Regime(s) of validity of sound-proof models

Sound-proof for small scales, compressible for large scales via scale-dependent time integration

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin

Regime(s) of validity of sound-proof models

Motivation

Stratification limit in the design-regime

Wave-breaking regime with strong stratification

Scale-dependent time-integrator

Thanks to ...

Ulrich Achatz	(Goethe-Universität, Frankfurt)
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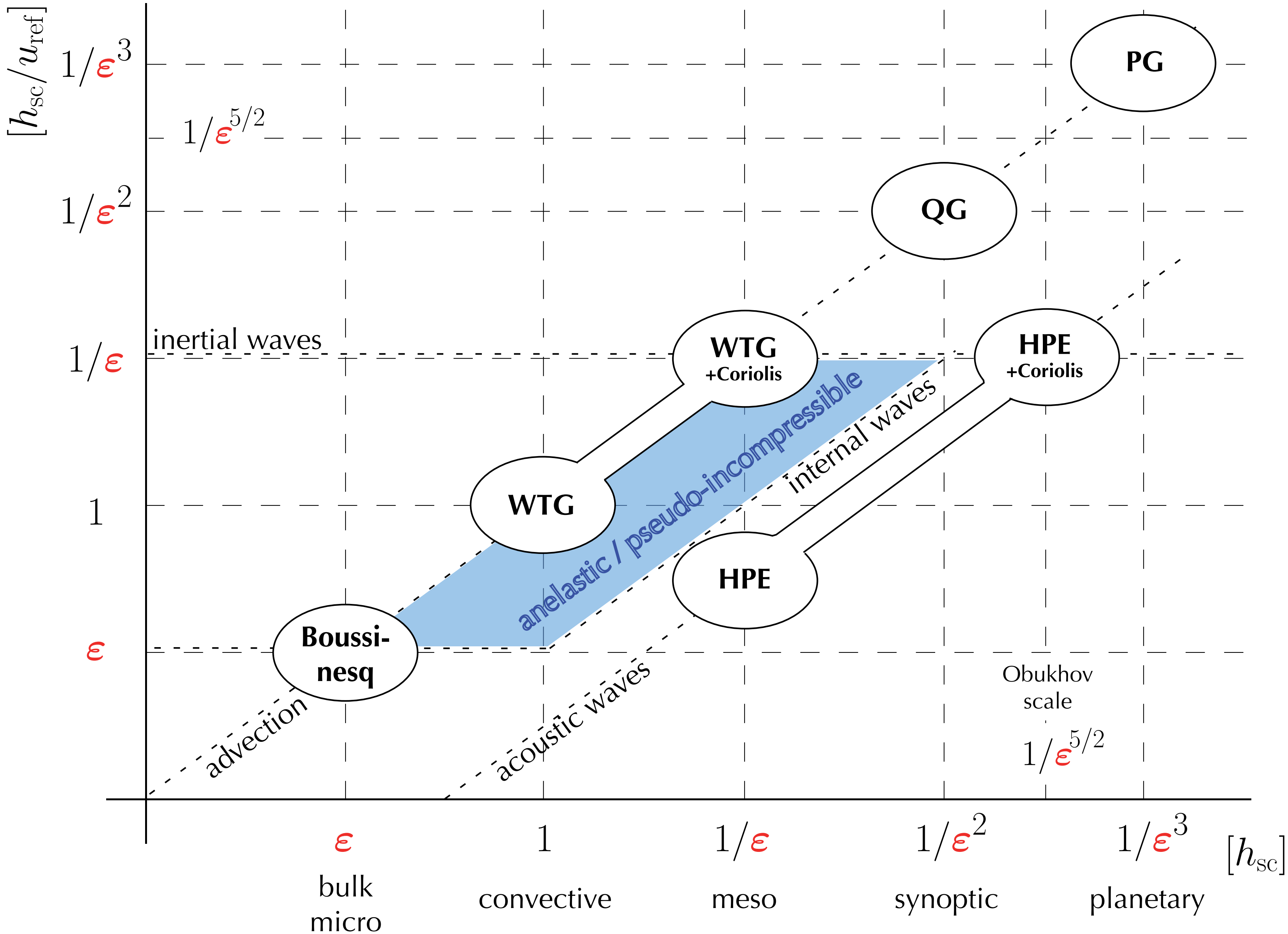
Regimes of Validity ... Motivation



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Regimes of Validity ... Motivation

Compressible flow equations

$$
\begin{aligned}
\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
(\rho w)_t \; + \nabla \cdot (\rho \boldsymbol{v} w) + P \pi_z &= -\rho g \\
\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

drop term for:

anelastic (approx.)

pseudo-incompressible

$$
P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k}\,, \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)
$$

Regimes of Validity ... Motivation

Anelastic

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho}\boldsymbol{v})_t + \nabla \cdot (\overline{\rho}\boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho}\nabla\pi = \frac{\theta'}{\overline{\theta}}\,\overline{\rho} g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P\boldsymbol{v}) = 0$$

$$\overline{\rho}(z)\theta = P\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla\pi = \frac{\theta'}{\overline{\theta}}\, g \boldsymbol{k}$$

$$\theta_t + \boldsymbol{v} \cdot \nabla\theta = 0$$

$$\theta' = \theta(z) - \overline{\theta}(z)$$

Pseudo-incompressible

baroclinic torque / modified divergence

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho\boldsymbol{v})_t + \nabla \cdot (\rho\boldsymbol{v} \circ \boldsymbol{v}) + \overline{P}\nabla\pi = \frac{\theta'}{\overline{\theta}}\,\rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P}\boldsymbol{v}) = 0$$

$$\rho(z)\theta = \overline{P}\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$(1/\theta)_t + \boldsymbol{v} \cdot \nabla(1/\theta) = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \underline{\theta\nabla\pi} = \frac{\theta'}{\overline{\theta}}\, g \boldsymbol{k}$$

$$\underline{\nabla \cdot (\overline{P}\boldsymbol{v})} = 0$$

relevant for deep atmospheres / large scales*

*see, e.g., Davies et al., QJR (2003), Smolarkiewicz & Dörnbrack, Int. J. Num. Meth. Fluids (2007)

Regimes of Validity ... Motivation

Anelastic

Boussinesq

$$\times \quad \nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$(\overline{\rho} \boldsymbol{v})_t + \nabla \cdot (\overline{\rho} \boldsymbol{v} \circ \boldsymbol{v}) + \overline{\rho} \nabla \pi = \frac{\theta'}{\overline{\theta}} \overline{\rho} g \boldsymbol{k}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

$$\overline{\rho}(z)\theta = P\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\nabla \cdot (\overline{\rho} \boldsymbol{v}) = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \pi = \frac{\theta'}{\overline{\theta}} g \boldsymbol{k}$$

$$\theta_t + \boldsymbol{v} \cdot \nabla \theta = 0$$

$$\theta' = \theta(z) - \overline{\theta}(z)$$

Pseudo-incompressible

zero-Mach, variable density flow eqs.

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \overline{P} \nabla \pi = \frac{\theta'}{\overline{\theta}} \rho g \boldsymbol{k}$$

$$\times \quad \nabla \cdot (\overline{P} \boldsymbol{v}) = 0$$

$$\rho(z)\theta = \overline{P}\,, \qquad \theta = \overline{\theta}(z) + \theta'$$

$$\rho_t + \boldsymbol{v} \cdot \nabla \rho = 0$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho} \nabla \pi = (\rho - \overline{\rho})\, g \boldsymbol{k}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

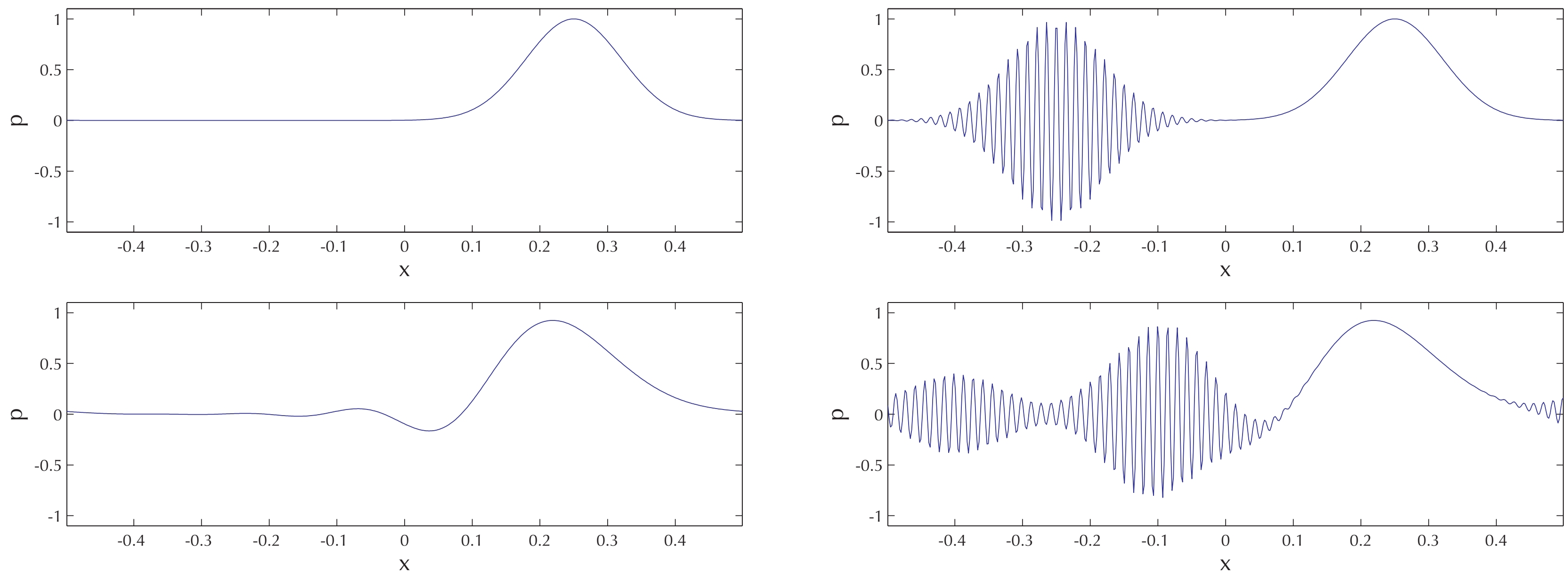
Small scale limits

Motivation ... Numerics

Why not simply solve the full compressible equations?

Linear Acoustics, simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***

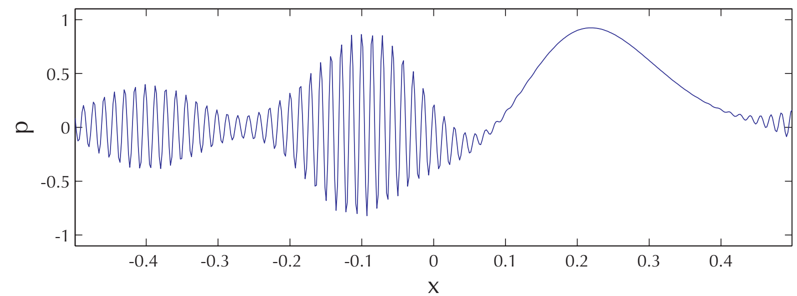


$t = 0$

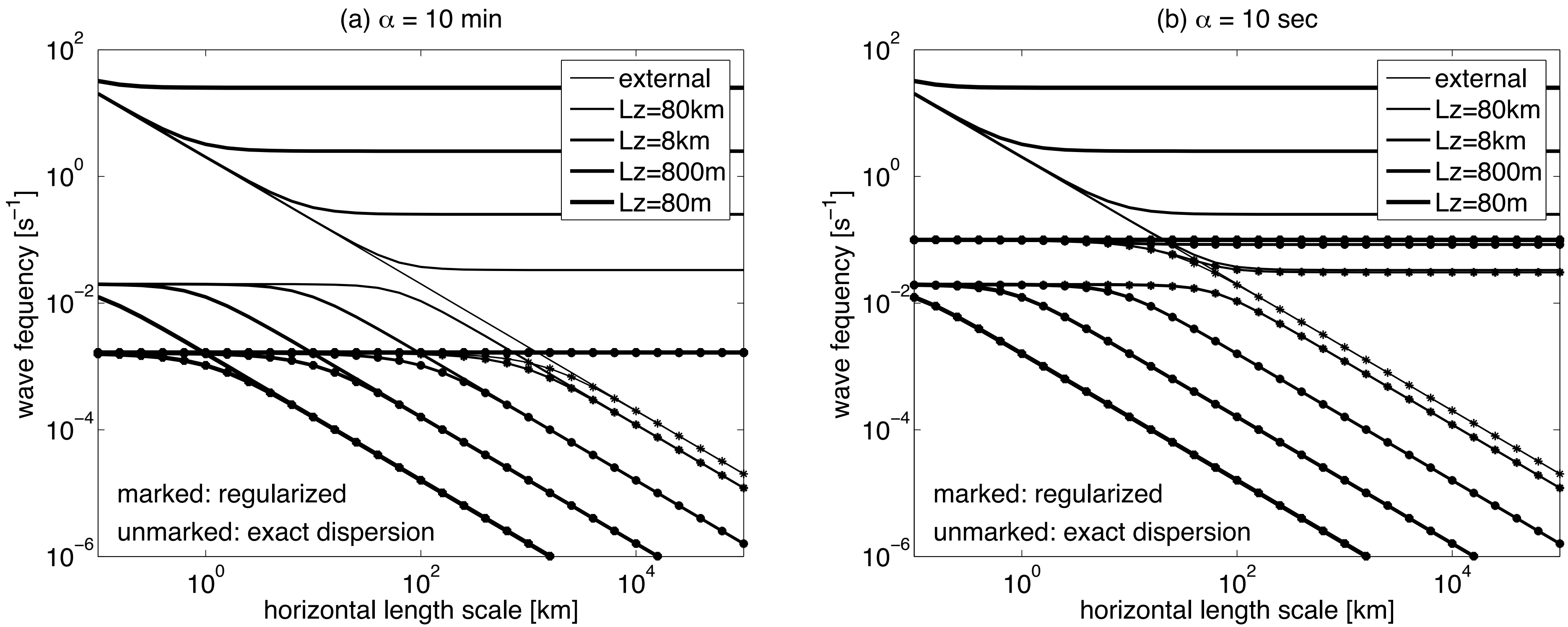
$t = 3$

Motivation ... Numerics

Why not simply solve the full compressible equations?



*



* adapted from Reich et al. (2007)

Goal

Compressible flow solver which

- properly handles long-wave dynamics
- defaults to **proper sound-proof limit** at small scales and for large time steps

Regime(s) of validity of sound-proof models

Motivation

Stratification limit in the design-regime

Wave-breaking regime with strong stratification

Scale-dependent time-integrator

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

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advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
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sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips (1962)

Regimes of Validity ... Design Regime

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\rm ref}}{h_{\rm sc}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\rm sc}}}{u_{\rm ref}}\sqrt{\dfrac{h_{\rm sc}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\rm sc}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\rm ref}/\rho_{\rm ref}}}{h_{\rm sc}} = \dfrac{\sqrt{g h_{\rm sc}}}{h_{\rm sc}}$	$\dfrac{\sqrt{g h_{\rm sc}}}{u_{\rm ref}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\rm sc}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Regimes of Validity ... Design Regime

Desirable:

1. **Sound-proof model** which

2. accurately represents the **(fast) internal waves**, and

3. remains accurate over **advective time scales**.

Regimes of Validity ... Design Regime

$$
\begin{aligned}
\tilde{\theta}_\tau \; + \; \frac{1}{\varepsilon^{\nu}} \, \tilde{w} \, \frac{d\widehat{\theta}}{dz} & & = -\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau \; + \; \frac{1}{\varepsilon^{\nu}} \, \frac{\tilde{\theta}}{\overline{\overline{\theta}}} \boldsymbol{k} \; & + \; \frac{1}{\varepsilon} \, \overline{\theta} \nabla \tilde{\pi} & = -\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\boldsymbol{v}} - \varepsilon^{1-\nu} \tilde{\theta} \nabla \tilde{\pi} \; \cdot \\
\tilde{\pi}_\tau \; & + \; \frac{1}{\varepsilon} \left(\gamma \Gamma \overline{\pi} \nabla \cdot \tilde{\boldsymbol{v}} + \tilde{w} \frac{d\overline{\pi}}{dz} \right) & = -\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\pi} - \gamma \Gamma \tilde{\pi} \nabla \cdot \tilde{\boldsymbol{v}}
\end{aligned}
$$

For the linear variable coefficient system:

✓ Conservation of weighted quadratic energy

✓ Control of time derivatives by initial data $(\tau = O(1))$

... consider internal wave scalings for $\tau = O(\varepsilon^{\nu})$:

$$
\vartheta = \frac{\tau}{\varepsilon^{\nu}} \, , \qquad \pi^* = \varepsilon^{\nu - 1} \tilde{\pi} \, ,
$$

Klainerman, Majda (1982) ... Dutrifoy, Majda, Schochet (2006,2008)

Regimes of Validity ... Design Regime

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_\vartheta + \tilde{w}\,\frac{d\overline{\theta}}{dz} = 0$$

$$\tilde{\boldsymbol{v}}_\vartheta + \frac{\tilde{\theta}}{\overline{\theta}^{\varepsilon}}\,\boldsymbol{k} + \overline{\theta}^{\varepsilon}\nabla\pi^* = 0$$

$$\varepsilon^{\mu}\,\pi^*_\vartheta + \left(\gamma\Gamma\overline{\pi}^{\varepsilon}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}^{\varepsilon}}{dz}\right) = 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix}\tilde{\theta}\\ \tilde{\boldsymbol{u}}\\ \tilde{w}\\ \pi^*\end{pmatrix}(\vartheta,\boldsymbol{x},z) = \begin{pmatrix}\Theta^*\\ \boldsymbol{U}^*\\ W^*\\ \Pi^*\end{pmatrix}(z)\;\exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Regimes of Validity ... Design Regime

Relation between compressible and pseudo-incompressible vertical modes

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}}}\frac{1}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^*$$

$\varepsilon^{\mu} = 0$: pseudo-incompressible case

regular Sturm-Liouville problem for internal wave modes *(rigid lid)*

$\varepsilon^{\mu} > 0$: compressible case

nonlinear Sturm-Liouville problem ...

$\frac{\omega^2/\lambda^2}{\overline{c}^{\varepsilon\,2}} = O(1)$: perturbations of pseudo-incompressible modes & EVals

Regimes of Validity ... Design Regime

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^{\varepsilon\,2}}}\frac{1}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}^{\varepsilon}\overline{P}^{\varepsilon}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^{\varepsilon\,2}} = O(1)\right)$

- pseudo-incompressible modes/EVals = compressible modes/EVals + $O(\varepsilon^{\mu})$ †
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

The anelastic and pseudo-incompressible models remain relevant for stratifications

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{sc}} \lesssim 40\ \mathrm{K}$$

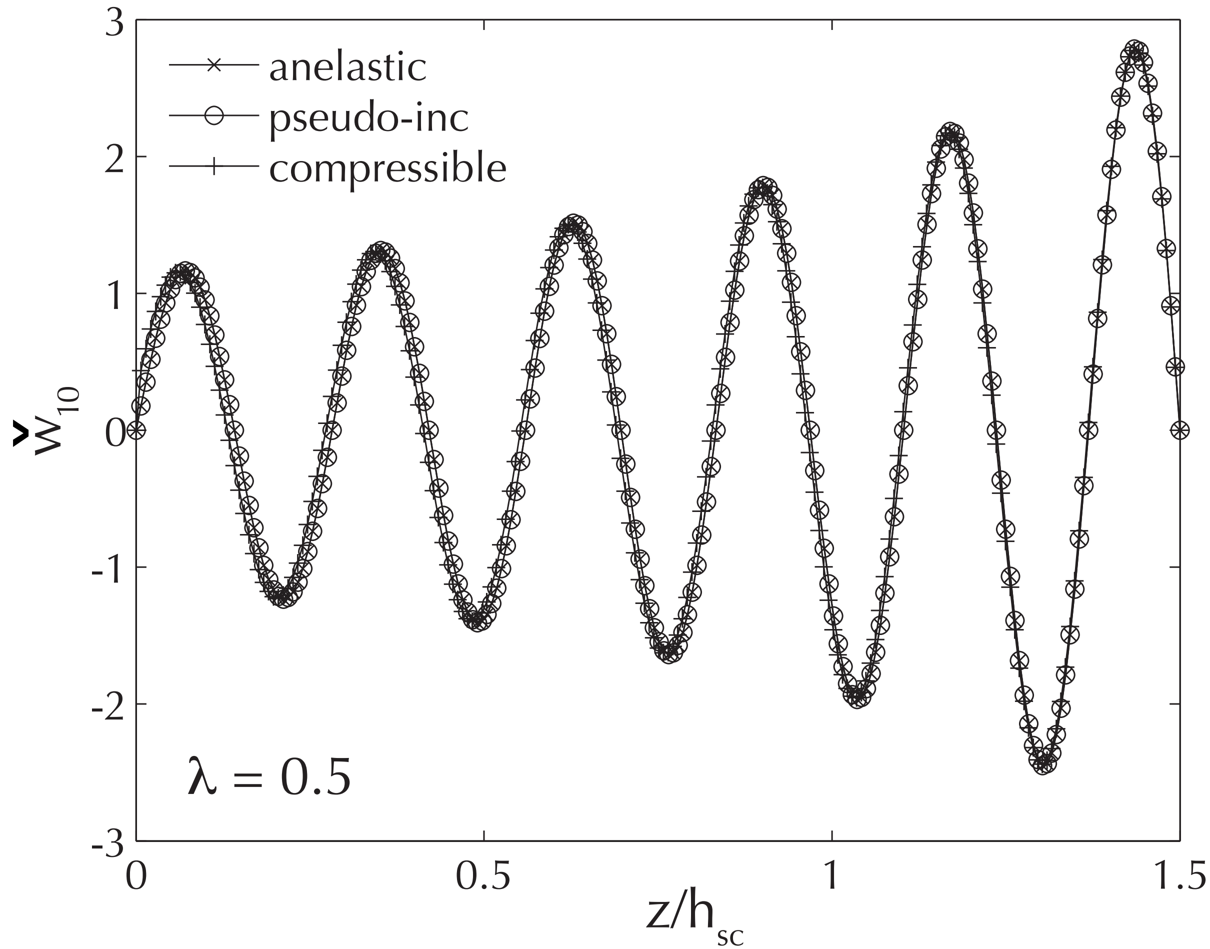
not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

R.K., U. Achatz et al., JAS, accepted (min. rev.)

† **rigorous proof with D. Bresch**

Regimes of Validity ... Design Regime

A typical vertical structure function ($L \sim \pi h_{\rm sc} \sim 30$ km)



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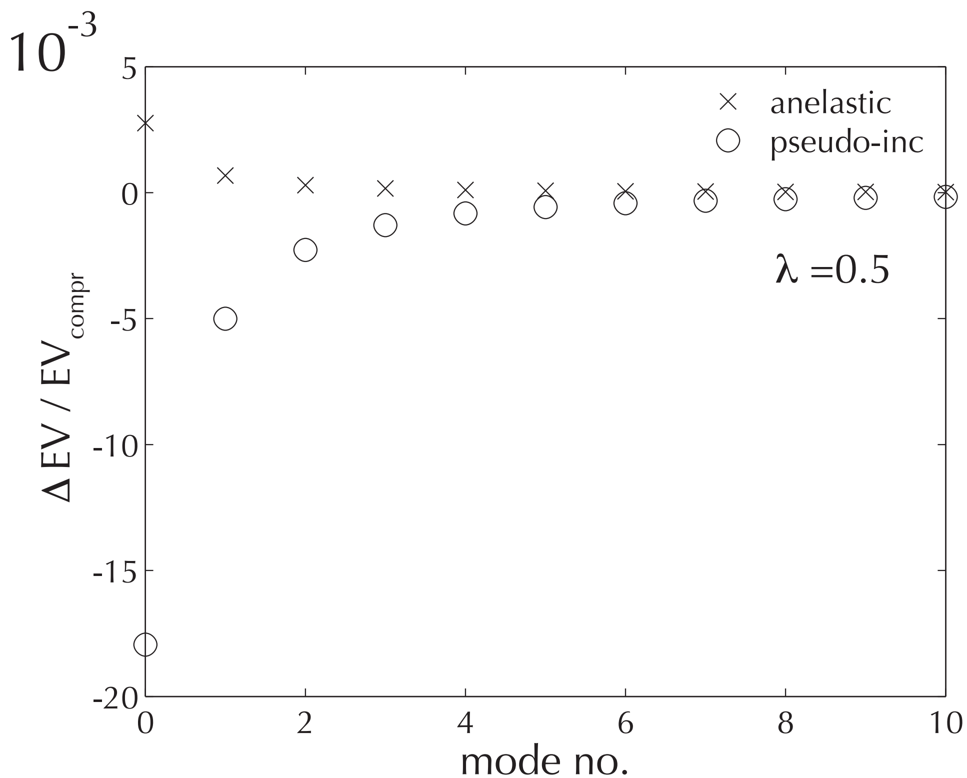
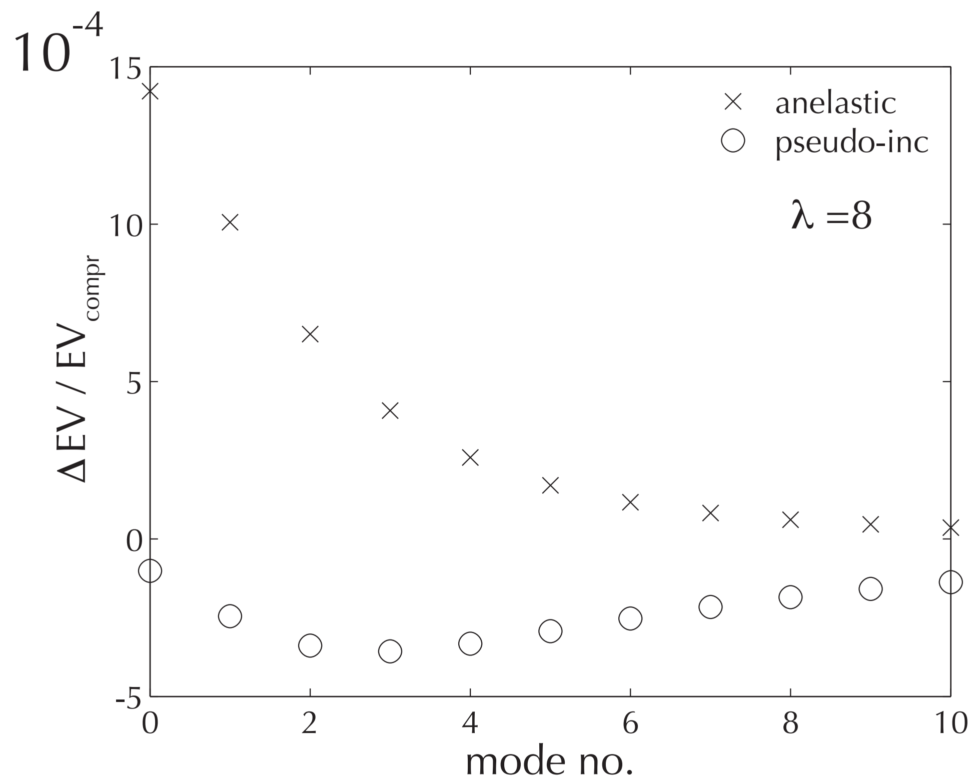
thanks to Dr. Veerle LeDoux, Ghent, for the SL-solver MATSLISE!

Regimes of Validity ... Design Regime

Relative eigenvalue errors

$$\frac{EV_{sproof} - EV_{compr}}{EV_{compr}}$$





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Regime(s) of validity of sound-proof models

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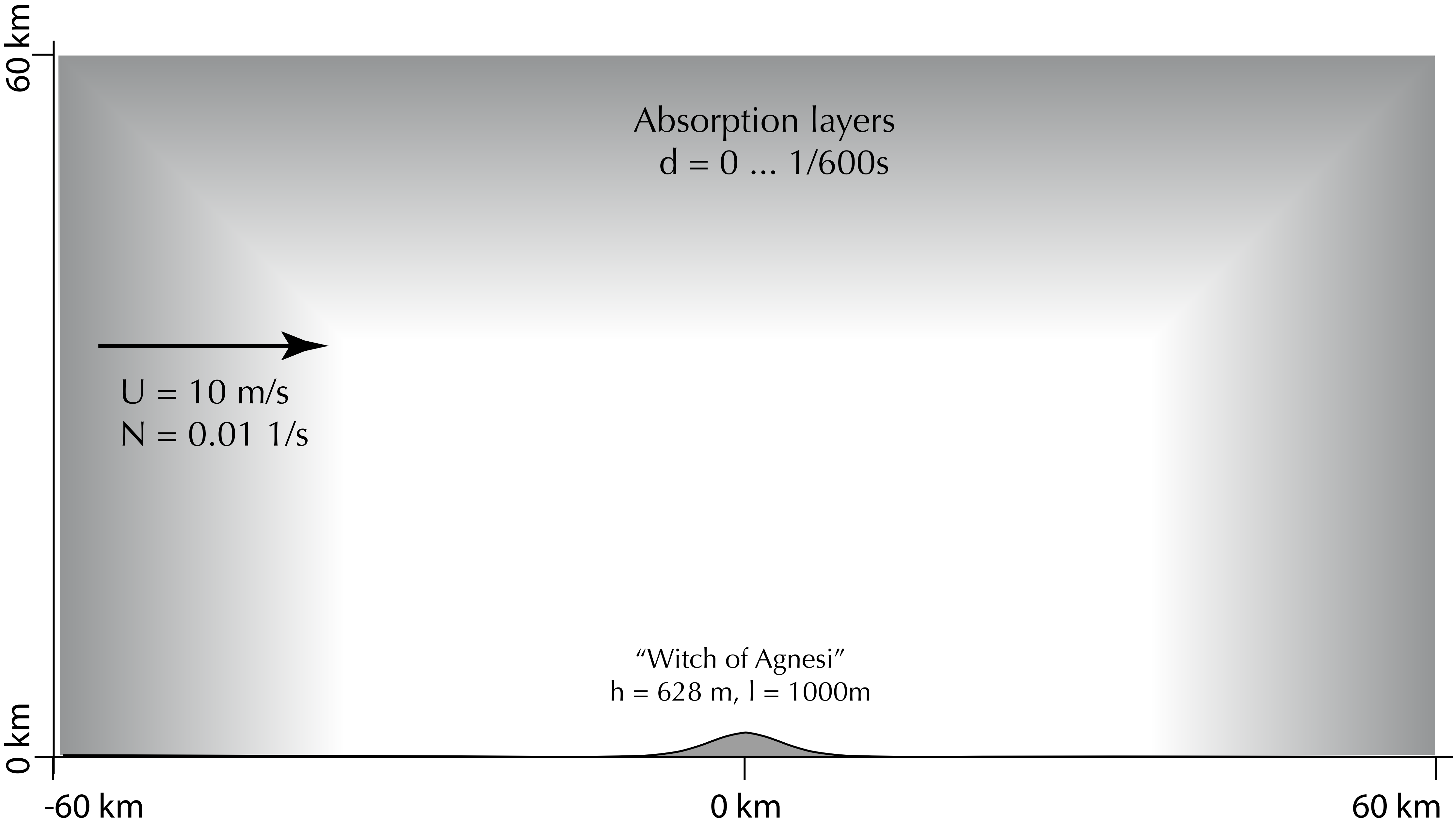
Stratification limit in the design-regime

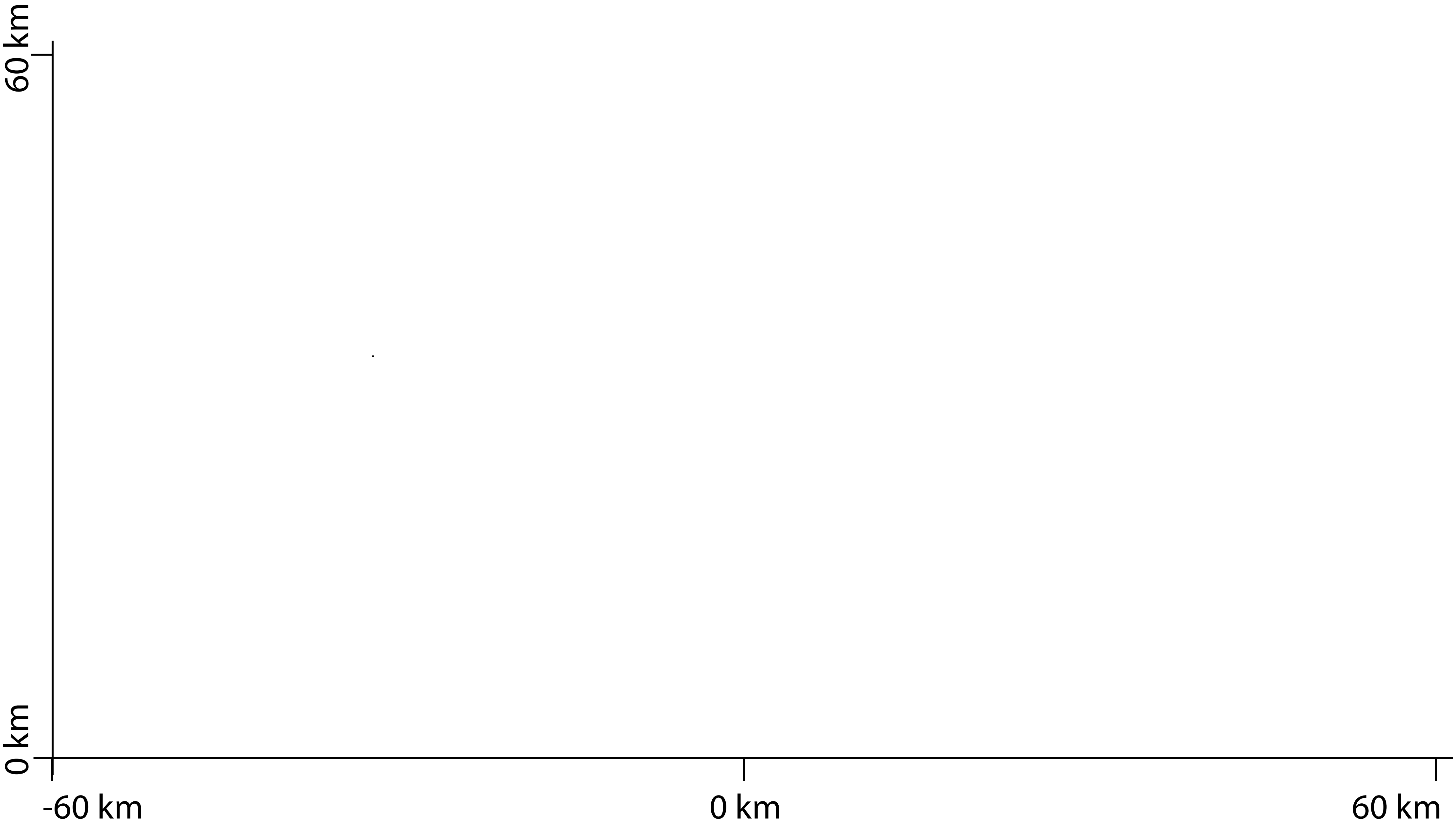
Wave-breaking regime with strong stratification

Scale-dependent time-integrator

Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Remarks

- Test case involves isothermal stratification
  at $\gamma - 1 = 1.067$: isothermal $\approx$ isentropic

- In the real atmosphere, internal waves tend to break in the stratosphere or higher up
  yet for $\gamma - 1 = O(1)$: isothermal $\not\approx$ isentropic

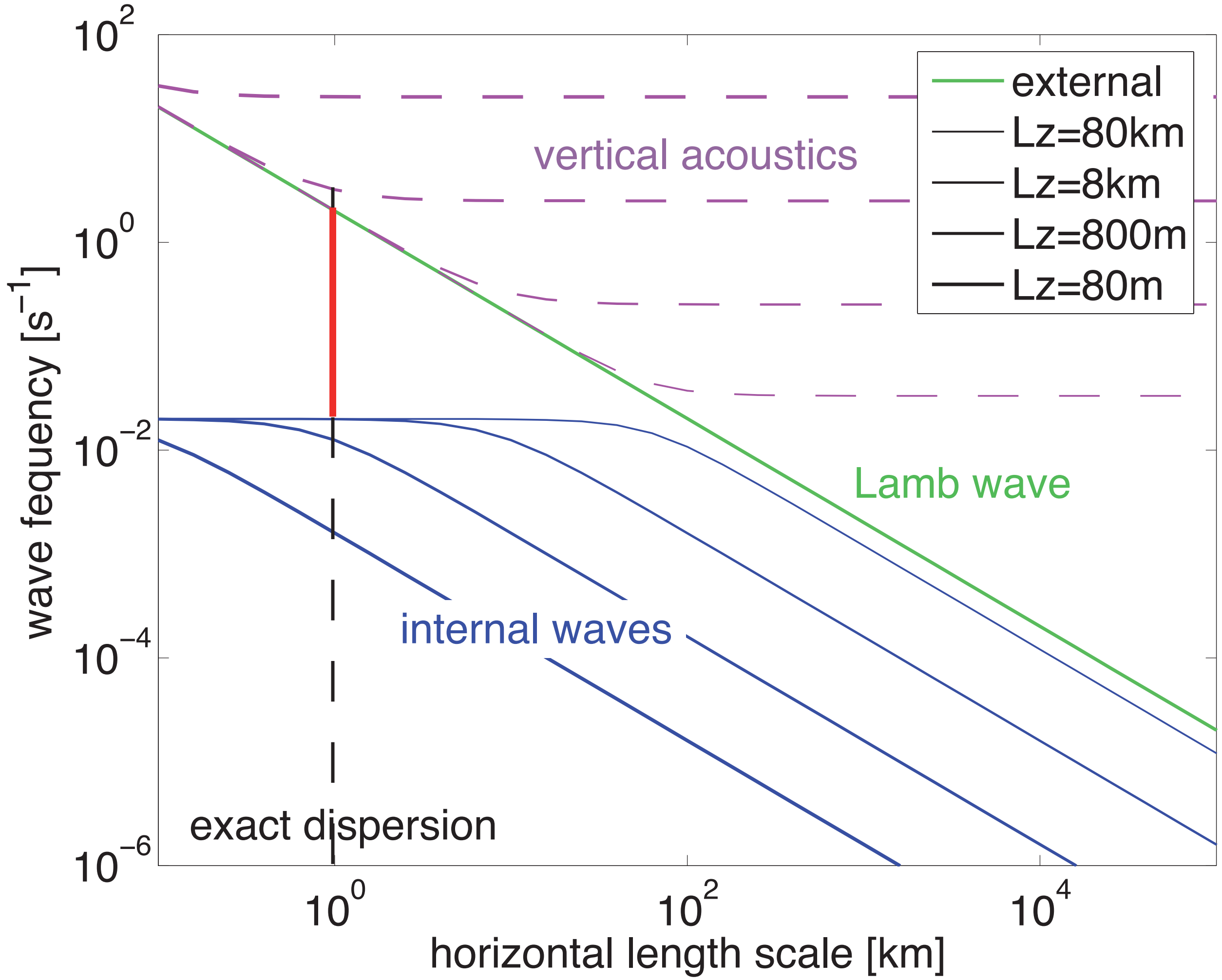
- Modifying Durran (1988)
  The pseudo-incompressible approximation should be valid whenever the time scale, T, of the relevant flow phenomena* satisfies $T \gg T_{\mathrm{sound}}$.

$$\Downarrow$$

Consider short-wavelength internal modes in a strongly stratified atmosphere

* Durran (1998): "lagrangian time scale"

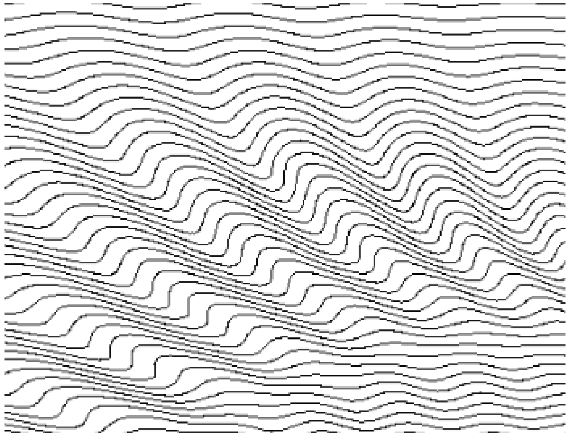
In fact there is a time scale gap for short wave lengths $L \sim 1\ \mathrm{km}$



Wave breaking regime, strong stratification

WKB theory:

- ~ 1 km wave packets
- modulated over ~ 10 km distances
- stratification of order $O(1)$
- scalings allow overturning of θ-contours



Expansion scheme:

$$U(t, \boldsymbol{x}, z; \boldsymbol{\varepsilon}) = \overline{U}(z) + U_1^{(0)} \exp\left(i\frac{\varphi^{\varepsilon}}{\boldsymbol{\varepsilon}}\right) + \boldsymbol{\varepsilon} \sum_{n=0}^{\mathbf{2}} U_n^{(1)} \exp\left(in\frac{\varphi^{\varepsilon}}{\boldsymbol{\varepsilon}}\right)$$

$$\varphi^{\varepsilon} = \varphi^{(0)} + \boldsymbol{\varepsilon}\varphi^{(1)} + o(\boldsymbol{\varepsilon})$$

$$\left(U_n^{(i)}, \varphi^{(i)}\right) \equiv \left(U_n^{(i)}, \varphi^{(i)}\right)(t, \boldsymbol{x}, z)$$

Wave breaking regime, strong stratification

Leading order: — classical Boussinesq / ray tracing theory

$$\underbrace{\begin{pmatrix} -i\hat{\omega} & 0 & 0 & ik \\ 0 & -i\hat{\omega} & -N & im \\ 0 & N & -i\hat{\omega} & 0 \\ ik & im & 0 & 0 \end{pmatrix}}_{M(\hat{\omega},k,m)} \begin{pmatrix} \hat{U}^{(0)} \\ \hat{W}^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}^{(2)} \end{pmatrix} = 0 \qquad \text{where} \qquad \left\{ \begin{aligned} \hat{\omega} &= -\frac{\partial \varphi^{(0)}}{\partial t} - k u_0^{(0)} \\ k &= \frac{\partial \varphi^{(0)}}{\partial \boldsymbol{x}} \\ m &= \frac{\partial \varphi^{(0)}}{\partial z} \end{aligned} \right.$$

Wave breaking regime, strong stratification

First order:

$$M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \dfrac{1}{\underline{N}} \dfrac{\Theta_1}{\overline{\theta}} \\ \overline{\theta} \Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \dfrac{1}{\underline{N}} \dfrac{\Theta_1}{\overline{\theta}} \\ \overline{\theta} \Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\dfrac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)} \dfrac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)} \dfrac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)} \dfrac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\dfrac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)} \dfrac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)} \dfrac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \dfrac{1}{N} \left[-\dfrac{\partial}{\partial \tau} \left(\dfrac{\Theta_1^{(1)}}{\theta^{(0)}} \right) - U_0^{(0)} \dfrac{\partial}{\partial \chi} \left(\dfrac{\Theta_1^{(1)}}{\theta^{(0)}} \right) \right] \\ -\dfrac{\partial U_1^{(0)}}{\partial \chi} - \dfrac{\partial W_1^{(0)}}{\partial \zeta} - \dfrac{1-\kappa}{\kappa} \dfrac{W_1^{(0)}}{\pi^{(0)}} \dfrac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

Wave breaking regime, strong stratification

First order:

$$M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

- First-order Hamilton-Jacobi-eqn. for $\varphi^{(1)}$

Wave breaking regime, strong stratification

First order: — **pseudo-incompressible** corrections

$$M^{(0)}\begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)}\begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}$$

- **pseudo-incompressible** wave action conservation law

Wave breaking regime, strong stratification

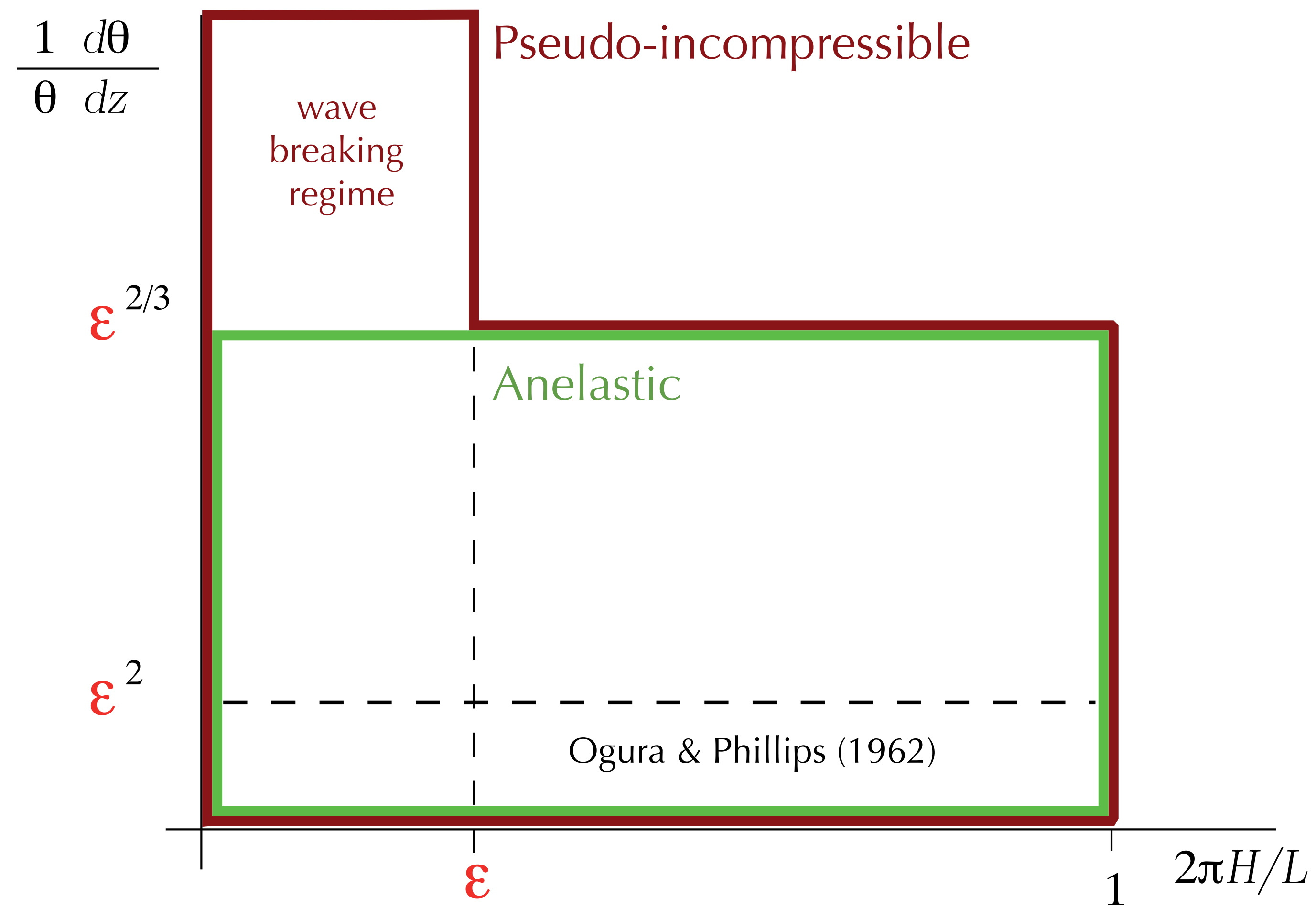
First order: — **pseudo-incompressible** corrections

$$
M^{(0)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(1)} + M^{(1)} \begin{pmatrix} U_1 \\ W_1 \\ \frac{1}{N}\frac{\Theta_1}{\overline{\theta}} \\ \overline{\theta}\Pi_1 \end{pmatrix}^{(0)} = \begin{pmatrix} -\frac{\partial U_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial U_1^{(0)}}{\partial \chi} - W_1^{(0)}\frac{\partial U_0^{(0)}}{\partial \zeta} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \chi} \\ -\frac{\partial W_1^{(0)}}{\partial \tau} - U_0^{(0)}\frac{\partial W_1^{(0)}}{\partial \chi} - \theta^{(0)}\frac{\partial \Pi_1^{(2)}}{\partial \zeta} \\ \frac{1}{N}\left[-\frac{\partial}{\partial \tau}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right) - U_0^{(0)}\frac{\partial}{\partial \chi}\left(\frac{\Theta_1^{(1)}}{\theta^{(0)}}\right)\right] \\ -\frac{\partial U_1^{(0)}}{\partial \chi} - \frac{\partial W_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{W_1^{(0)}}{\pi^{(0)}}\frac{\partial \pi^{(0)}}{\partial \zeta} \end{pmatrix}
$$

$+$ **Nonlinear Effects:**

Explicit solutions for all higher-oder modes $\sim \exp\left(i\, n\, \varphi^{(1)}/\varepsilon\right)$, $\qquad (n = 1, 2, \ldots)$

Regimes ... Summary



The pseudo-incompressible model wins by a small margin

Regimes ... Summary

Compressible flow equations

$$\boldsymbol{\rho_t} + \nabla\cdot(\rho\boldsymbol{v}) \;=\; 0$$

$$(\rho\boldsymbol{u})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{u}) + P\nabla_{\|}\pi \;=\; 0$$

$$(\rho w)_t \;+ \nabla\cdot(\rho\boldsymbol{v}w) + P\pi_z \;=\; -\rho g$$

$$\boldsymbol{P_t} + \nabla\cdot(P\boldsymbol{v}) \;=\; 0$$

drop term for:

anelastic (approx.)

pseudo-incompressible

$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k}\,, \quad (\boldsymbol{u}\cdot\boldsymbol{k}\equiv 0)$$

Regime(s) of validity of sound-proof models

- Motivation
- Stratification limit in the design-regime
- Wave-breaking regime with strong stratification

Scale-dependent time-integrator

Thanks to ...

Omar Knio	(Johns Hopkins University, Baltimore)
Piotr Smolarkiewicz	(NCAR, Boulder)
Stefan Vater	(FU Berlin)

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Numerics

Why not simply solve the full compressible equations?

Competing approaches: **model codes**

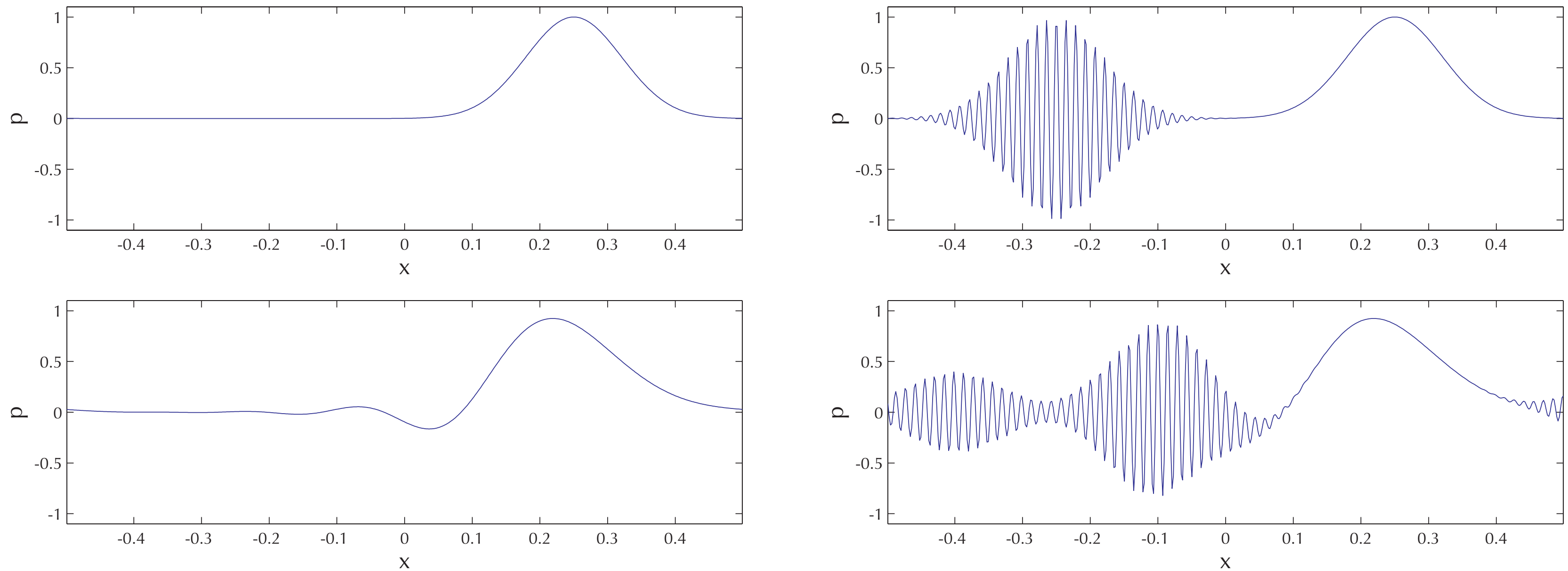
- Split-explicit / multi-rate methods, e.g.,
  - Runge-Kutta (slow) + forward-backward (fast), e.g.,
    Wicker & Skamarock, MWR, (98), ... ; *MM5, LM, WRF ...*
  - Multirate infinitesimal schemes, peer methods
    Wensch et al., BIT, (09); *ASAM, ...*
- Semi-implicit / linearly implicit schemes
  - explicit advection, damped 2nd or 1st-order schemes for fast modes, e.g.,
    Robert, Japan Met. J., (69), ... ; *UKMO, ...*
  - linearly implicit Rosenbrock-type methods, e.g.,
    Reisner et al., MWR, (05), ...; *ASAM, LANL Hurricane model, ...*
- Fully implicit integration

Numerics

Why not simply solve the full compressible equations?

Simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***



$t = 0$

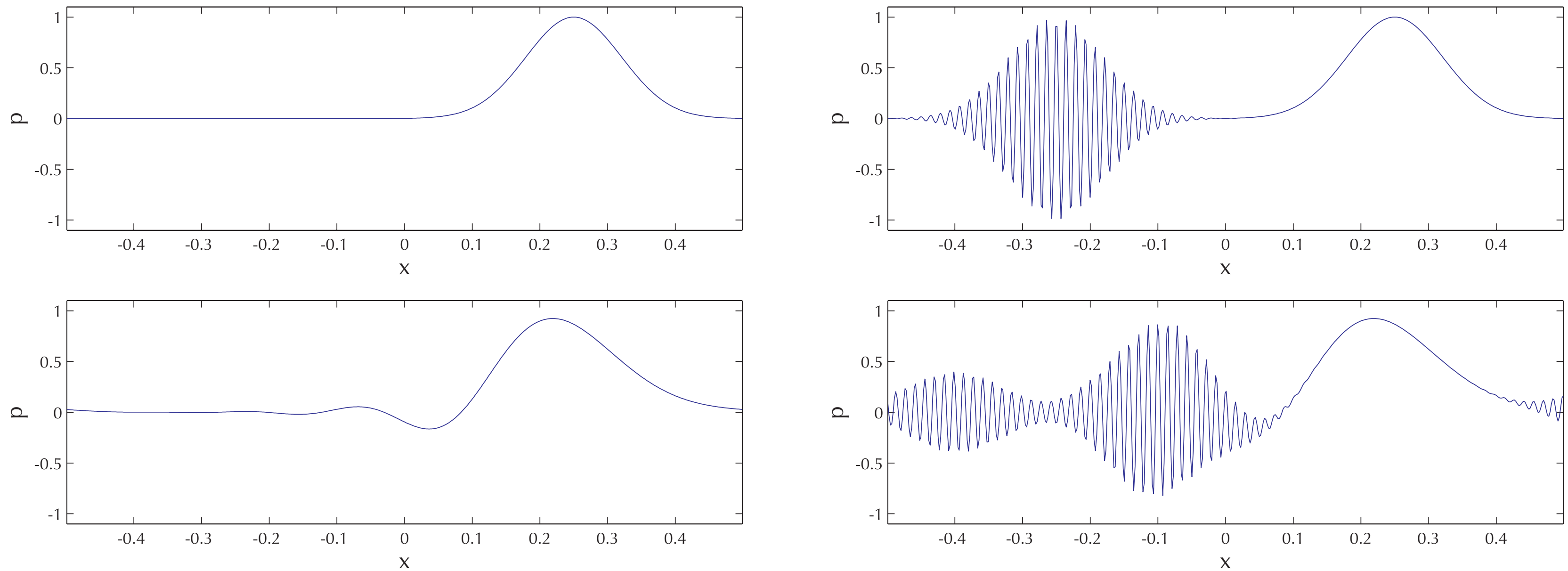
$t = 3$

Numerics

Why not simply solve the full compressible equations?

Simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 *grid pts.,* $\mathrm{CFL} = 10$*)*



$t = 0$

$t = 3$

Ideas:

- Slave short waves $(c\Delta t/\ell > 1)$ to long waves $(c\Delta t/\ell \leq 1)$
- with pseudo-incompressible limit behavior

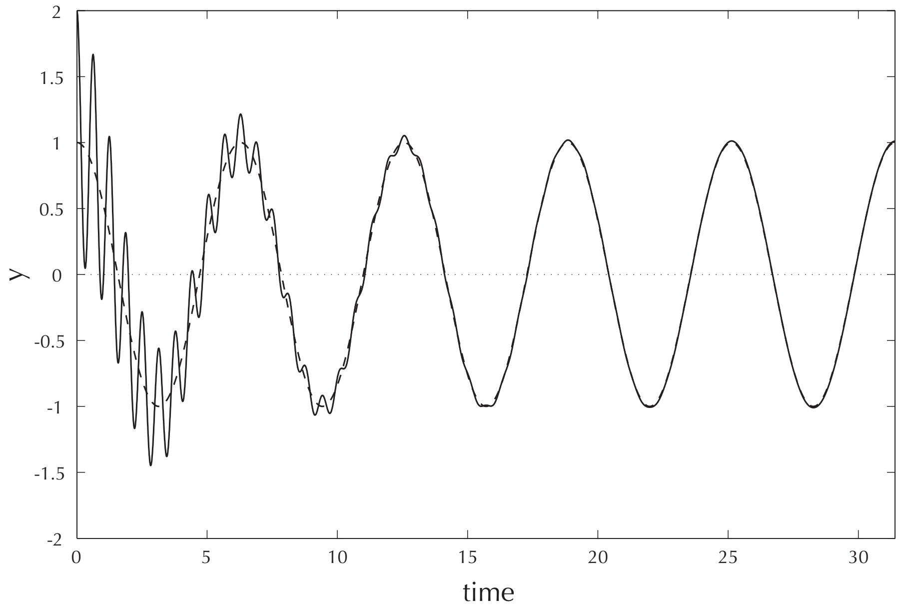
"super-implicit" scheme

non-standard multi grid

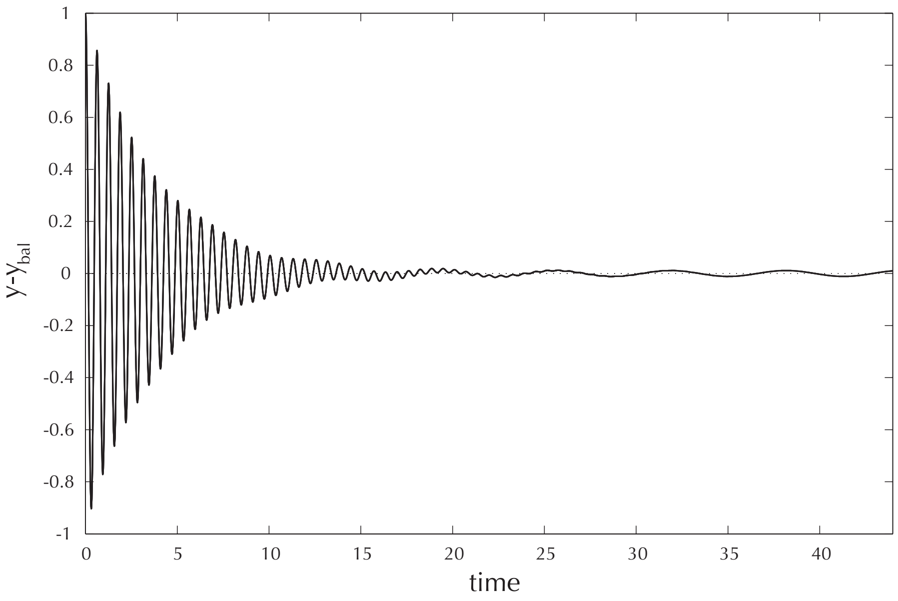
projection method

Numerics

$$\varepsilon \ddot{y} + \varepsilon \kappa \dot{y} + y = \cos(t)\,, \qquad \begin{cases} y(0) = 1 + a \\ \dot{y}(0) = 0 \end{cases}, \qquad (\varepsilon = 0.01)$$



$y(t)$



$y(t) - \cos(t)$

Numerics

$$\varepsilon\ddot{y} + \varepsilon\kappa\dot{y} + y = \cos(t)$$

Slow-time asymptotics for $\varepsilon \ll 1$:

$$y(t) = y^{(0)}(t) + \varepsilon y^{(1)}(t) + \ldots\,, \qquad \begin{aligned} y^{(0)}(t) &= \cos(t) \\ y^{(1)}(t) &= -(\ddot{y}^{(0)} + \kappa\dot{y}^{(0)})(t) \end{aligned}$$

Associated "super-implicit" discretization *(extreme BDF)*:

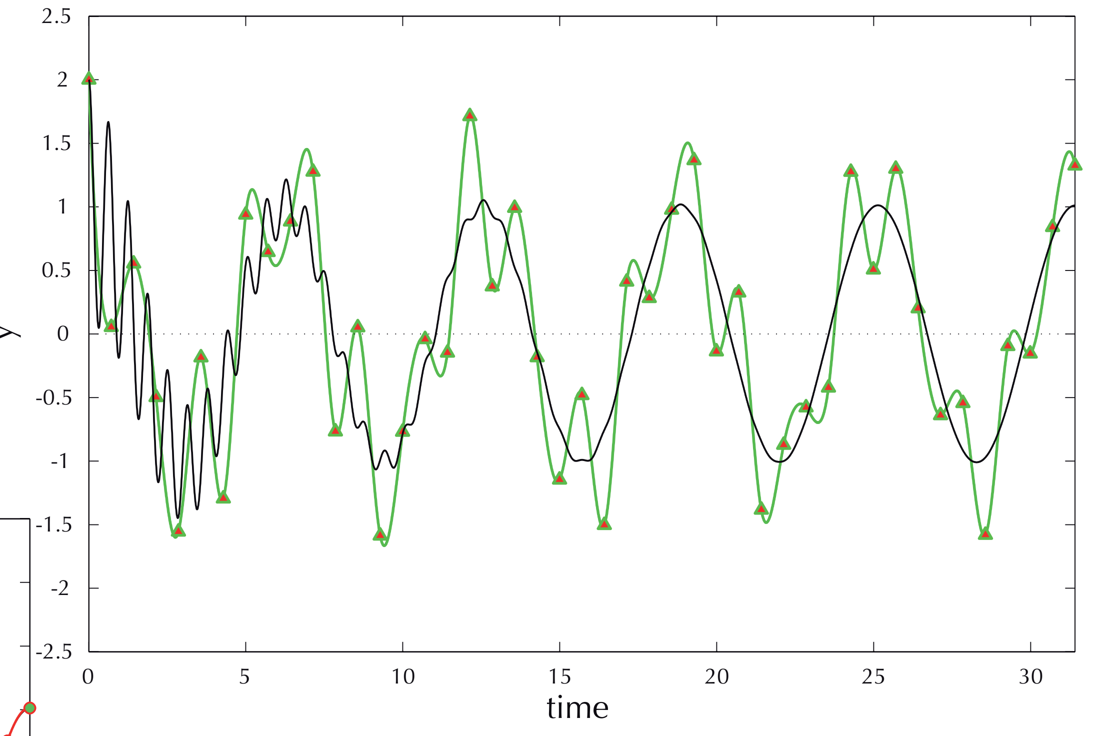
$$\begin{aligned} y^{n+1} &= \cos(t^{n+1}) - \varepsilon\left[(\delta_t + \kappa)\ \dot{y}\right]^{*,n+1} \\ \dot{y}^{n+1} &= \frac{1}{\Delta t}\left(y^{n+1} - y^n + \frac{1}{2}\left(y^{n+1} - 2y^n + y^{n-1}\right)\right) \end{aligned}$$

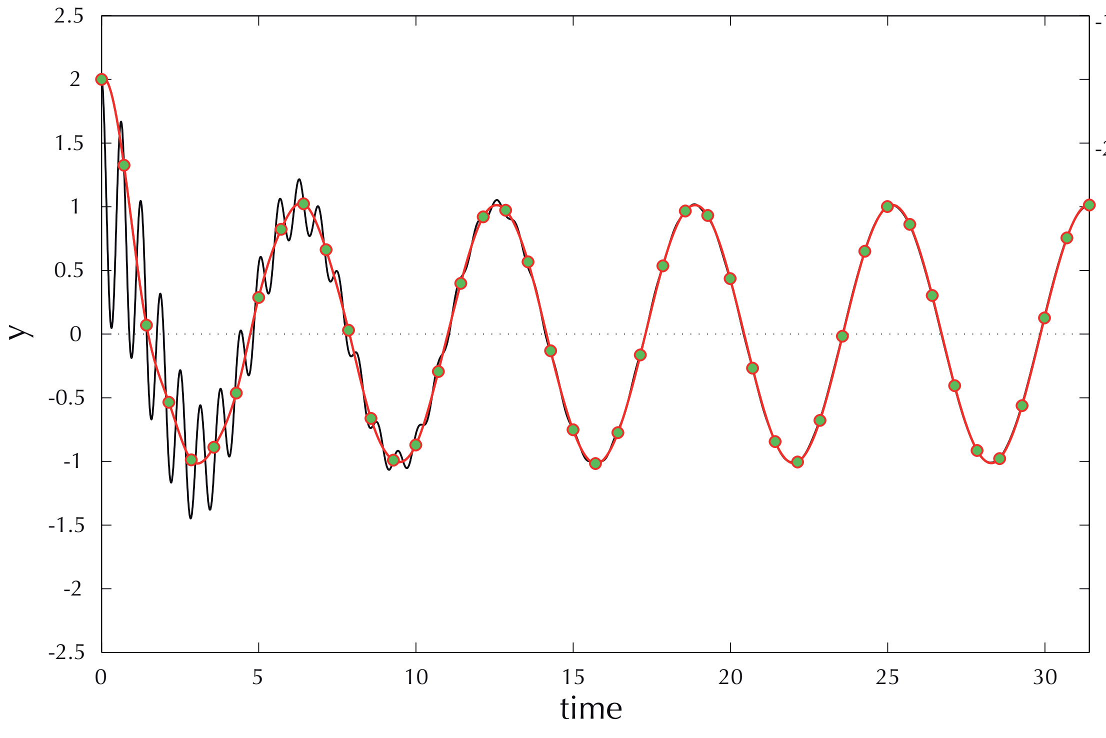
where

$$\begin{aligned} u^{*,n+1} &= 2u^n - u^{n-1} \\ (\delta_t u)^{*,n+1} &= \frac{1}{\Delta t}\left(u^n - u^{n-1} + \frac{3}{2}(u^n - 2u^{n-1} + u^{n-2})\right) \end{aligned}$$

Numerics

Implicit midpoint rule $\Delta t = 7\sqrt{\varepsilon}$

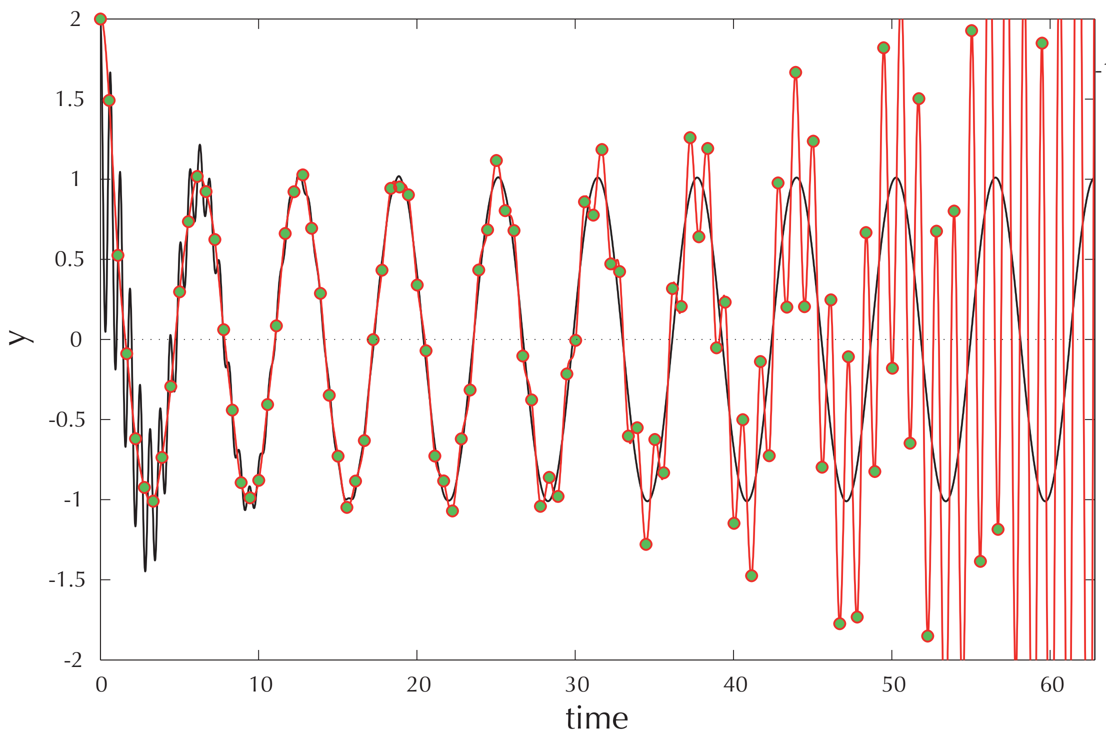


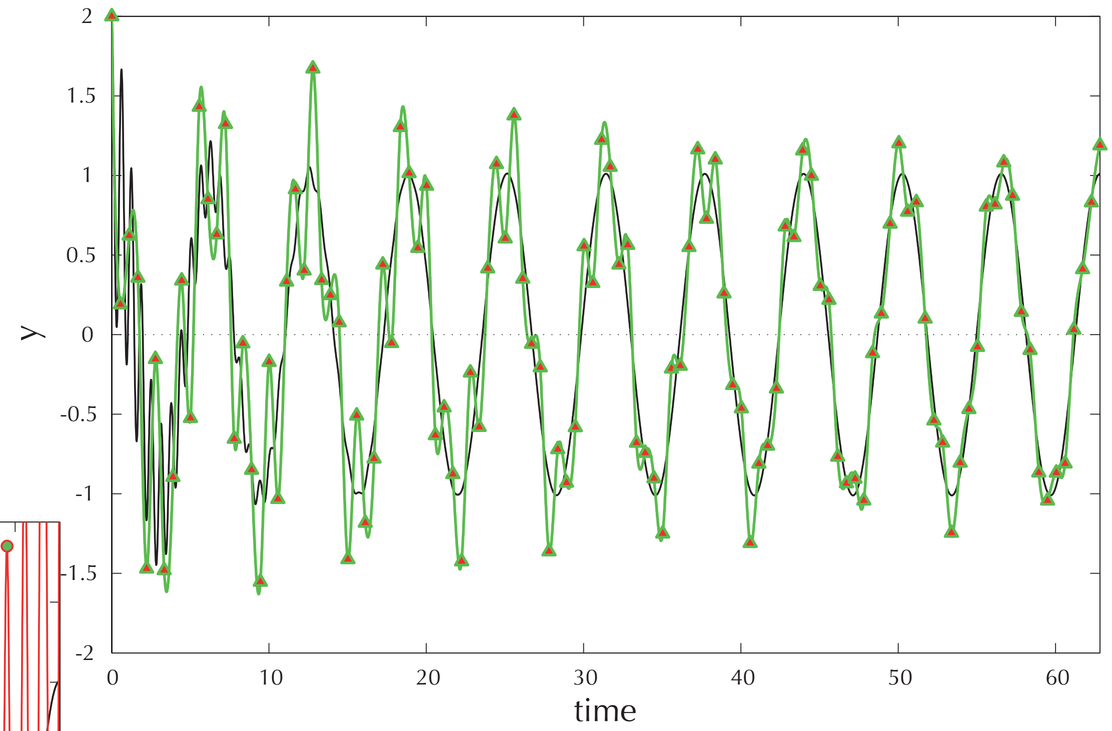


$\Delta t = 7\sqrt{\varepsilon}$ Super-implicit scheme

Numerics

Implicit midpoint rule $\Delta t = 5.55\sqrt{\varepsilon}$



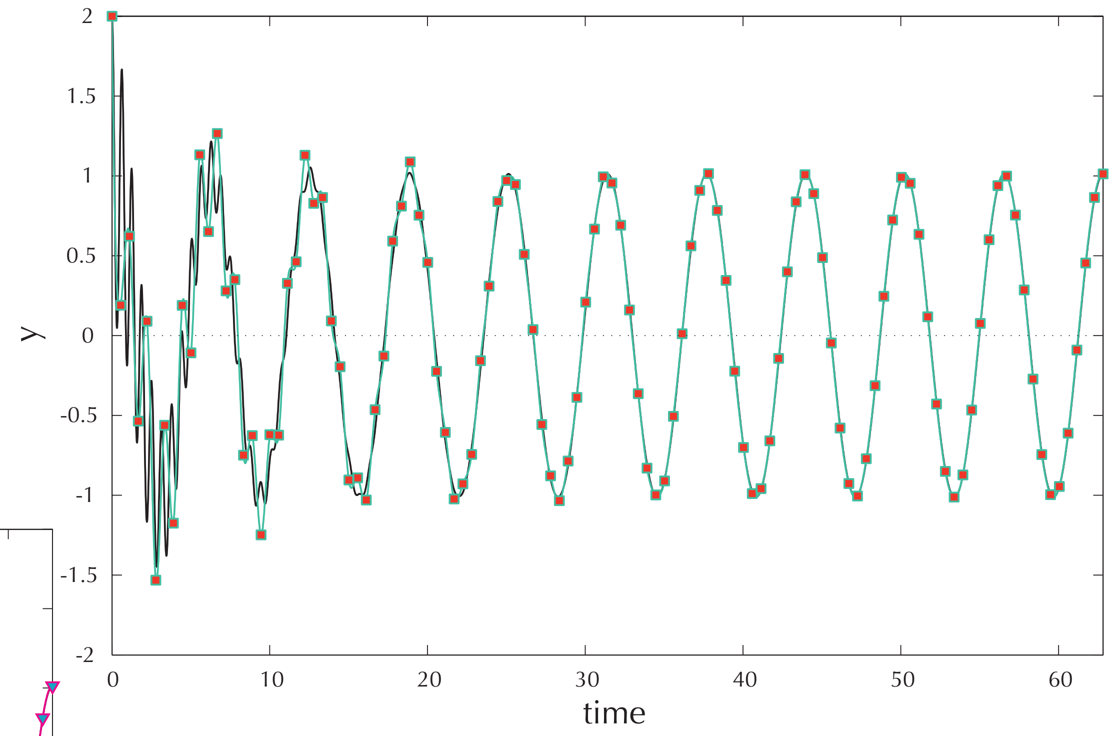


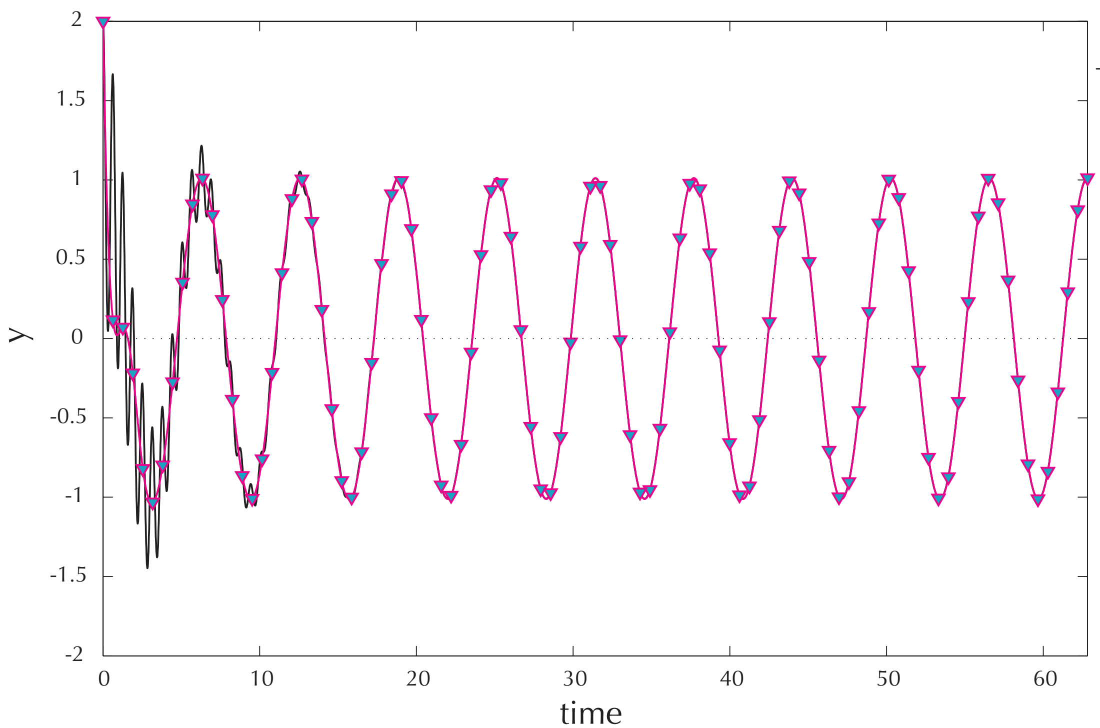
$\Delta t = 5.55\sqrt{\varepsilon}$ Super-implicit scheme

Numerics

Blended scheme $\Delta t = 5.55\sqrt{\varepsilon}$

$$\Delta y\big|_{\mathrm{BL}} = \eta\, \Delta y\big|_{\mathrm{IMP}} + (1-\eta)\, \Delta y\big|_{\mathrm{SU}}$$





$\Delta t = 5.55\sqrt{\varepsilon}$ BDF2 – for comparison

Numerics

Compressible flow equations:

$$\begin{aligned}
\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + P \nabla \pi &= -\rho g \boldsymbol{k} \\
\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}$$

$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R$$

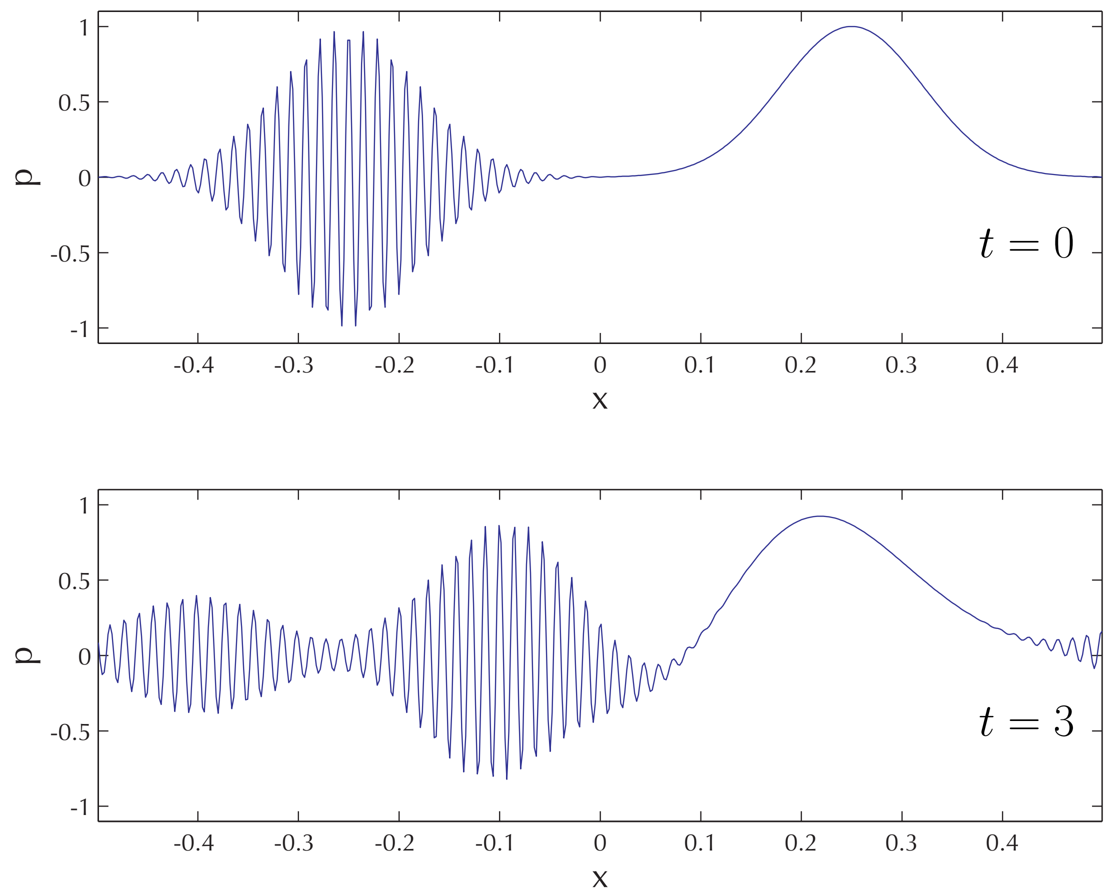
Numerics

For starters: 1D Linear acoustics:

$$u_t + p_x = 0$$
$$p_t + c^2 u_x = 0$$

Desired:

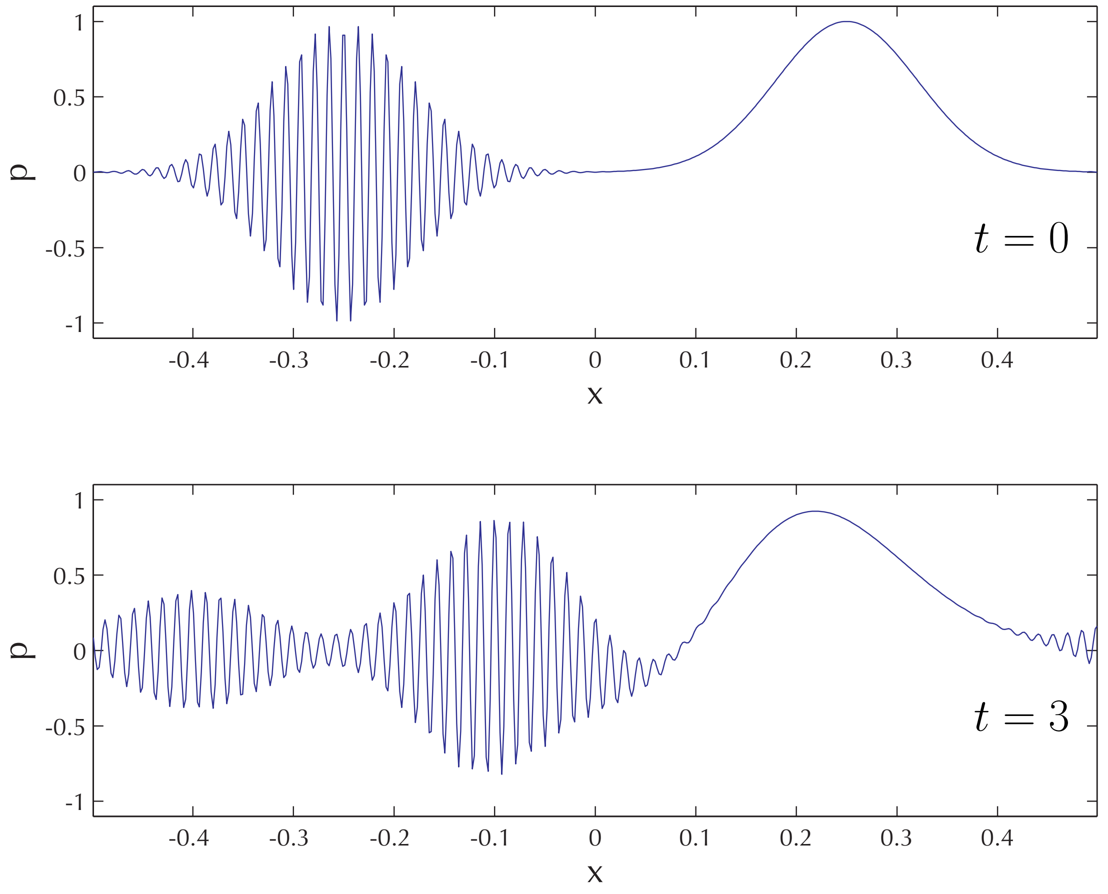
- remove underresolved modes
- minimize dispersion for marginally resolved modes



Numerics

1D Linear acoustics:

$$u_t + p_x = 0$$
$$p_t + c^2 u_x = 0$$



Desired:

- remove underresolved modes
- minimize dispersion for marginally resolved modes

Strategy:

scale-dependent IMP-SU-Blended scheme via **multi grid**

Numerics

Implicit mid-point rule for linear acoustics

$$\frac{u^{n+1}-u^n}{\Delta t}+\frac{\partial}{\partial x}p^{n+\frac{1}{2}}=0\,,\qquad \frac{p^{n+1}-p^n}{\Delta t}+c^2\,\frac{\partial}{\partial x}u^{n+\frac{1}{2}}=0$$

with

$$X^{n+\frac{1}{2}}=\frac{1}{2}\left(X^{n+1}+X^n\right)$$

Implicit problem for half-time fluxes

$$u^{n+\frac{1}{2}}=u^n-\frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}\,,\qquad p^{n+\frac{1}{2}}=p^n-\frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}}$$

Eliminate $u^{n+\frac{1}{2}}$

$$\left(1-\frac{c^2\Delta t^2}{4}\frac{\partial^2}{\partial x^2}\right)p^{n+\frac{1}{2}}=p^n-\frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^n$$

Numerics

Implicit mid-point rule $\Rightarrow$ **super-implicit**

$$u^{n+\frac{1}{2}} = u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}}$$

$$\underline{p^{n+\frac{1}{2}}} = \underline{p^n} - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}}$$

key step:

$$\begin{aligned} u^{n+\frac{1}{2}} &= u^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}p^{n+\frac{1}{2}} \\ &= - \frac{c^2\Delta t}{2}\frac{\partial}{\partial x}u^{n+\frac{1}{2}} - \frac{\Delta t}{2}\left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}} \end{aligned}$$

Pressure **"projection"** equation

$$\frac{c^2\Delta t}{2}\frac{\partial^2}{\partial x^2}p^{n+\frac{1}{2}} = c^2\frac{\partial}{\partial x}u^n + \left(\frac{\partial p}{\partial t}\right)^{\mathbf{BD},n+\frac{1}{2}}$$

Scale-dependence via multi-grid

$$p = \sum_{j=1}^{J} p^{(j)}$$

where

$$p^{(j)} = (1 - P \circ R)\ R^{j-1} p \qquad \text{with} \qquad \begin{array}{ll} R: & \text{MG restriction} \\ P: & \text{MG prolongation} \end{array}$$

scale-dependent blending

$$u^{n+\frac{1}{2}} = u^{n} - \frac{\Delta t}{2} \frac{\partial}{\partial x} p^{n+\frac{1}{2}}$$

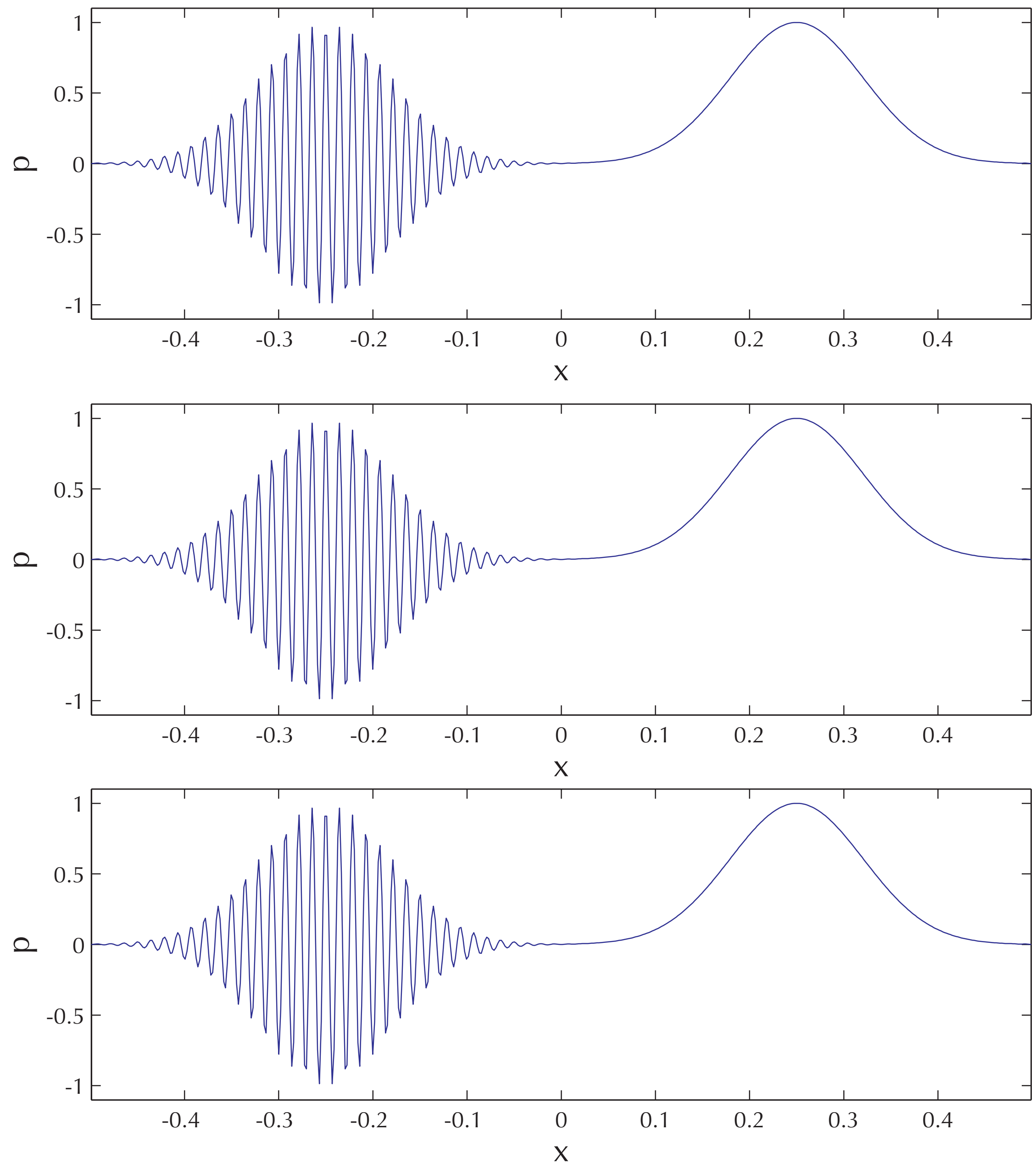
$$\sum_j \boldsymbol{\eta}^{(j)} {p^{(j)}}^{n+\frac{1}{2}} = \sum_j \boldsymbol{\eta}^{(j)} {p^{(j)}}^{n} - \frac{c^2 \Delta t}{2} \frac{\partial}{\partial x} u^{n+\frac{1}{2}} - \sum_j (1 - \boldsymbol{\eta}^{(j)}) \frac{\Delta t}{2} \left(\frac{\partial p^{(j)}}{\partial t}\right)^{\mathbf{BD}, n+\frac{1}{2}}$$

Numerics

implicit midpoint

new scheme

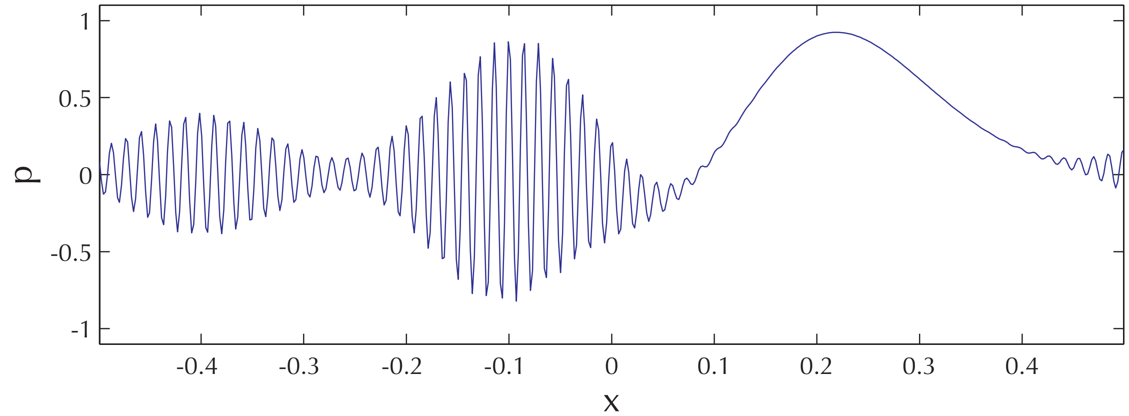
BDF2



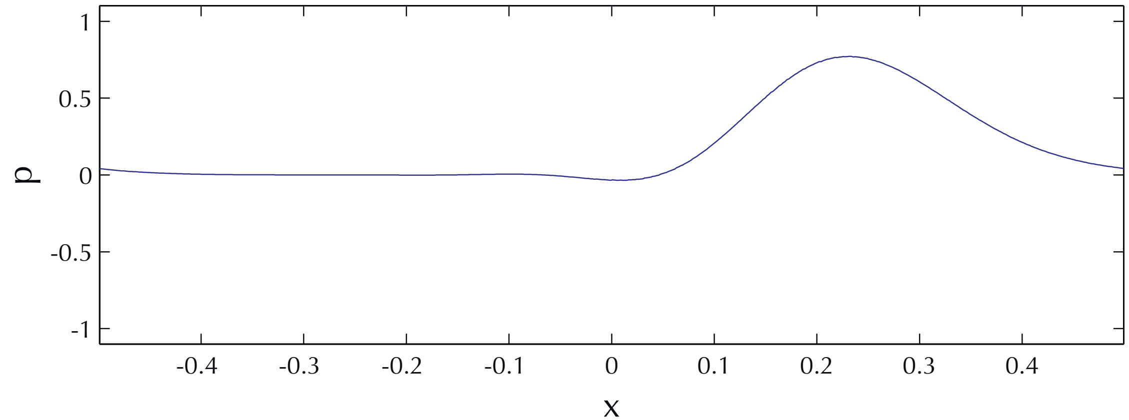
$t = 0$

Numerics

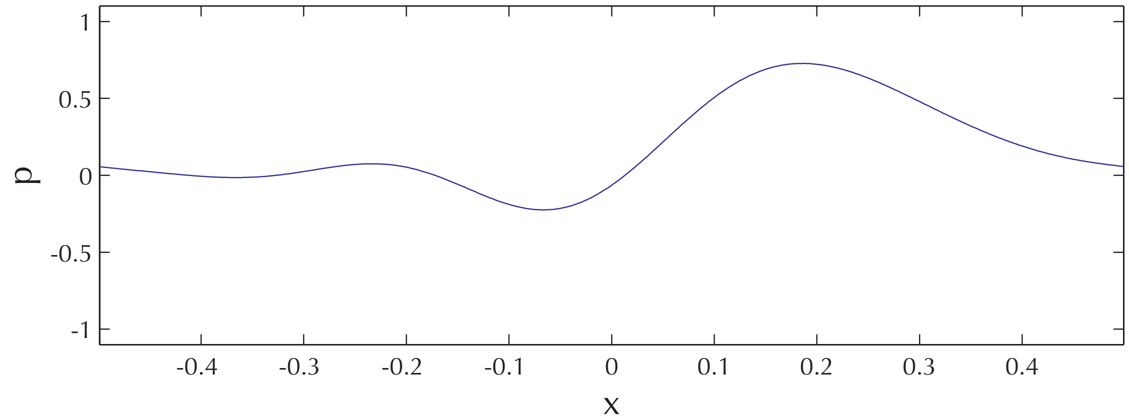
implicit midpoint



new scheme



BDF2

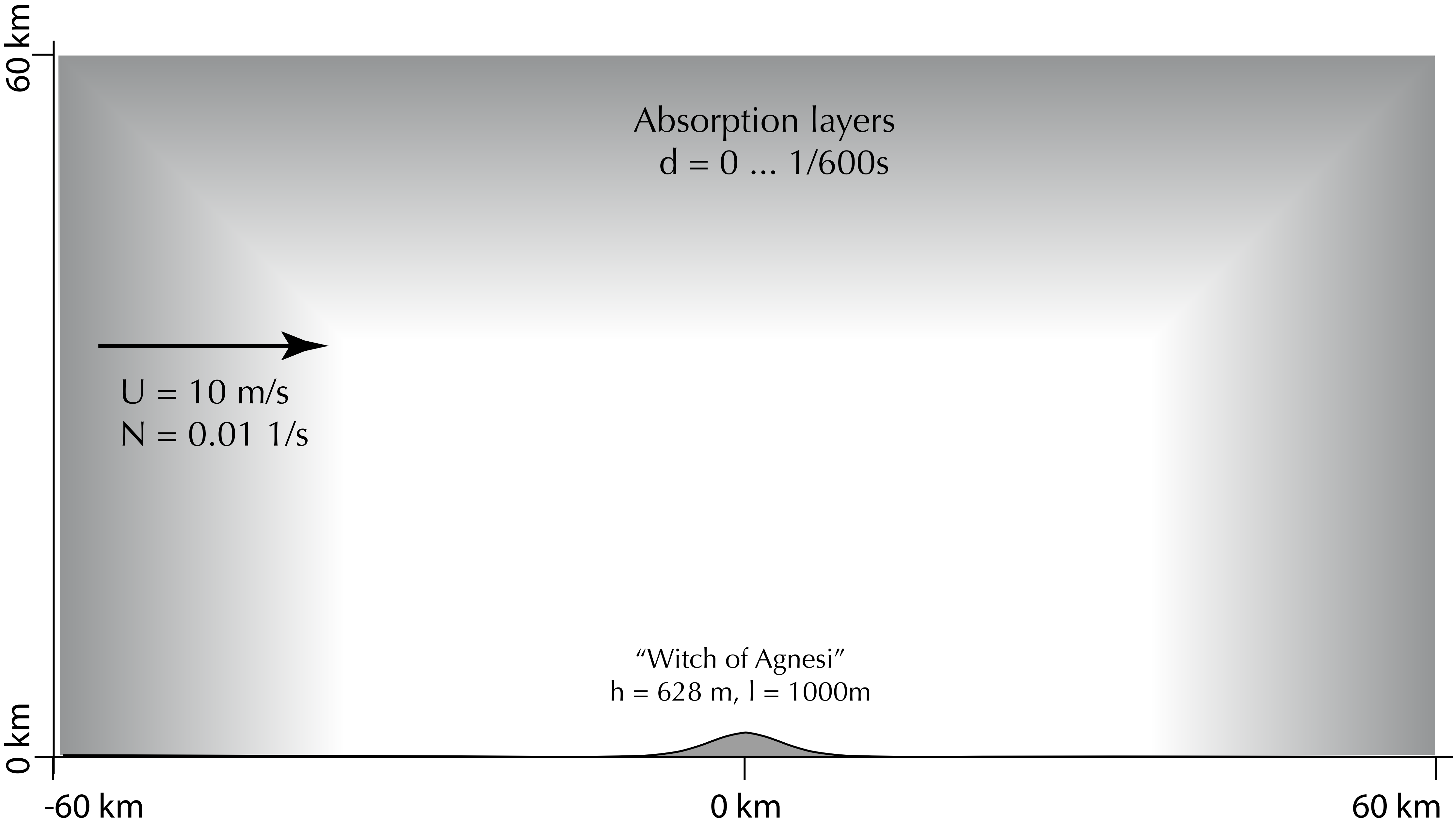


$t = 3$

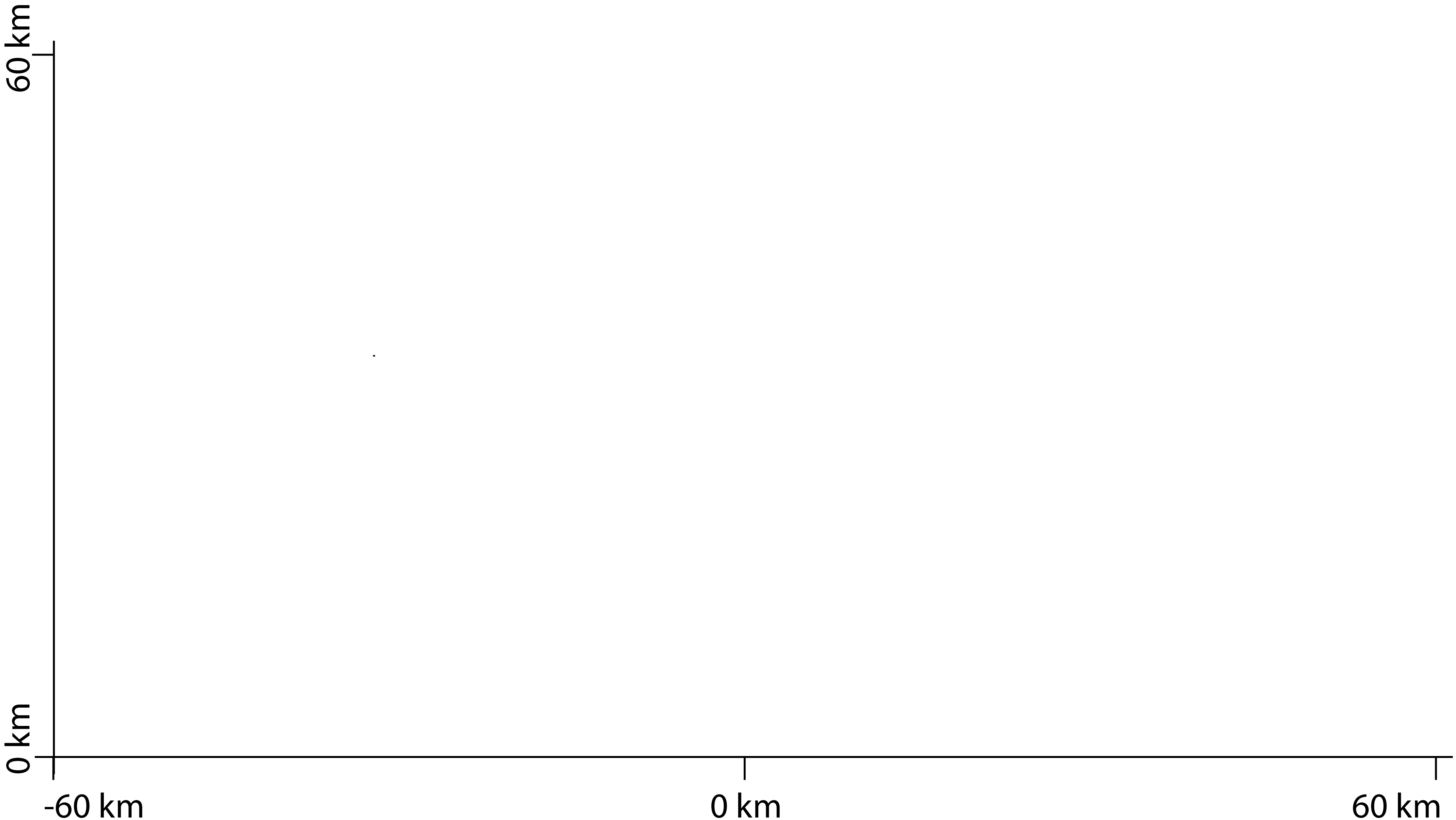
Preliminary results with implicit midpoint

(without IMP-SU–blending)

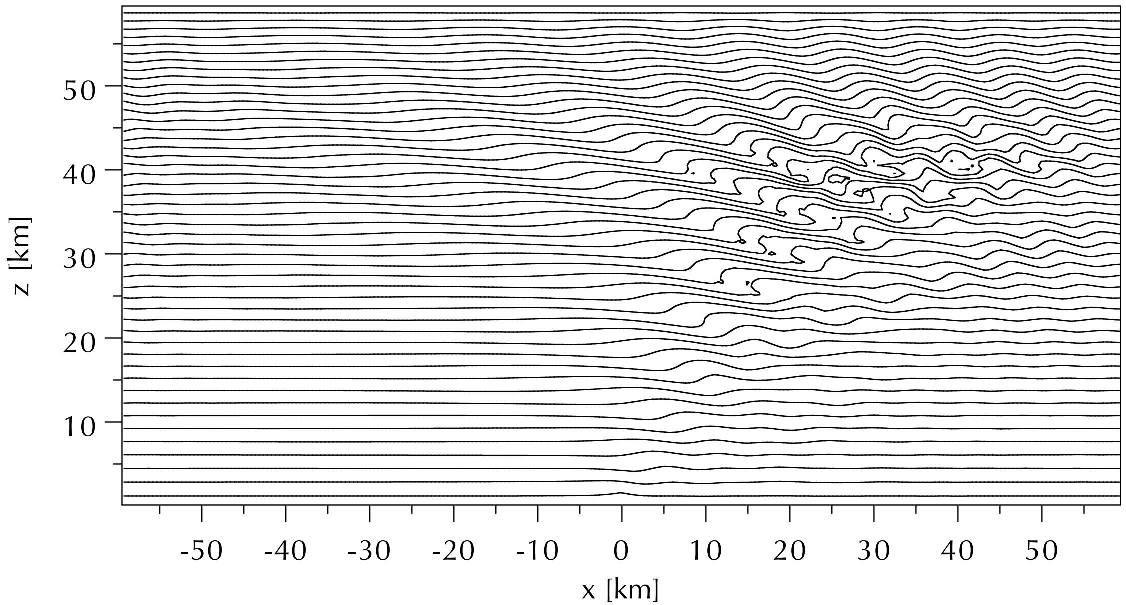
Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



pseudo-incompressible

3 hours

sharpened van Leer's limiter

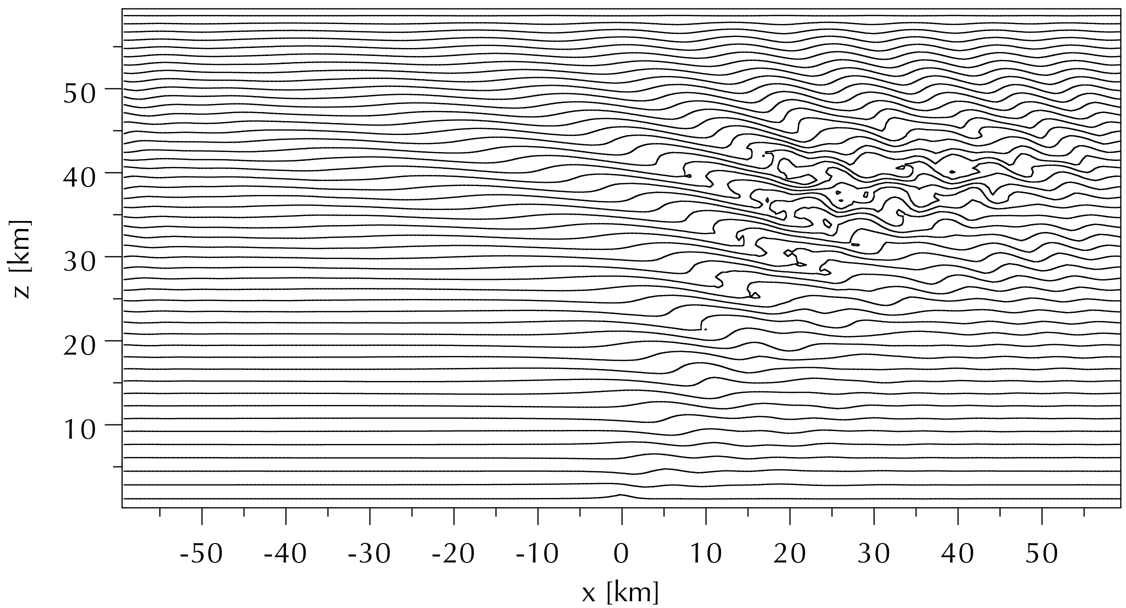
$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-3}$

anelastic

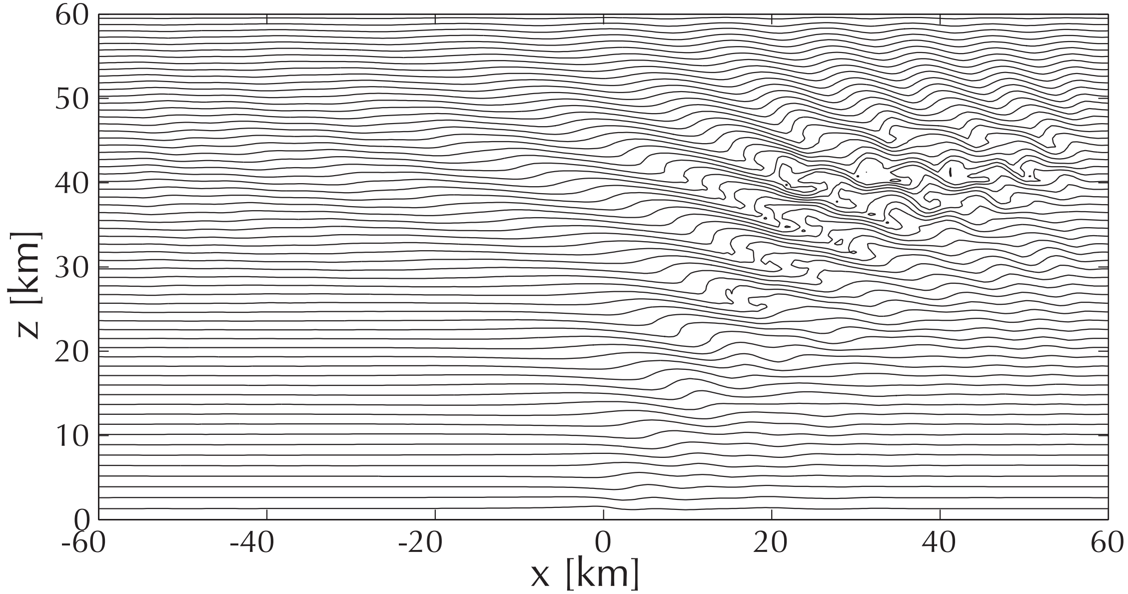
3 hours

sharpened van Leer's limiter

$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-3}$



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



pseudo-incompressible

3 hours

sharpened van Leer's limiter

$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-4}$

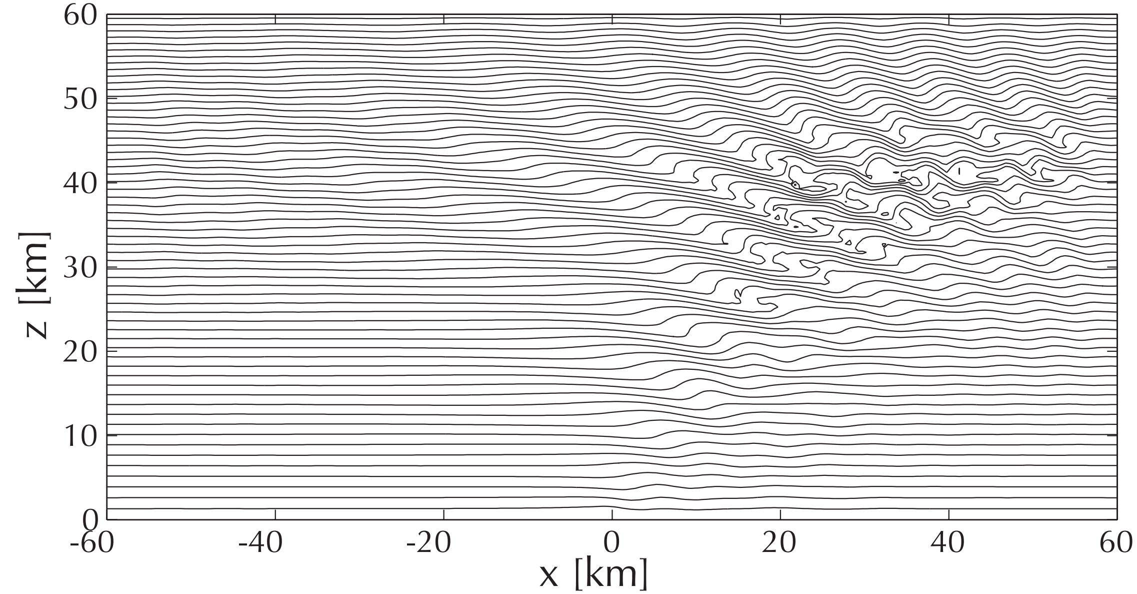
Compressible Euler eqs.

3 hours

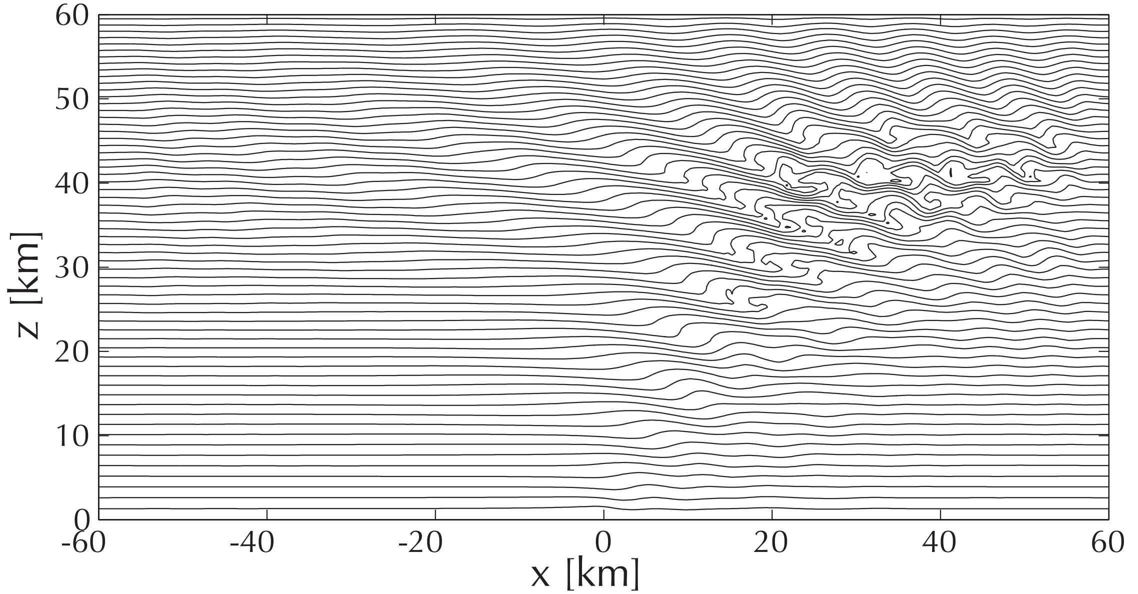
sharpened van Leer's limiter

$\Delta t \cdot \text{residual} < 10^{-4}$

$\text{CFL}_{\text{adv}} = 1.0$



Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



pseudo-incompressible

3 hours

sharpened van Leer's limiter

$\Delta t \, \nabla \cdot (P\boldsymbol{v}) < 10^{-4}$

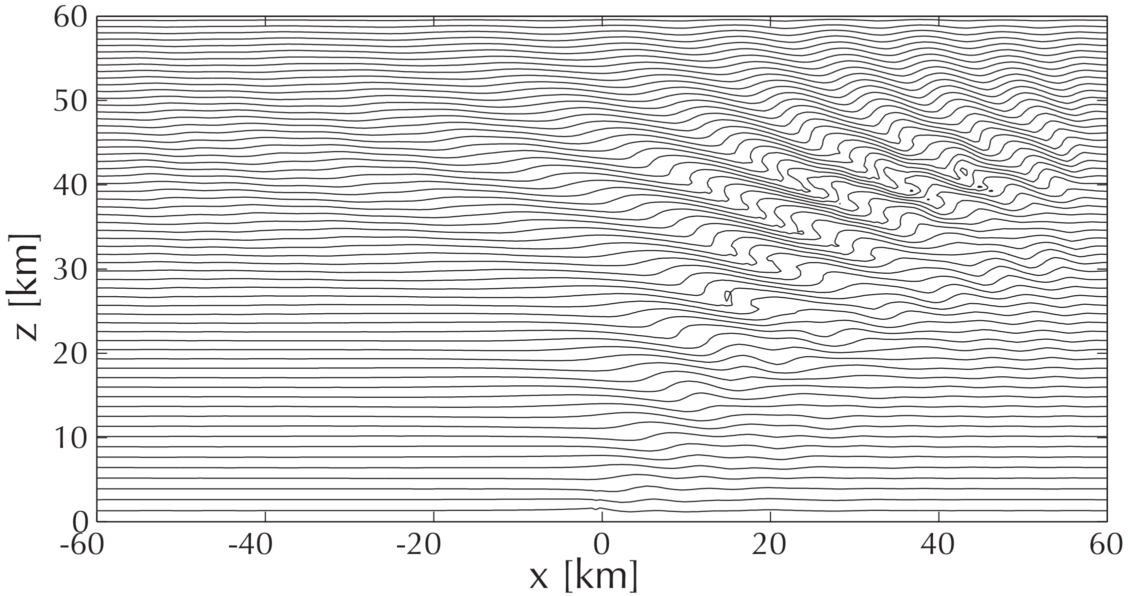
Compressible Euler eqs.

3 hours

sharpened van Leer's limiter

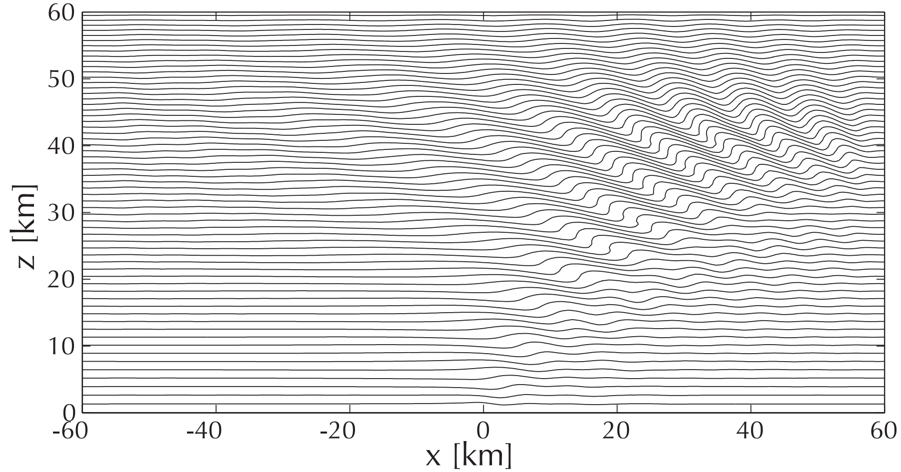
$\Delta t \cdot \mathrm{residual} < 10^{-4}$

$\mathbf{CFL_{ac} = 2.0}$

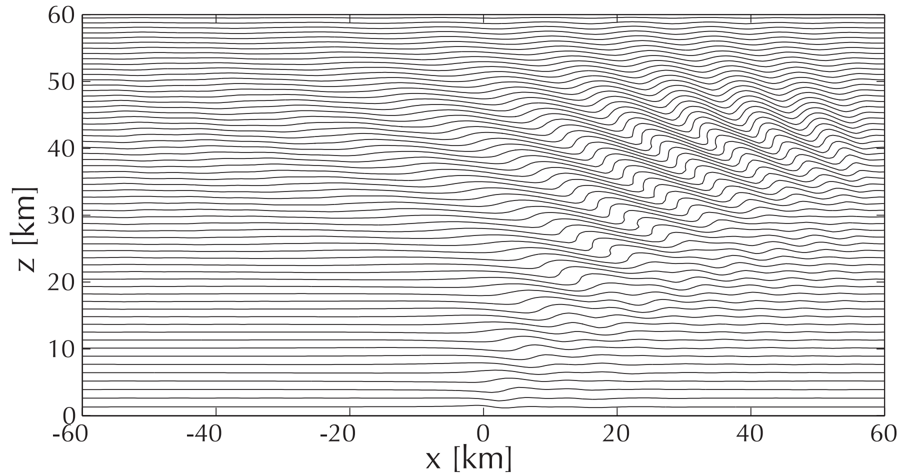


Results at time $t = 2\,h$

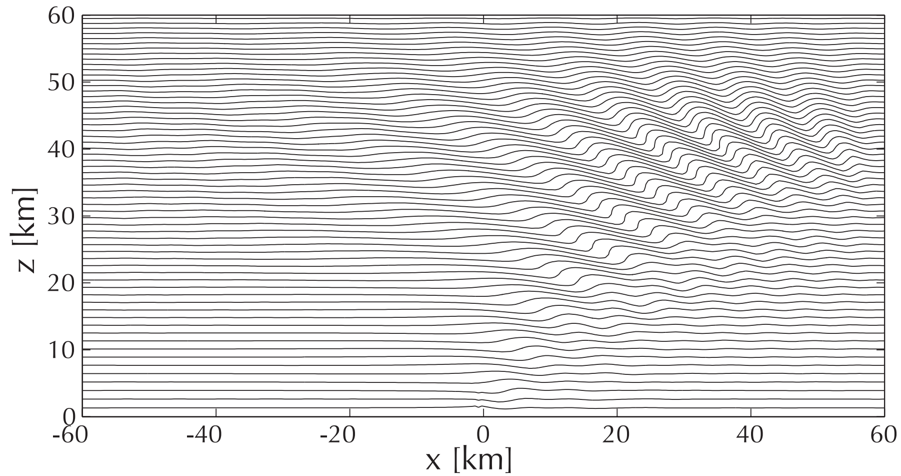
pseudo-incompressible



compressible, $\mathrm{CFL}_{\mathrm{adv}} = 1$



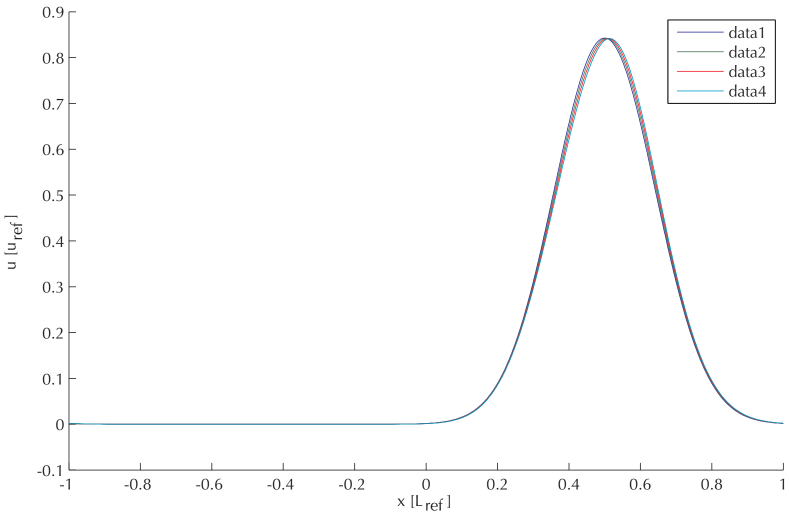
compressible, $\mathrm{CFL}_{\mathrm{ac}} = 2$

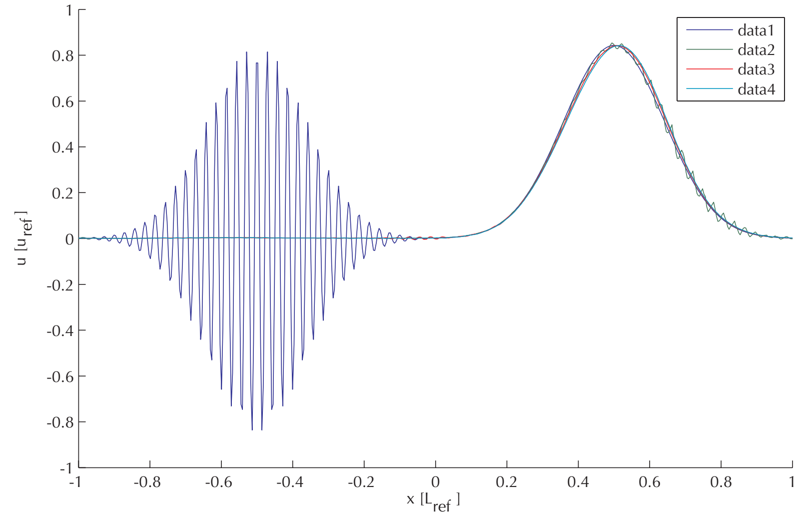


1D Acoustic test revisited, compressible Euler $(\mathrm{Mach} = 10^{-4})$

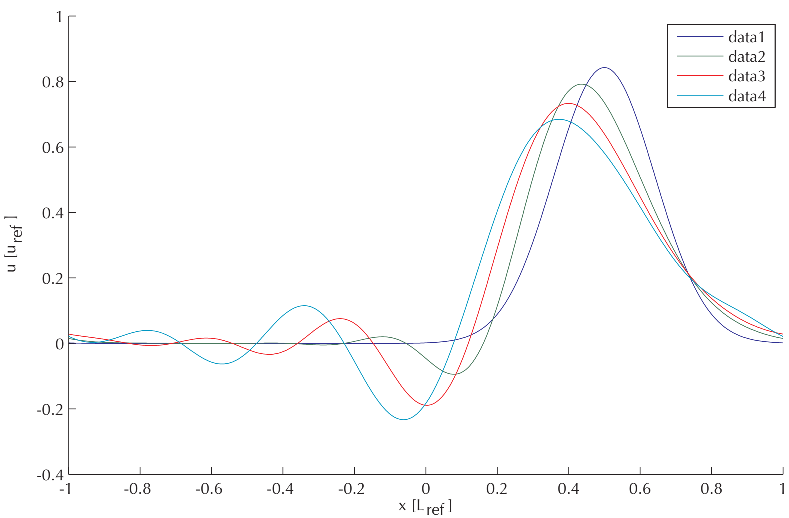
single pulse data

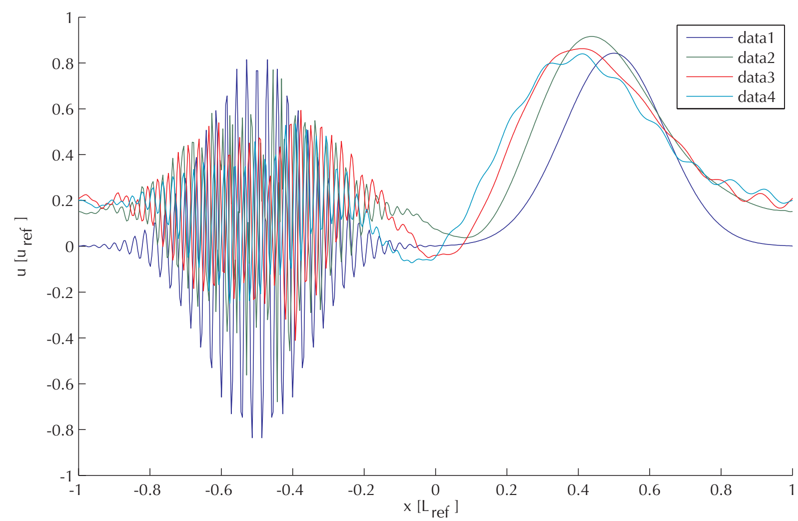
multiscale data





$\mathrm{CFL_{ac}} = 1.0$





$\mathrm{CFL_{ac}} = 10.0$

Summary

Regime(s) of validity of sound-proof models

- Motivation
- Stratification limit in the design-regime
- Wave-breaking regime with strong stratification

Scale-dependent time-integrator