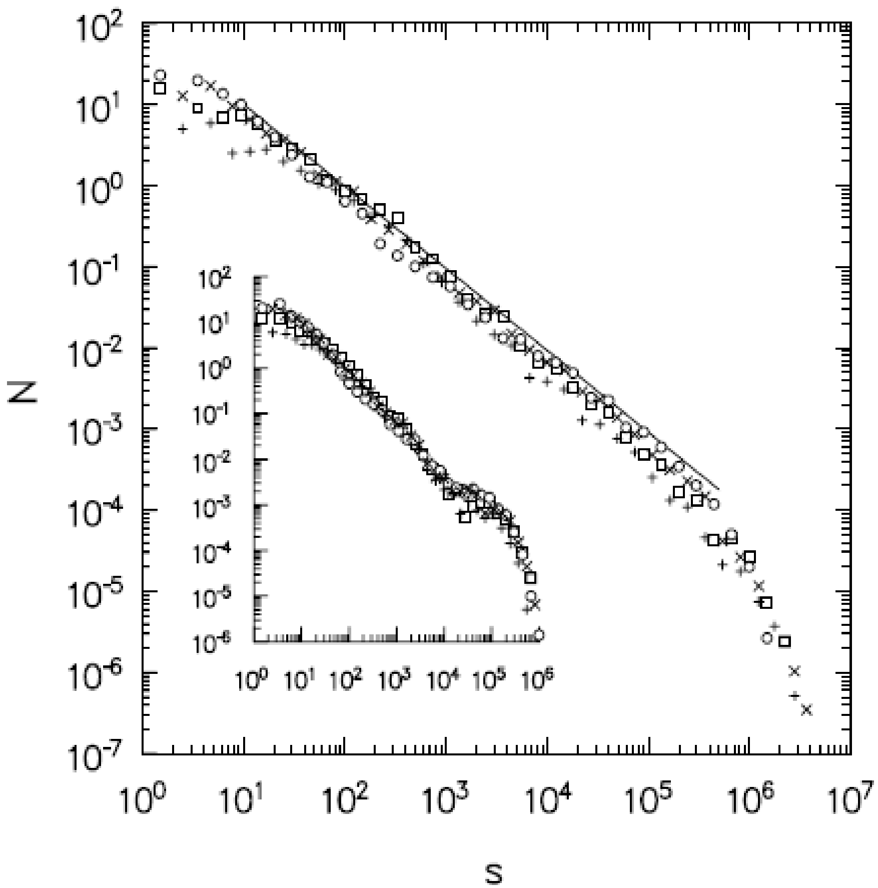


Figure 4. Rain-size (main graph) and drought-duration (inset) distributions in systems of vortices of equal strength, $T \simeq 0.85\tau_c$. $\circ$: $N_V = 10$; $\times$: $N_V = 20$; $\Box$: $N_V = 50$; $+$: $N_V = 100$. The vortex intensity K is scaled $\sim 1/\sqrt{N_V}$ in these studies. The straight lines have slopes of -1.01 (rain size) and -1.13 (drought).

This simple two-dimensional model yields power-law distributions but does not reproduce the observed exponents (1.36 and 1.42 for rain intensity and drought duration, resp.)
It does raise the possibility that the observed power laws are due to chaotic advection

SUMMARY

Power-law distributions are observed in many natural and social systems

SOC provides a mechanism for generating scale-invariant behavior without parameter tuning

The essential ingredients are: (1) a system of many coupled nonlinear elements having a threshold for activity; (2) a slow loss mechanism; (3) an even slower external drive

In stochastic models, SOC works by forcing the system to an absorbing-state critical point

In some instances, alternatives to SOC have been proposed


Thanks!