Large scale coherent structures and turbulence in quasi-2D hydrodynamic models

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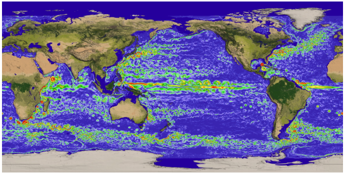
IPAM, 3-7 May 2010

Outline

Large scale coherent structures in geophysical flows coexist with turbulence

Zonal turbulence on Jupiter (NASA)

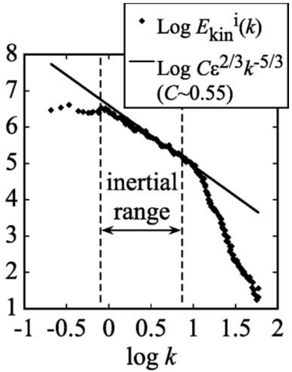


Simulation of Earths oceans (Earth Simulator Center/JAMSTEC)

Turbulence in 3D

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$



- Reynolds number $R = \frac{LU}{\nu}$.
- Energy injected into large eddies.
- Energy removed from small eddies at viscous scale.
- Energy transferred by interaction between scales.
- Concept of *inertial range* and *energy cascade*.

Kolmogorov theory of 3D turbulence

K41 : As $R \to \infty$, small scale statistics depend only on the scale, k, and the energy flux, ϵ. Dimensionally :

$$E(k) = c\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \qquad \text{Kolmogorov spectrum}$$

Fluctuations characterised by structure functions:

$$S_n(r) = (\delta u_L)^n = \left\langle \left((\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})) \cdot \frac{\vec{r}}{r} \right)^n \right\rangle .$$

Scaling form in stationary state:

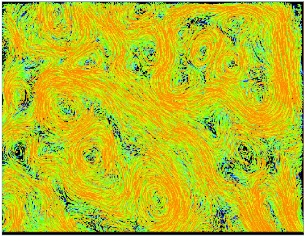
$$\lim_{r\to 0} \lim_{\nu \to 0} \lim_{t\to\infty} S_n(r) = C_n \, (\epsilon \, r)^{\zeta_n} .$$

Third order structure function is special. In inertial range:

$$S_3(r) = -\frac{4}{5}\,\epsilon\, r \qquad \zeta_3 = 1 \text{ (exact)}.$$

Turbulence in two dimensions

Draining soap film (W. Goldburg et al., Pittsburgh)



Electromagnetically driven fluid layer (R. Ecke et al., Los Alamos)

Conservation Laws in two-dimensional hydrodynamics

Streamfunction formulation of 2D NS : $\mathbf{u} = (\partial_y \psi, -\partial_x \psi)$.

Vorticity : $\omega = \nabla \times \mathbf{u} = \Delta\psi \, \hat{\mathbf{z}}$

Advective derivative : $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}$

N-S equation: $\frac{d}{dt}(\Delta\psi) = \nu\Delta(\Delta\psi) - \alpha\Delta\psi + f(x,t)$

There are *two* inviscid invariants :

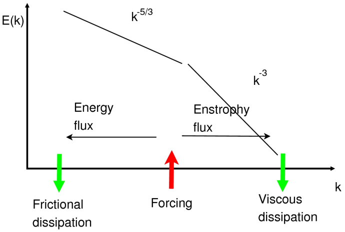
Total energy : $E = \frac{1}{2}\int |\nabla\psi|^2 \, d\mathbf{x} = \frac{1}{2}\int k^2 \, |\psi_k|^2 \, d\mathbf{k}$

Total enstrophy : $H = \frac{1}{2}\int \omega^2 d\mathbf{x} = \frac{1}{2}\int k^4 \, |\psi_k|^2 \, d\mathbf{k}$

This profoundly affects the physics.

Inverse Cascade in Two Dimensional Turbulence

A K41–like phenomenological theory of 2-D turbulence was developed by Kraichnan, Leith and Batchelor in 1967-68.

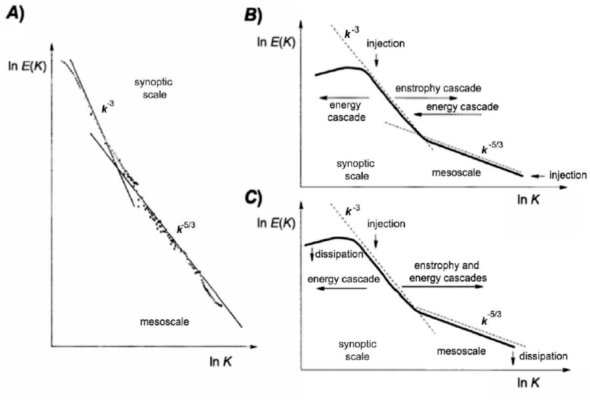


- For a stationary state, E and H must cascade in opposite directions in k.
- H goes to *smaller* scales with $E(k) = c_\eta\, \eta^{2/3}\, k^{-3}$.
- E goes to *larger* scales with $E(k) = c_\epsilon\, \epsilon^{2/3}\, k^{-5/3}$.
- *Inverse cascade* is a new phenomenon.

Analogue of 4/5-Law (Lindborg 1999):

$$S_3(r) = 2\,\epsilon\, r$$

The Atmospheric Energy Spectrum on Earth



Nastrom and Gage, 1985.

How should we interpret the spectra?

We must at least establish the direction of the energy flux:

$$S_3(r) = \langle (\delta\, u_L)^3 \rangle$$

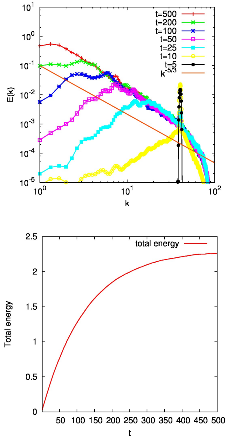
Analogue of $4/5$-Law (Lindborg 1999):

$$S_3(r) = 2\,\epsilon\, r$$

Positive $S_3(r)$ corresponds to an inverse cascade. This is unclear from the atmospheric data:

- Lindborg 1999 : "$S_3(r)$ is negative".
- Cho and Lindborg 2001 : "$S_3(r)$ is positive".
- Falkovich et al. 2008 : "$S_3(r)$ is very sensitive to presence of coherent structures".

Development of the Inverse Cascade

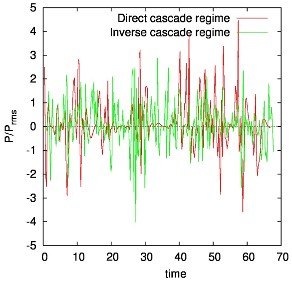


- Peak of the energy spectrum moves to larger scales leaving $k^{-5/3}$ spectrum behind it.
- Timescale slows down as cascade proceeds.
- Growth eventually saturates due to the dissipation of energy near the spectral peak due to friction.
- Vorticity movie

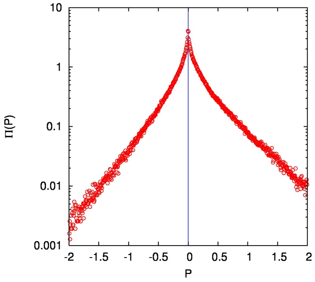
Statistics of Power Input

Why is the direction of the energy flux not immediately obvious? Consider the local power injected into the system :

$$P(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, t).$$



Timeseries of $P(t)$



PDF, $\Pi(P)$ with Gaussian forcing

Strong intermittency of energy injection?

Yes, but not for a very interesting reason:

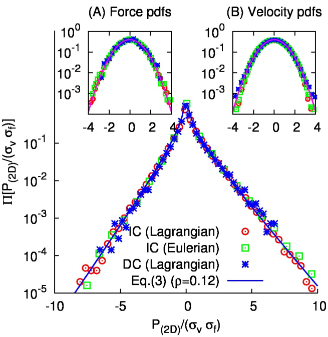
Distribution of products of normals

If x_1 and x_2 are joint normally distributed random variables with mean zero, variances σ_1^2 and σ_2^2 and correlation coefficient ρ,

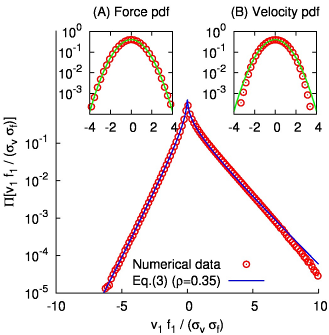
$$\mathbb{P}_{XY}(z = x_1 x_2) = \frac{e^{\frac{\rho z}{(1-\rho^2)\sigma_1\sigma_2}}}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \mathrm{K}_0\left(\frac{|z|}{(1-\rho^2)\sigma_1\sigma_2}\right)$$

- $\langle z \rangle = \rho\,\sigma_1\sigma_2$ must be equal to the total energy dissipation rate in the steady state.
- Degree of asymmetry is controlled by the dissipation rate.
- Deviations from the product distribution have dynamical significance, not the distribution itself.

Comparison with numerical measurements



2 D



3 D

Gallavotti-Cohen fluctuation relation

An amusing aside: the product distribution satisfies the fluctuation relation: Consider the sum of n samples from $\mathbb{P}_{XY}$:

$$M_n = \frac{1}{n}\sum_{i=1}^{n} z_i \qquad \text{where } z_i \sim \mathbb{P}_{XY}(z)$$

Large deviation principle:

$$\mathbb{P}(M_n > x) \asymp \mathrm{e}^{-nI(x)}.$$

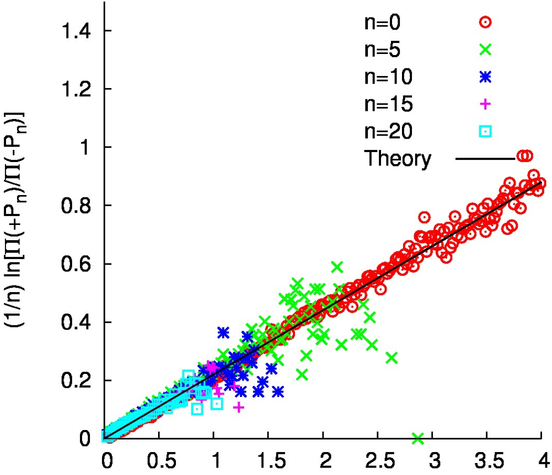
The rate function, $I(x)$, is explicitly computable.

$$\frac{\mathbb{P}(M_n > X)}{\mathbb{P}(M_n < -X)} = e^{-n(I(X)-I(-X))}, \tag{1}$$

where:

$$I(x) - I(-x) = -\Sigma x \quad \text{with } \Sigma = \frac{2\rho}{(1-\rho^2)\sigma_1\sigma_2}.$$

Gallavotti-Cohen fluctuation relation



Scale-by-Scale Energy Balance

Inertial range energy flux is a third moment not a product. How do its statistics look compared to the injected flux?
Filter streamfunction, $\psi(\mathbf{x})$, at scale l:

$$\bar{\psi}(\mathbf{x}) = \int d\mathbf{y}\, \psi(\mathbf{y})\, G_l(\mathbf{x}-\mathbf{y})$$

Large scale energy, $E_l = \left|\nabla\bar{\psi}(\mathbf{x})\right|^2$, satisfies

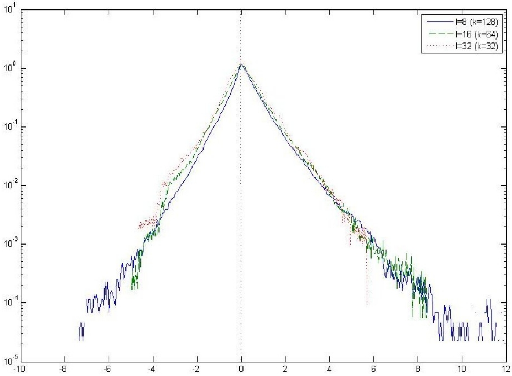
$$\frac{\partial E_l}{\partial t} + \nabla\cdot\mathbf{S} = \Pi_l$$

$\mathbf{S}$ is the spatial energy flux and

$$\Pi_l = \bar{\psi}\,\nabla\cdot(\overline{\mathbf{u}\,\nabla^2\psi} - \bar{\mathbf{u}}\,\nabla^2\bar{\psi})$$

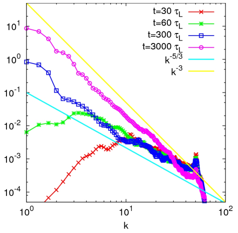
is the scale to scale energy transfer.

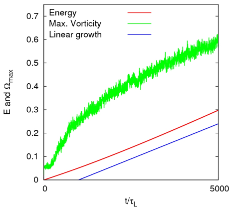
PDF of Π_l from numerical data (inverse cascade)



PDFs of Π_l for $l = 8l_f$, $l = 16l_f$ and $l = 32l_f$ where l_f is the forcing scale. PDFs have been rescaled with their variances.

Energy “Condensation”: coherent structures as a finite size effect

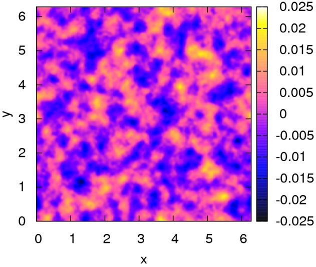




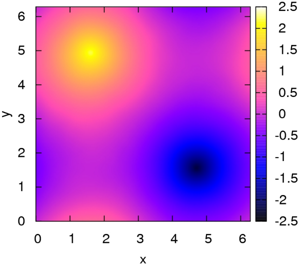
- In absence of friction, inverse cascade is blocked at scale L at $t = t^*$. After t^*, energy accumulates at large scales. (Smith & Yakhot 1991)
- Before t^*, $E(k) \sim k^{-5/3}$ at large scales (Kraichnan).
- After t^*, observe a gradual cross-over to $E(k) \sim k^{-3}$ at large scales.
- What is the x-space picture?

Real space view

- Large scale coherent vortices at the scale of the box.
- Organisation of smaller scale fluctuations
- Separation of time-scales.



Stream function before condensation



Stream function after condensation

Laboratory experiments of Shats et al.

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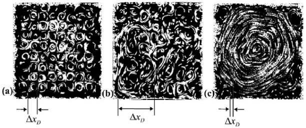

FIG. 2. Evolution of turbulence in a thin layer of electrolyte in a cell during spectral condensation. Trajectories of the tracer particles averaged over 12 frames of recorded video are shown. (a) The initial (linear) stage, $t=3$ s. (b) The inverse cascade stage, $t=25$ s. (c) The condensate stage, $t=60$ s. Δx_D represents the spatial scale of the trace particle transport during three stages of the flow evolution.

Laboratory experiments on driven fluid layers by M. Shats' group clearly demonstrate condensation phenomenon. (Also numerics by Clercx et al.)

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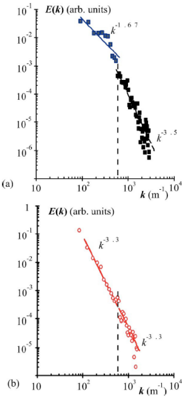
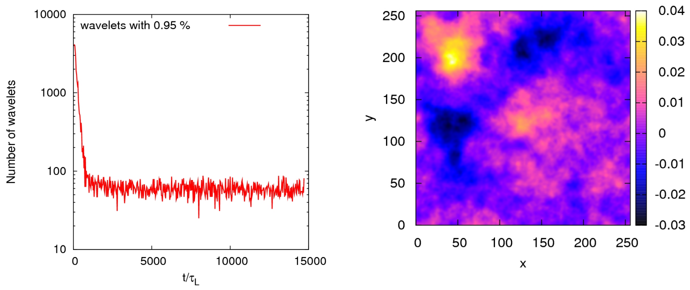


FIG. 4. (Color online) Energy spectra of the fluid velocities (a) during the inverse cascade stage of the flow development and (b) after the condensate has formed. The injection scale k_i is shown by the vertical dashed line.

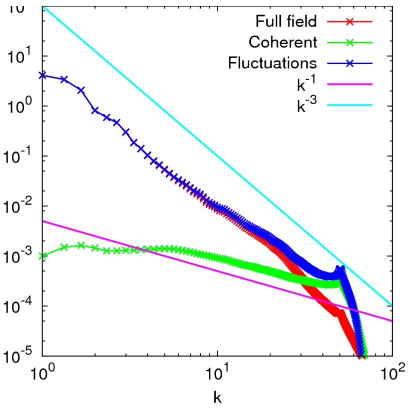
Extracting Coherent Flow using Wavelets

Spectral representation is not useful for splitting the coherent flow from the fluctuations: use wavelets.



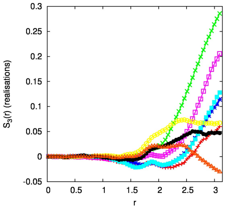
Coherent component defined to contain 0.95 of the total energy.

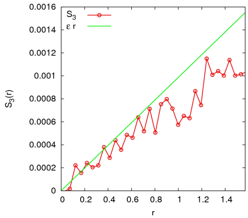
Decomposed spectrum



- Large scale k^{-3} spectrum is entirely the signature of the coherent structure.
- Nothing to do with cascades.
- Background turbulent fluctuations have a k^{-1} spectrum.
- Lesson: don't over-emphasise spectral representation!

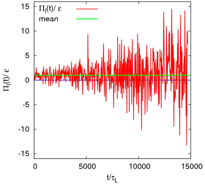
What about confirming the direction of energy transfer?



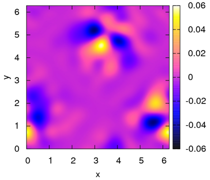


- Computation of $S_3(r)$ is dominated by the presence of the coherent part.
- Time required to average out coherent flow is very long.
- Very sensitive to windowing.
- Conclude $S_3(r)$ is not an good diagnostic of the energy cascade.

Scale-to-scale energy transfer in presence of coherent flow



Energy transfer $l = 0.4$.



Snapshot of $\Pi_l(\mathbf{x})$.

- Measurement of the total scale-to-scale energy transfer in the inertial range shows strong signature of the coherent structure.
- Quadrupolar stucture of the coherent energy transfer reflects the fact that a symmetric vortex cannot sustain a flux due to “depletion” of nonlinearity.
- Movie

Conclusions

- The inverse cascade is a robust mechanism for the organisation of large scale structures from small scale fluctuations in two dimensional turbulence (and in more geophysically realistic models).
- Cartoon view of a smooth river of energy flowing through scales is misleading. Fluctuations are huge.
- The XY-product distribution is the benchmark distribution for understanding fluctuations in composite quantities like fluxes, *not* the normal distribution.
- Finite size effects may lead to strong organisation of the large scales (”condensation”).
- Diagnostics developed for h.i.t. (eg structure functions, spectra) do not work well in the presence of such coherent structures but others exist.

References

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