A hierarchy of models, from planetary to mesoscale, within a single switchable numerical framework

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Outline of the talk

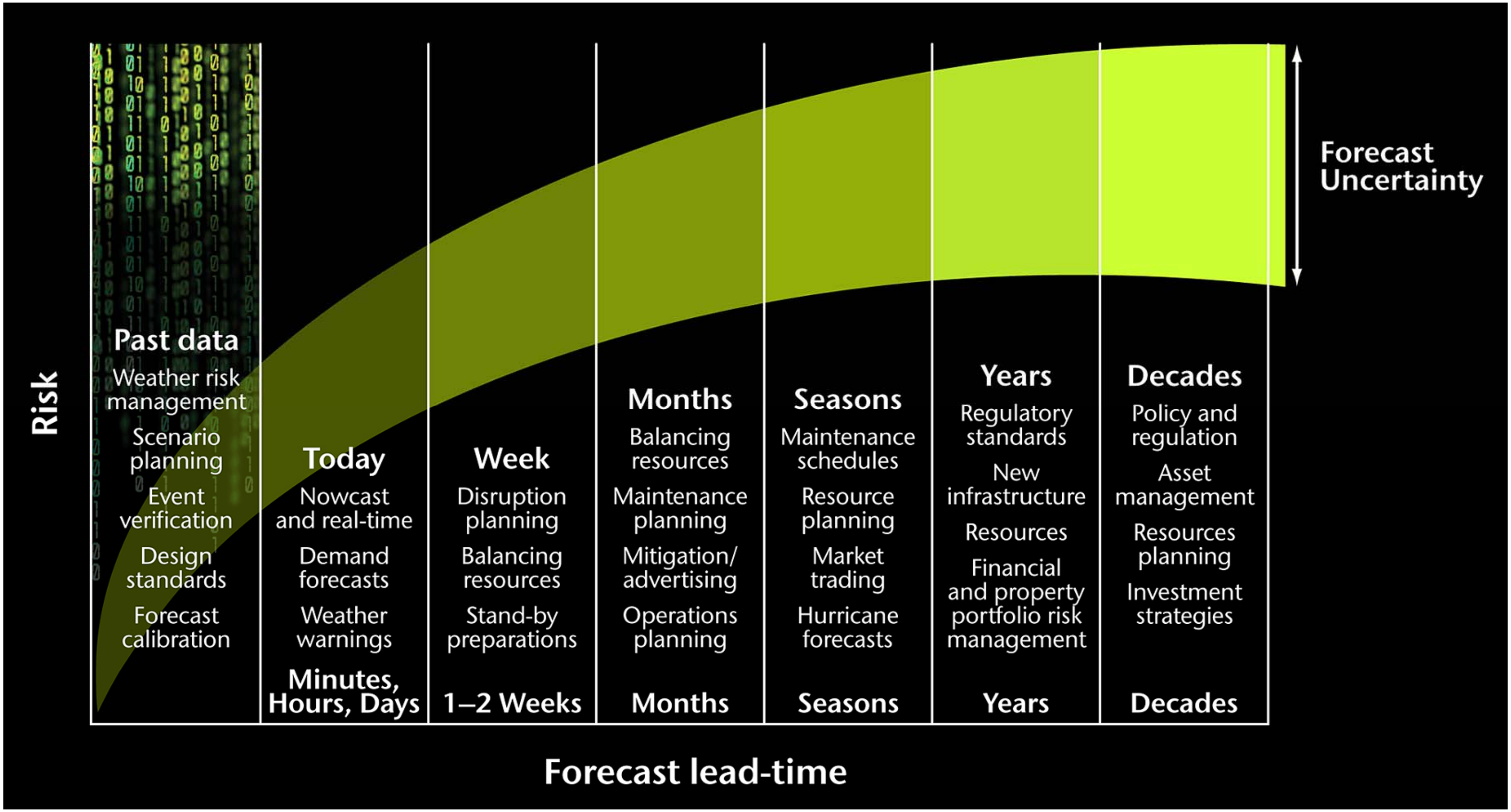
- Overview of Met Office's Unified Model
- Spherical Geopotential Approximation
- Relaxing that approximation
- A method of implementation
- Alternative view of Shallow-Atmosphere Approximation
- Application to departure points

Met Office's Unified Model

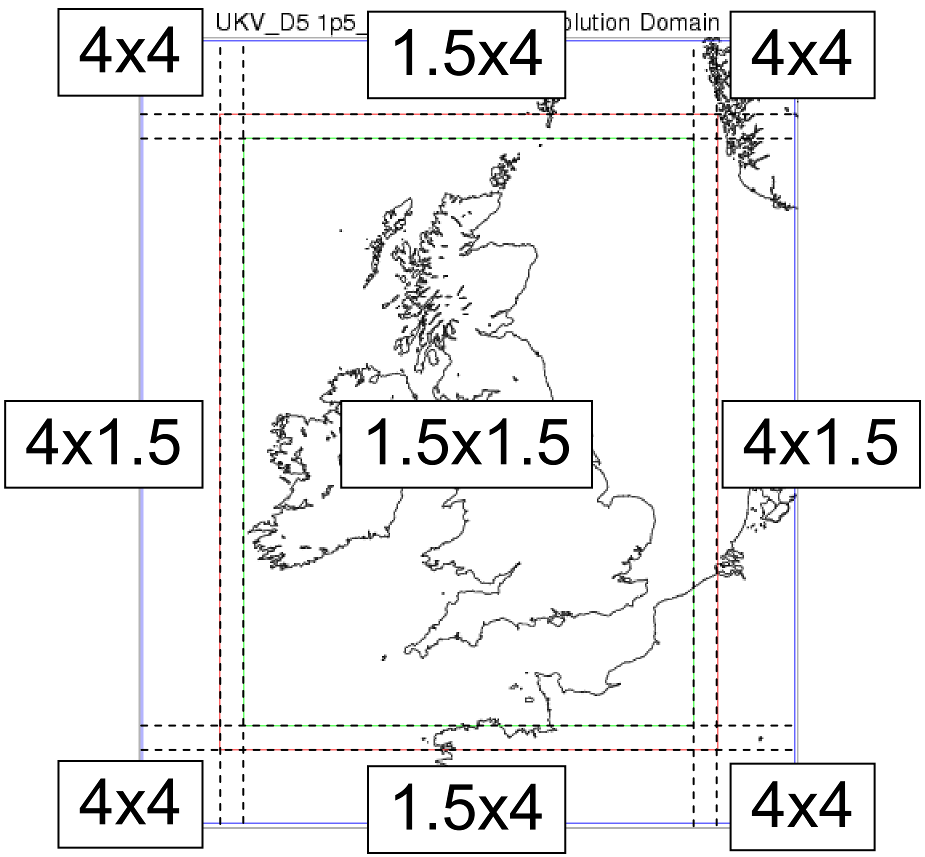
Unified Model (UM) in that *single* model for:

- Operational forecasts at
  - Mesoscale (resolution approx. 12km → 4km → 1km)
  - Global scale (resolution approx. 25km)
- Global and regional climate predictions (resolution approx. 100km, run for 10-100 years)
- + Research mode (1km - 10m) and single column model
- 20 years old this year

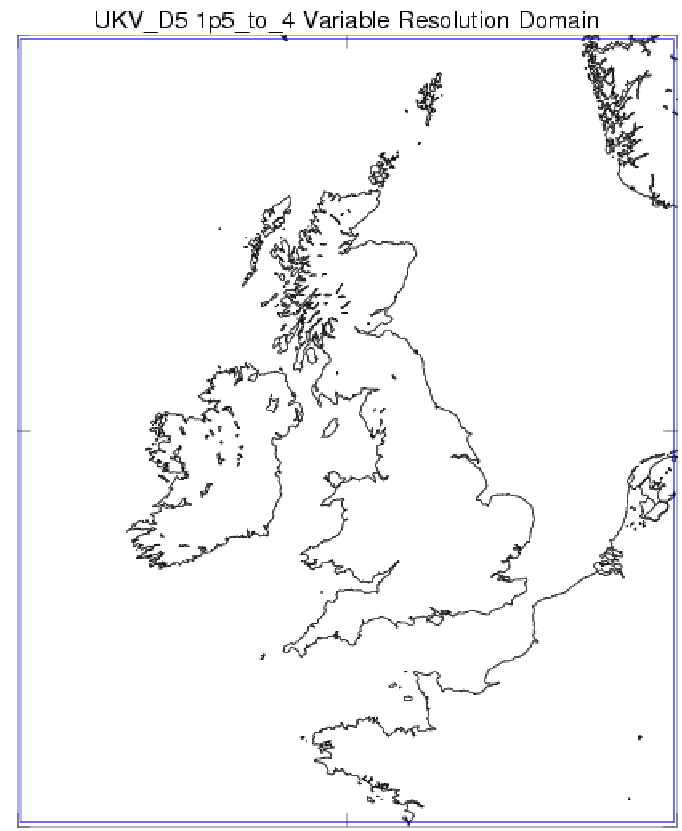
Timescales & applications



UKV Domain



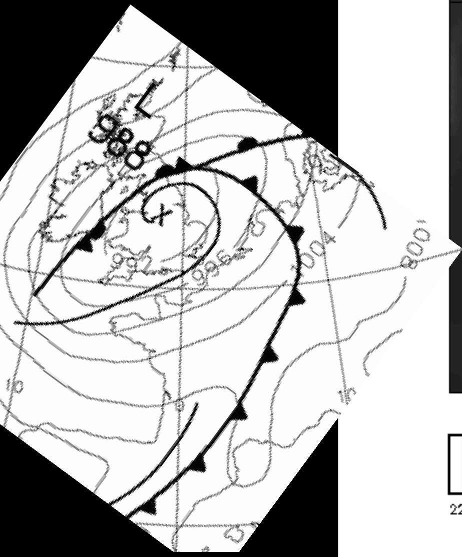
UKV_D5 1p5_
olution Domain
4x4
1.5x4
4x4
4x1.5
1.5x1.5
4x1.5
4x4
1.5x4
4x4

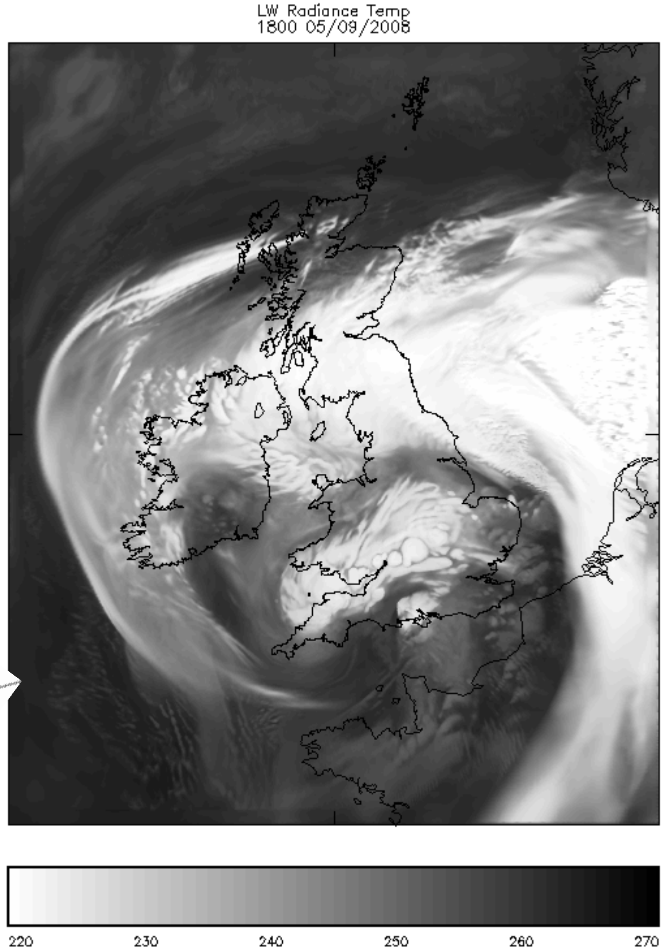


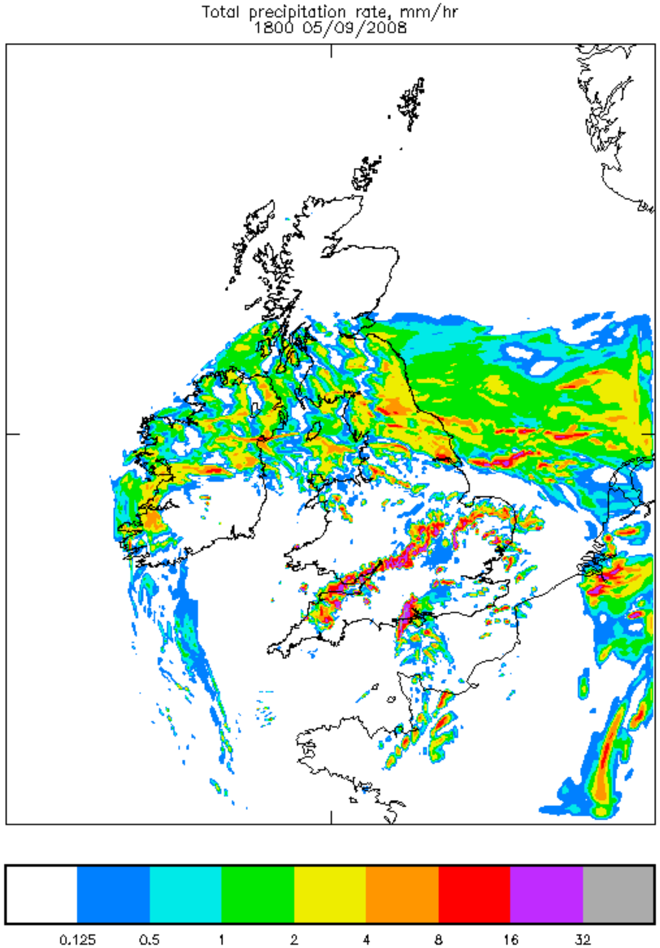
UKV_D5 1p5_to_4 Variable Resolution Domain

The ‘Morpeth Flood’, 06/09/2008

1.5 km L70
Prototype UKV
From
15 UTC 05/09
12 km

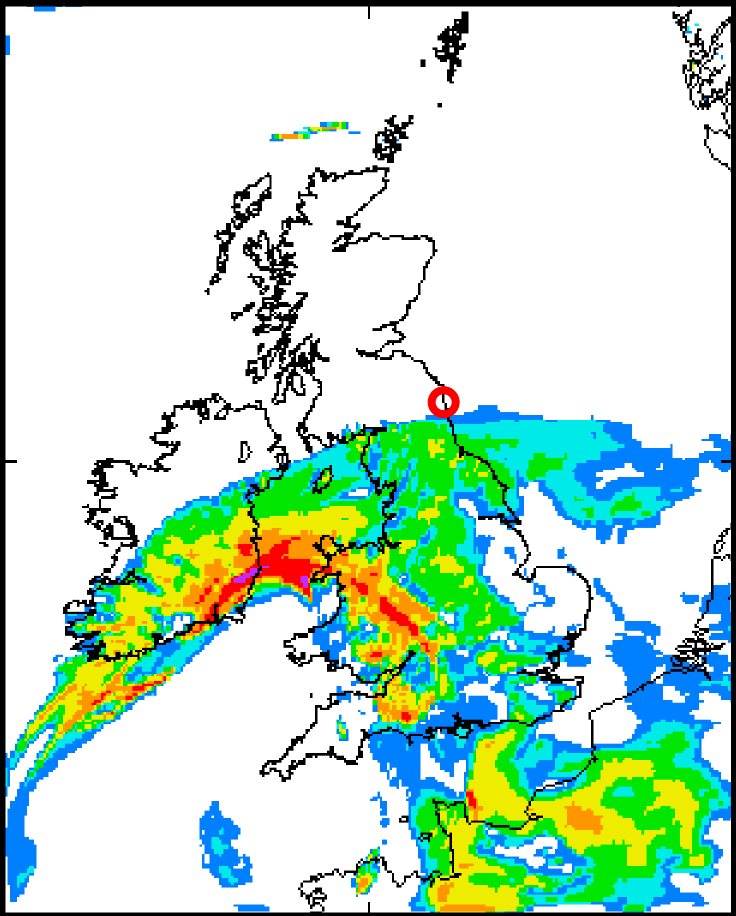




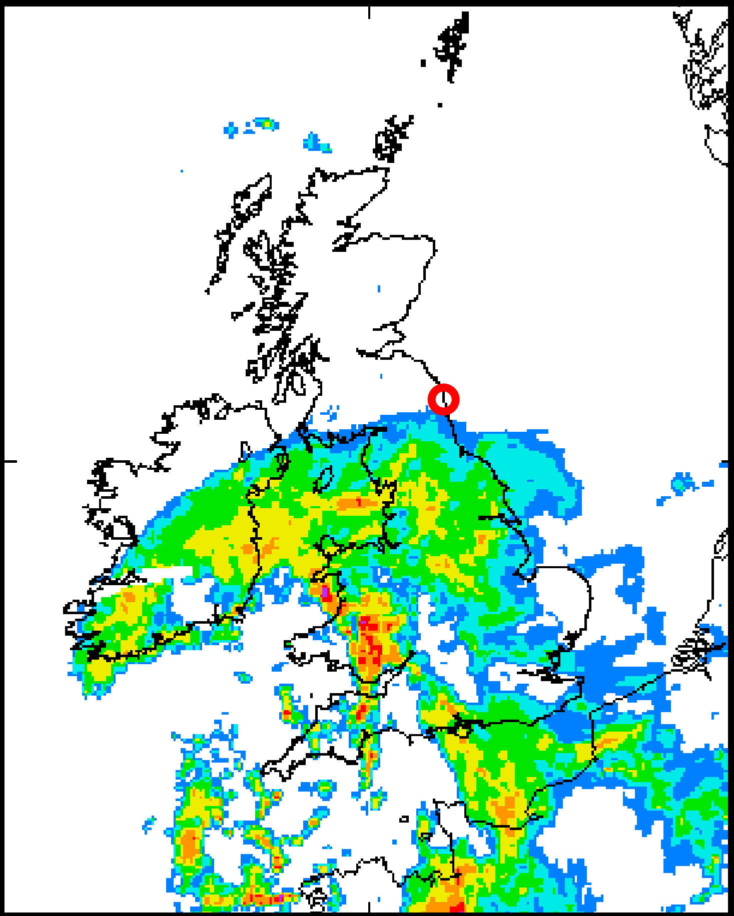


0600 UTC

2009: Clark et al, Morpeth Flood



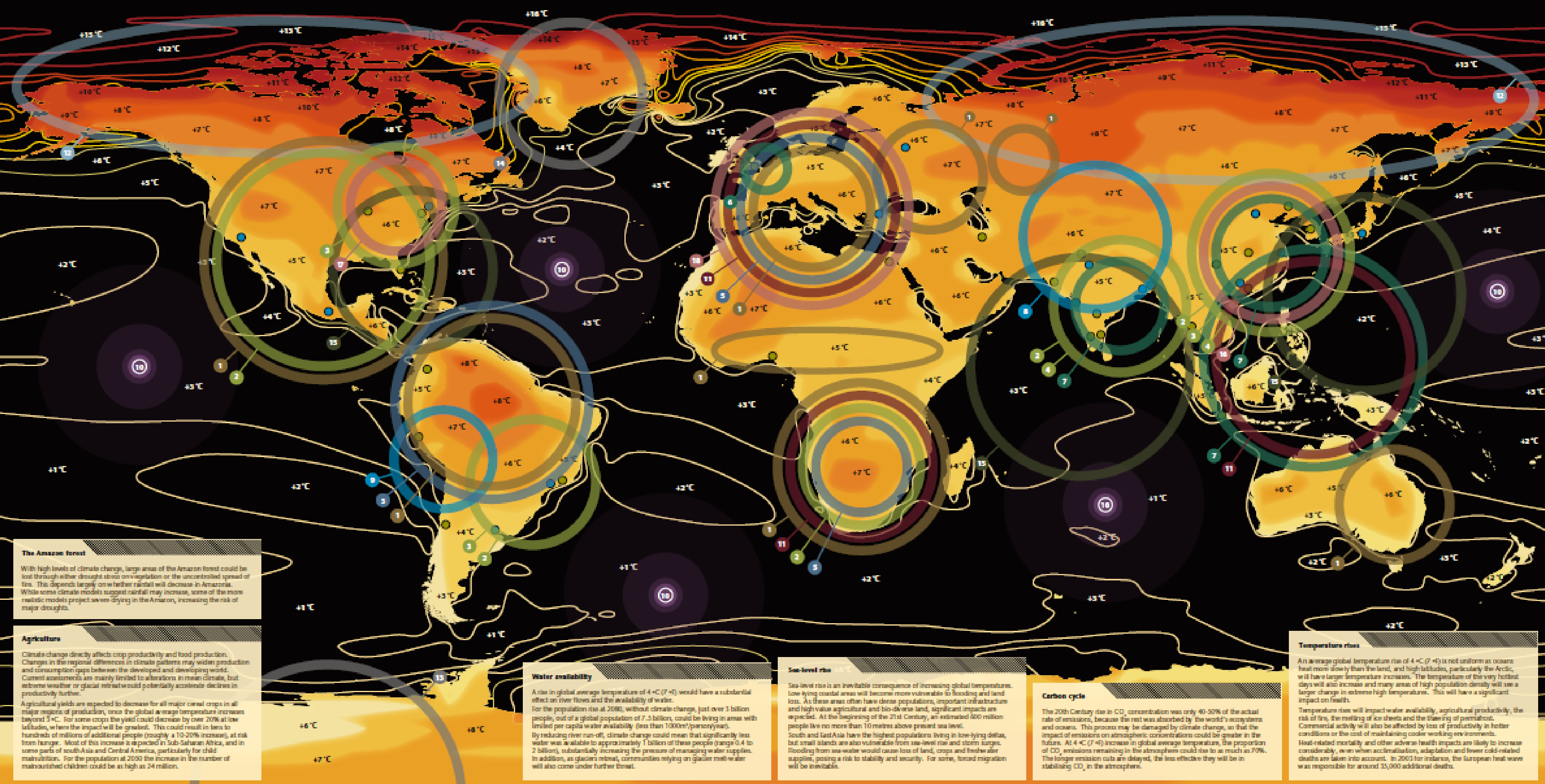
UKV 4-20hr forecast 1.5km gridlength convection permitting model



UK radar network

The impact of a global temperature rise of 4 °C (7 °F)

Met Office Hadley Centre | QUEST Quantifying and Understanding the Earth System | Walker Institute | Tyndall°Centre for Climate Change Research | HM Government



The Amazon forest

With high levels of climate change, large areas of the Amazon forest could be lost through either drought stress on vegetation or the uncontrolled spread of fire. This depends largely on whether rainfall will decrease in Amazonia. While some climate models suggest rainfall may increase, some of the more realistic models project severe drying in the Amazon, increasing the risk of major droughts.

Agriculture

Climate change directly affects crop productivity and food production. Changes in the regional differences in climate patterns may widen production and consumption gaps between the developed and developing world. Current assessments are mainly limited to alterations in mean climate, but extreme weather or glacial retreat would potentially accelerate declines in productivity further.

Agricultural yields are expected to decrease for all major cereal crops in all major regions of production, once the global average temperature increases beyond 3 °C. For some crops the yield could decrease by over 20% at low latitudes, where the impact will be greatest. This could result in tens to hundreds of millions of additional people (roughly a 10-20% increase), at risk from hunger. Most of this increase is expected in Sub-Saharan Africa, and in some parts of south Asia and Central America, particularly for child malnutrition. For the population at 2050 the increase in the number of malnourished children could be as high as 24 million.

Water availability

A rise in global average temperature of 4 °C (7 °F) would have a substantial effect on river flows and the availability of water.

For the population rise at 2080, without climate change, just over 3 billion people, out of a global population of 7.3 billion, could be living in areas with limited per capita water availability (less than 1000m³/person/year).

By reducing river run-off, climate change could mean that significantly less water was available to approximately 1 billion of these people (range 0.4 to 2 billion), substantially increasing the pressure of managing water supplies. In addition, as glaciers retreat, communities relying on glacier melt-water will also come under further threat.

Sea-level rise

Sea-level rise is an inevitable consequence of increasing global temperatures. Low-lying coastal areas will become more vulnerable to flooding and land loss. As these areas often have dense populations, important infrastructure and high value agricultural and bio-diverse land, significant impacts are expected. At the beginning of the 21st Century, an estimated 600 million people live no more than 10 metres above present sea level.

South and East Asia have the highest populations living in low-lying deltas, but small islands are also vulnerable from sea-level rise and storm surges. Flooding from sea-water would cause loss of land, crops and freshwater supplies, posing a risk to stability and security. For some, forced migration will be inevitable.

Carbon cycle

The 20th Century rise in CO_2 concentration was only 40-50% of the actual rate of emissions, because the rest was absorbed by the world's ecosystems and oceans. This process may be damaged by climate change, so that the impact of emissions on atmospheric concentrations could be greater in the future. At 4 °C (7 °F) increase in global average temperature, the proportion of CO_2 emissions remaining in the atmosphere could rise to as much as 70%. The longer emission cuts are delayed, the less effective they will be in stabilising CO_2 in the atmosphere.

Temperature rises

An average global temperature rise of 4 °C (7 °F) is not uniform as oceans heat more slowly than the land, and high latitudes, particularly the Arctic, will have larger temperature increases. The temperature of the very hottest days will also increase and many areas of high population density will see a larger change in extreme high temperatures. This will have a significant impact on health.

Temperature rises will impact water availability, agricultural productivity, the risk of fire, the melting of ice sheets and the thawing of permafrost. Commercial activity will also be affected by loss of productivity in hotter conditions or the cost of maintaining cooler working environments.

Heat-related mortality and other adverse health impacts are likely to increase considerably, even when acclimatisation, adaptation and fewer cold-related deaths are taken into account. In 2003 for instance, the European heat wave was responsible for around 35,000 additional deaths.

1. High forest-fire danger projected to affect every populated continent. Regions moving into the high-danger category include: large areas of the United States; Mexico; South America, east of the Andes; southern and east Africa; the Sahel; eastern and southern Australia and southern Europe.
2. Maize and wheat yields reduced by up to 40% at low latitudes.
3. Soybean yield could decrease in all regions of production, including North and South America, southern and eastern Asia.
4. Decrease in rice yield of up to 30% in China, India, Bangladesh and Indonesia.
5. Water resources affected by up to 70% reduction in run-off around the Mediterranean, southern Africa and large areas of South America.
6. Sea-level rise combined with storm surges could pose a serious threat to people and assets in the Netherlands and south-eastern parts of the UK.
7. Sea levels could rise as much as 80 cm by the end of the century. Longer term, 4 °C (7 °F) would result in a much higher rise in sea level.
   Sea level increases are likely to be even greater at low latitudes, disproportionately affecting tropical islands and low-lying regions such as Bangladesh.
   For the population at 2075, a mean sea-level rise of 53 cm means that up to an additional 150 million people per year would be flooded due to extreme sea levels. Three-quarters of these people live in Asia. Up to 56 million people would be flooded along the Indian Ocean coast, 25 million along the east Asian coast and 33 million people would be flooded along the South-East Asian coast.
   Other vulnerable regions include Africa, Caribbean islands, Indian Ocean islands and Pacific small islands.
8. Half of all Himalayan glaciers significantly reduced by 2050, even at a global average temperature rise below 4 °C.
   The Indus river basin obtains 70% of its summer flow from glacial melt.
   In China, 23% of the population lives in the western regions where glacial melt provides the principal dry season water source.
9. Complete disappearance of glaciers from many regions in South America.
   In Peru's Cordillera Blanca summer run-off from glaciers reduced by up to 69% as the glacial area falls by 75%.
10. Marine ecosystems could be fundamentally altered by ocean acidification which would have a significant impact on fisheries. This could cause substantial loss in revenue and jobs. The loss of coral reef habitats due to acidification may seriously affect many commercial fish species and could prove disastrous for coastal communities relying on subsistence fishing of reef species.
11. Drought events occur twice as frequently across southern Africa, South-East Asia and the Mediterranean basin.
12. Almost complete disappearance of near-surface permafrost from Northern Siberia. Reduction of permafrost in Canada and Alaska. Infrastructure built on the permafrost foundation at risk.
13. It is not known how stable the West Antarctic Ice Sheet is, or whether a 4 °C (7 °F) global temperature rise will send it into irreversible decline. If this ice sheet did melt it would contribute a further 3.3 metres to long-term sea-level rise globally.
14. Greenland Ice Sheet has a 60% likelihood of irreversible decline. This would result in a very long-term sea-level rise of up to 7 metres globally.
15. Tropical cyclones could be more intense and destructive. Global population increases, particularly in coastal areas, and sea-level rise mean greater cyclone and hurricane related losses, disruptions to infrastructure and loss of life as a result of storm surges. For major cyclone disasters flooding from storm surges has been the primary cause of death.
16. Hottest days of the year could be as much as 6 °C (11 °F) warmer over highly populated areas of eastern China.
17. Hottest days of the year could become as much as 10-12 °C (18-22 °F) warmer over eastern North America, affecting Toronto, Chicago, Ottawa, New York and Washington DC.
18. Hottest days of the year across Europe could be as much as 8 °C (14 °F) warmer.

Disease patterns have changed with an overall increase in diarrhoea, vector-borne disease such as malaria and dengue fever, malnutrition and the health impacts of weather events such as flooding and drought.

Change in temperature from pre-industrial climate

+ °Celsius	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
+ °Fahrenheit	2	4	5	7	9	11	13	14	16	18	20	22	23	25	27	29

City populations: 5 – 10 Million; 10 – 20 Million

Source: UN Statistics Division Demographic Yearbook 2007

The underlying equations...

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Traditional Spherically Based Equations

$$\frac{D_r u}{Dt} - \frac{uv\tan\phi}{r} - (2\Omega\sin\phi)\,v + \frac{c_{pd}\theta_v}{r\cos\phi}\frac{\partial\Pi}{\partial\lambda} = \boxed{-\frac{uw}{r} - (2\Omega\cos\phi)\,w} + S^u$$

$$\frac{D_r v}{Dt} + \frac{u^2\tan\phi}{r} + (2\Omega\sin\phi)\,u + \frac{c_{pd}\theta_v}{r}\frac{\partial\Pi}{\partial\phi} = \boxed{-\frac{vw}{r}} + S^v$$

$$\boxed{\frac{D_r w}{Dt}} + c_{pd}\theta_v\frac{\partial\Pi}{\partial r} + \frac{\partial\Phi}{\partial r} = \boxed{\frac{u^2+v^2}{r} + (2\Omega\cos\phi)\,u + S^w}$$

Red = Shallow-Atmosphere Approx (eliminate boxed terms)

Blue = Hydrostatic Approx (eliminate boxed term)

Green = Spherical Geopotential Approx ($\partial\Phi/\partial r = g(r)$ only)

The Spherical Geopotential Approximation

- **All geopotentials - including the Earth's surface - are represented by concentric *spheres***
- **Apparent gravity acts towards the centre of the Earth**
- **To prevent spurious vorticity sources/sinks *g* cannot vary with latitude (White et al., 2005)**
- **Nearly all numerical models of the global atmosphere are based on the SGA**

Why relax the SGA?

- Good to get Earth's shape right(ish) in models!
- Good to include the small ($\approx$ 0.5%) increase of g from Equator to Poles without spurious vorticity sources/sinks
- Most likely to be evident in long term climate simulations
- Good to quantitatively test the accuracy of the spherical geopotential approximation
- Important though that signal is not influenced by different numerics

The Shape of the Earth

- **The Earth is approximately an oblate spheroid:**
  - **Equatorial radius**

$$a = 6378\,\mathrm{km}$$

  - **Polar radius**

$$c = 6357\,\mathrm{km}$$

- **Ellipticity small**

$$\varepsilon \equiv \frac{a-c}{a} \approx \frac{1}{298}$$

- **But** $a - c = 21$ **km, more than twice the height of Everest**

Geopotential near rotating fluid spheroid

- **Classical problem (Clairaut in 18th century) to account for the observed increase of g with latitude**

- **Re-examined by White et al. (2008) to find geopotentials**
- **Two small parameters:**

$$\varepsilon \equiv \frac{a-c}{a} \approx \frac{1}{298}$$

$$m \equiv \frac{\mathbf{\Omega}^2 a^3}{\gamma M_E} \sim \left| \frac{\text{centrifugal force}}{\text{Newtonian gravity}} \right| \approx \frac{1}{289},$$

- **To $O(\varepsilon, m)$ geopotentials are spheroids**

$$x^2 + z^2 \left[1 + (2\epsilon - m) \frac{a^2}{R^2} + m \frac{R^3}{a^3} \right] = R^2$$

Variation of Gravity

- **Near to the surface** $R = a$

$$\frac{g_P}{g_{Eq}} = 1 + \frac{5}{2}m - \epsilon$$

- **Earth's *actual mass distribution*** $m = \epsilon$ **(to within 3%):**

$$\boxed{\frac{g_P}{g_{Eq}} = 1 + \frac{3}{2}\epsilon} \Rightarrow 0.5\% \ \text{ increase}$$

- **For *uniform mass distribution*** $m = 4\epsilon/5$ **(Newton)**

$$\boxed{\frac{g_P}{g_{Eq}} = 1 + \epsilon}$$

captures 2/3rds of the variation

Spheroidal Coordinates...

Confocal Oblate Spheroids

- **"Oblate spheroidal coordinates" of e.g. Gill (1982) and Gates (2004) are in fact *confocal* oblate spheroidal coordinates:**

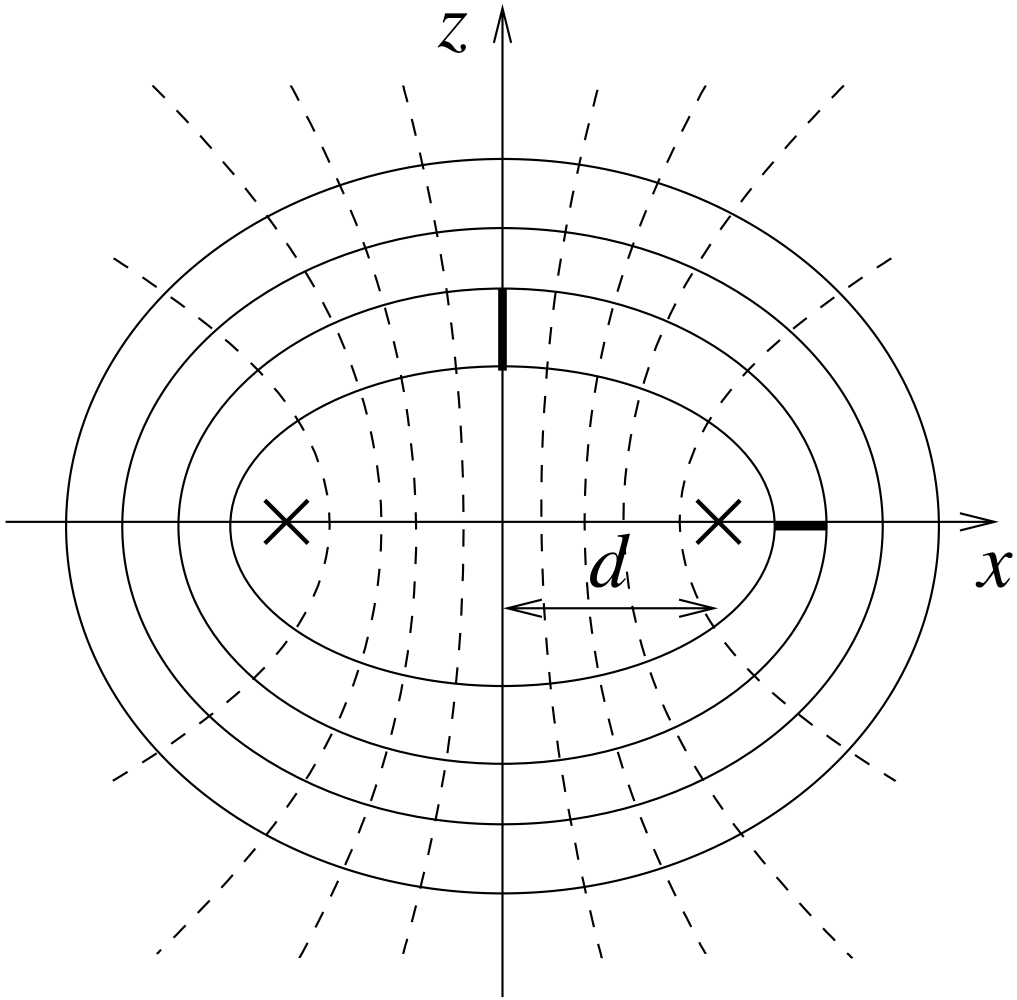
$$\frac{x^2}{\cosh^2\eta} + \frac{z^2}{\sinh^2\eta} = d^2$$

with d fixed, η variable

- **Match to Earth's surface $\Rightarrow$**

$$\boxed{\frac{g_P}{g_{Eq}} = \tanh\eta_0 \approx 1 - \epsilon \neq 1 + \frac{3}{2}\epsilon!}$$

- **Separation along minor axis *greater* than along major axis**



CONFOCAL ELLIPSES and HYPERBOLAS

Similar ellipses

Equation is:

$$x^2 + (1+\mu)\, z^2 = \mathtt{r}^2$$

Now μ fixed, $\mathtt{r}$ variable

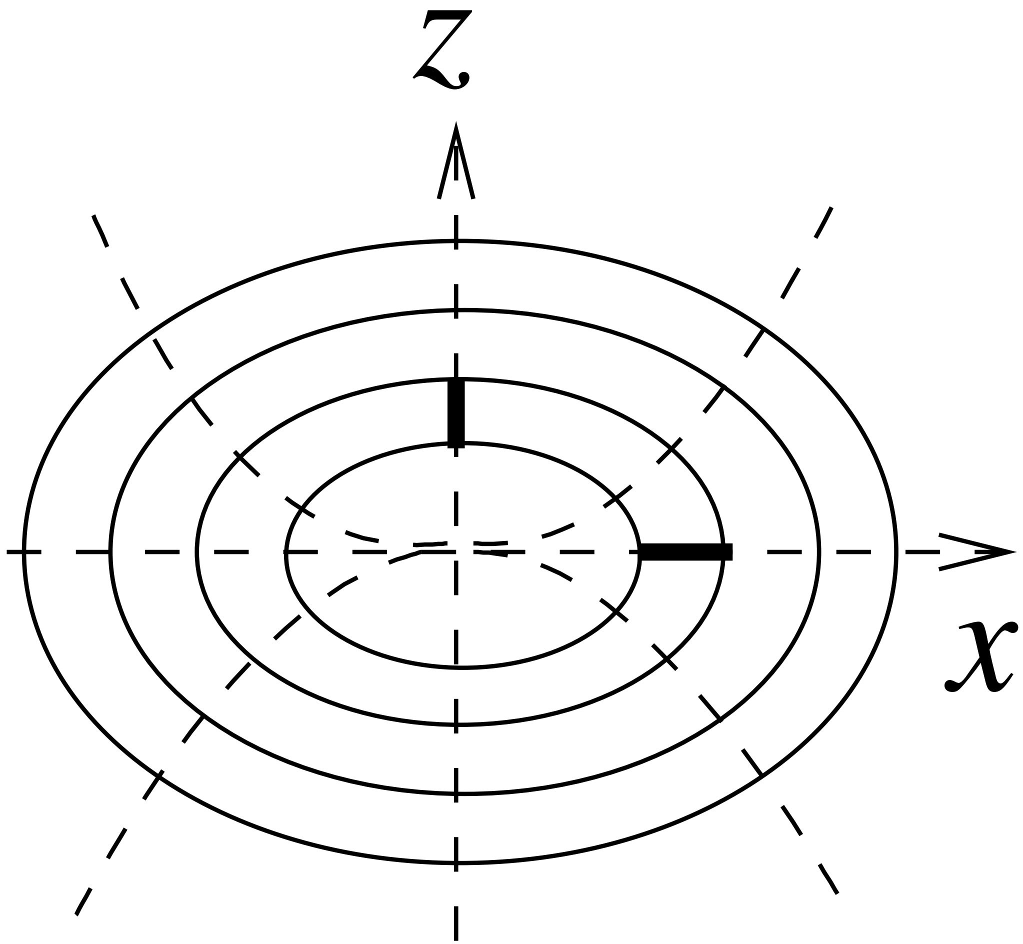
Match to Earth's surface $\Rightarrow$

$$\mu = 2\epsilon$$

and

$$\boxed{\frac{g_P}{g_{Eq}} = (1+\mu)^{1/2} \approx 1 + \epsilon}$$

- **Sign right!**
- **Captures 2/3rds actual variation for actual Earth**
- **Exact for an Earth with uniform mass distribution**



SIMILAR ELLIPSES

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Implementation...

Semi-Lagrangian scalar advection

Consider

$$\frac{DF}{Dt} = R$$

Integrate *along trajectory*:

$$\int_t^{t+\Delta t} \frac{DF}{Dt} dt = \frac{F^{t+\Delta t}(x) - F^t(x - U\Delta t)}{\Delta t} = \int_t^{t+\Delta t} R dt \approx \frac{R^{t+\Delta t}(x) + R^t(x - U\Delta t)}{2}$$

i.e.

$$\left(F - \frac{\Delta t}{2} R\right)_A^{t+\Delta t} = \left(F + \frac{\Delta t}{2} R\right)_D^t$$

Semi-Lagrangian vector advection

For the vector equation

$$\frac{D\mathbf{v}}{Dt} = \mathbf{S}$$

Integrate *along trajectory* to obtain, as before:

$$\left(\mathbf{v} - \frac{\Delta t}{2}\mathbf{S}\right)_A^{t+\Delta t} = \left(\mathbf{v} + \frac{\Delta t}{2}\mathbf{S}\right)_D^{t}$$

All well and good...

But what of components?

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The Problem of Curvilinear Coordinates...

The familiar problem that

$$\mathbf{i}.\frac{D\mathbf{v}}{Dt} \equiv \mathbf{i}.\frac{D}{Dt}\left(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}\right) \neq \frac{D\left(\mathbf{i}.\mathbf{v}\right)}{Dt} \quad \left[= \frac{Du}{Dt}\right]$$

translates into the SL equivalent that

$$\boxed{\mathbf{i}_A.\left(\mathbf{v}_D\right) \neq \left(\mathbf{i}_A.\mathbf{v}_A\right)_D \quad [= u_D]}$$

In fact

$$\mathbf{v}_D = u_{\mathcal{D}}\mathbf{i}_D + v_{\mathcal{D}}\mathbf{j}_D + w_{\mathcal{D}}\mathbf{k}_D$$

so that, e.g.,

$$\mathbf{i}_A.\left(\mathbf{v}_D\right) = u_{\mathcal{D}}\left(\mathbf{i}_A.\mathbf{i}_D\right) + v_{\mathcal{D}}\left(\mathbf{i}_A.\mathbf{j}_D\right) + w_{\mathcal{D}}\left(\mathbf{i}_A.\mathbf{k}_D\right)$$

The Matrix Formulation

- **Extending this to all three directions:**

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_D = \mathrm{M} \begin{pmatrix} u_{\mathcal{D}} \\ v_{\mathcal{D}} \\ w_{\mathcal{D}} \end{pmatrix}$$

where

$$\mathrm{M} \equiv \begin{pmatrix} \mathbf{i}_A.\mathbf{i}_D & \mathbf{i}_A.\mathbf{j}_D & \mathbf{i}_A.\mathbf{k}_D \\ \mathbf{j}_A.\mathbf{i}_D & \mathbf{j}_A.\mathbf{j}_D & \mathbf{j}_A.\mathbf{k}_D \\ \mathbf{k}_A.\mathbf{i}_D & \mathbf{k}_A.\mathbf{j}_D & \mathbf{k}_A.\mathbf{k}_D \end{pmatrix}$$

- M **transforms**
  - **from:** **vector components in the *departure-point* frame**
  - **to:** **vectors in the *arrival-point* frame**

The Consequence...

Component form of momentum equations can be written as:

$$\left(\mathbf{v} - \frac{\Delta t}{2}\mathbf{S}\right)_{\mathcal{A}}^{t+\Delta t} = \mathbf{M}\left(\mathbf{v} + \frac{\Delta t}{2}\mathbf{S}\right)_{\mathcal{D}}^{t}$$

where

$$\mathbf{X}_{\mathcal{A}} \equiv \left(X_{\mathcal{A}},\, Y_{\mathcal{A}},\, Z_{\mathcal{A}} \right)^T$$

$$\mathbf{X}_{\mathcal{D}} \equiv \left(X_{\mathcal{D}},\, Y_{\mathcal{D}},\, Z_{\mathcal{D}} \right)^T$$

- **No explicit metric terms**
- **No singularity at the pole**

M for Spherical Polar Coordinates

$$M_{11} = \cos\Delta\lambda, \quad M_{12} = \sin\phi_D \sin\Delta\lambda, \quad M_{13} = -\cos\phi_D \sin\Delta\lambda$$

$$M_{21} = -\sin\phi_A \sin\Delta\lambda$$

$$M_{22} = \cos\phi_A \cos\phi_D + \sin\phi_A \sin\phi_D \cos\Delta\lambda$$

$$M_{23} = \cos\phi_A \sin\phi_D - \sin\phi_A \cos\phi_D \cos\Delta\lambda$$

$$M_{31} = \cos\phi_A \sin\Delta\lambda$$

$$M_{32} = \sin\phi_A \cos\phi_D - \cos\phi_A \sin\phi_D \cos\Delta\lambda$$

$$M_{33} = \sin\phi_A \sin\phi_D + \cos\phi_A \cos\phi_D \cos\Delta\lambda$$

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Where have the Metric Terms Gone?

- **SL form holds for finite displacements**
- **Reduces to Eulerian form as displacement ($\Delta\lambda \equiv \lambda_A - \lambda_D$) and $\Delta t \rightarrow 0$:**
- **In this limit:**

$$\sin\Delta\lambda \rightarrow \Delta\lambda, \quad \frac{\Delta\lambda}{\Delta t} \rightarrow \frac{u_{\mathcal{A}}}{r_A \cos\phi_A}, \quad u_{\mathcal{D}} \rightarrow u_{\mathcal{A}}, \quad v_{\mathcal{D}} \rightarrow v_{\mathcal{A}}$$

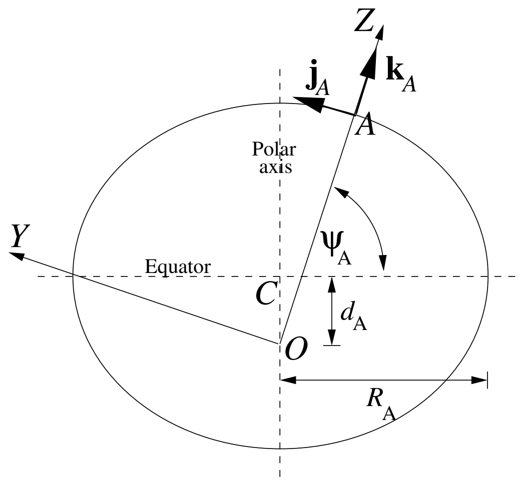
and so:

$$\boxed{M_{12} v_{\mathcal{D}} = v_{\mathcal{D}} \sin\phi_D \sin\Delta\lambda \rightarrow \frac{u_{\mathcal{A}} v_{\mathcal{A}} \tan\phi_A}{r_A} \Delta t}$$

- **Each off-diagonal element of M generates one metric term**

Similar Oblate Spheroidal Form

- **Basic derivation is the same**
- **"Trick": use secondary meridional coordinate, ψ, *geographic latitude* (not orthogonal to vertical coordinate)**



- **Significantly M has *exactly* the same form but with $\phi \rightarrow \psi$**
- **Contrast Eulerian formulation where (for COS at least) two extra metric terms appear**

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Shallow-atmosphere...

Shallow-Atmosphere Approximation

$$\frac{D_r u}{Dt} - \frac{uv\tan\phi}{r} - (2\Omega\sin\phi)\,v + \frac{c_{pd}\theta_v}{r\cos\phi}\frac{\partial\Pi}{\partial\lambda} = \boxed{-\frac{uw}{r} - (2\Omega\cos\phi)\,w} + S^u$$

$$\frac{D_r v}{Dt} - \frac{u^2\tan\phi}{r} + (2\Omega\sin\phi)\,u + \frac{c_{pd}\theta_v}{r}\frac{\partial\Pi}{\partial\phi} = \boxed{-\frac{vw}{r}} + S^v$$

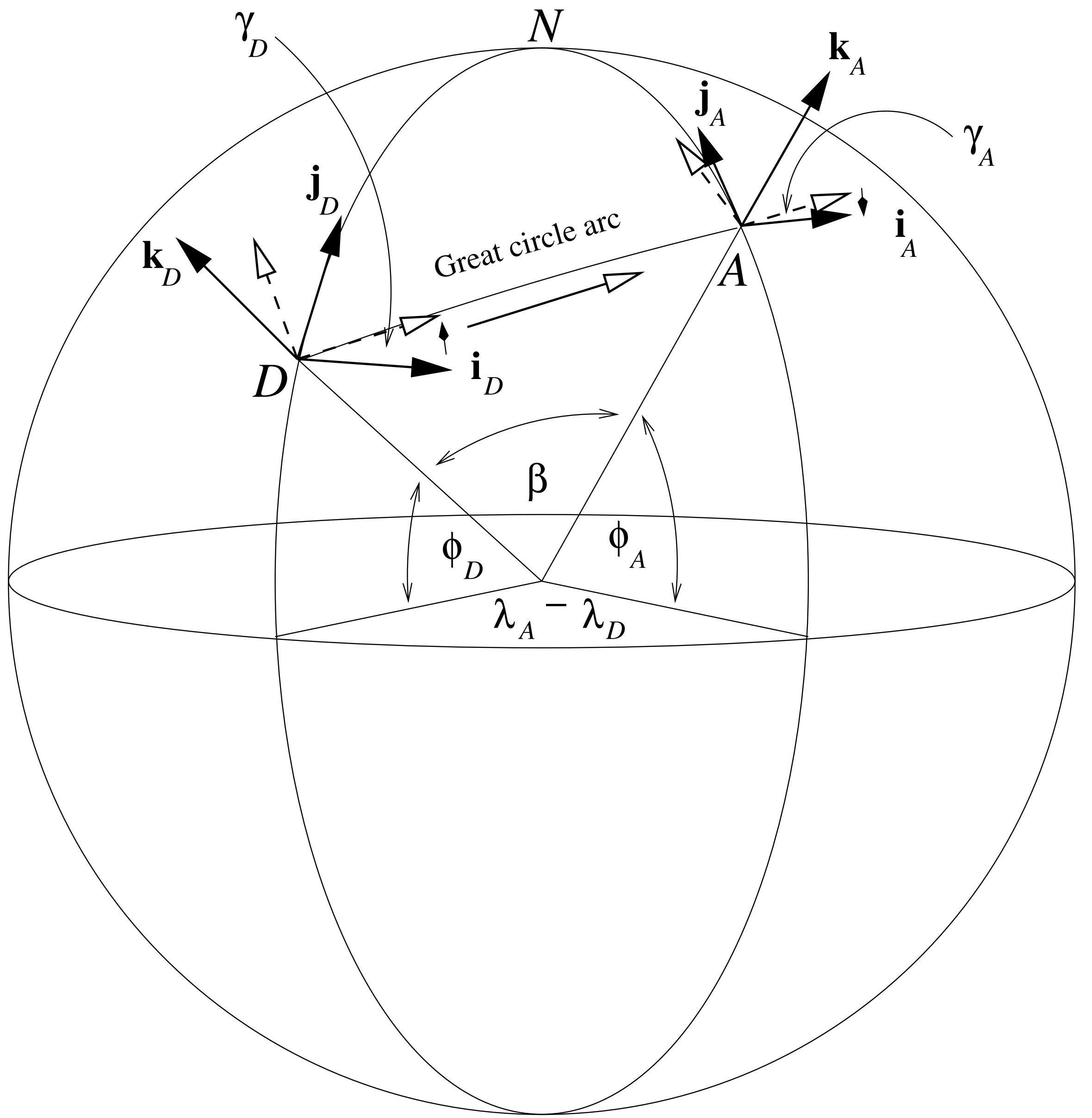
$$\boxed{\frac{D_r w}{Dt}} + c_{pd}\theta_v\frac{\partial\Pi}{\partial r} + \frac{\partial\Phi}{\partial r} = \boxed{\frac{u^2+v^2}{r} + (2\Omega\cos\phi)\,u + S^w}$$

- **Traditional approach:**
  - **Eliminate all boxed red terms**
  - **Replace r with a in all algebraic expressions**

Shallow-atmosphere - a different perspective

- M **is a rotation matrix**
- **Can be factored in infinite number of ways**
- **One is of particular interest:**

1. **Rotate departure-point UVT about radial direction to line up** i **with Great Circle connecting** r_D **with** r_A
2. **Rotate UVT along Great Circle arc about new** j **direction**
3. **Rotate UVT about new radial direction to line up** i **direction**



A PARTICULAR DECOMPOSITION OF $\mathbb{M}$

Shallow-atmosphere - a different perspective

- **Result is** $\mathrm{M} = \mathrm{ACD}$
- **Shallow-atmosphere approximation is obtained by** $\mathrm{C} \rightarrow \mathrm{I}$
- $\mathbf{M} \rightarrow \mathbf{AD} \equiv \mathbf{Q}$**:**

$$\mathbf{Q} = \begin{pmatrix} p & q & 0 \\ -q & p & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with

$$p = \frac{M_{11} + M_{22}}{1 + M_{33}}$$

and

$$q = \frac{M_{12} - M_{21}}{1 + M_{33}}$$

giving the same result as in Temperton et al. (2001)

Departure points...

Rotation Matrix for Departure Points?

- **Departure points now take on greater significance**

- **And they are also governed by a vector equation**

$$\frac{D\mathbf{x}}{Dt} = \mathbf{v}$$

- **So can we apply consistent approach (and avoid polar singularity issues)?**

- **Discrete vector form is**

$$\mathbf{x}_D = \mathbf{x}_A - \frac{\Delta t}{2}\left(\mathbf{v}_A + \mathbf{v}_D\right)$$

Rotation Matrix for Departure Points II

As before component form can be written as

$$\mathbf{M}\mathbf{x}_{\mathcal{D}} = \mathbf{x}_{\mathcal{A}} - \frac{\Delta t}{2}\left(\mathbf{v}_{\mathcal{A}} + \mathbf{M}\mathbf{v}_{\mathcal{D}}\right)$$

Or, since $\mathbf{x}_{\mathcal{A}} = \left(0,\ 0,\ r_A\right)^T$ **and** $\mathbf{x}_{\mathcal{D}} = \left(0,\ 0,\ r_D\right)^T$**:**

$$M_{13} r_{\mathcal{D}} = -\frac{\Delta t}{2}\left[u_{\mathcal{A}} + \left(M_{11} u_{\mathcal{D}} + M_{12} v_{\mathcal{D}} + M_{13} w_{\mathcal{D}}\right)\right]$$
$$M_{23} r_{\mathcal{D}} = -\frac{\Delta t}{2}\left[v_{\mathcal{A}} + \left(M_{21} u_{\mathcal{D}} + M_{22} v_{\mathcal{D}} + M_{23} w_{\mathcal{D}}\right)\right]$$
$$M_{33} r_{\mathcal{D}} = r_A - \frac{\Delta t}{2}\left[w_{\mathcal{A}} + \left(M_{31} u_{\mathcal{D}} + M_{32} v_{\mathcal{D}} + M_{33} w_{\mathcal{D}}\right)\right]$$

A Local Cartesian Transform Approach

The result can be cast into a local Cartesian form:

$$X_{DA} = -\frac{\Delta t}{2}\left(U_{\mathcal{A}} + U_{DA}\right)$$

$$Y_{DA} = -\frac{\Delta t}{2}\left(V_{\mathcal{A}} + V_{DA}\right)$$

$$Z_{DA} = r_A - \frac{\Delta t}{2}\left(W_{\mathcal{A}} + W_{DA}\right)$$

where

$$\mathbf{X}_{DA} \equiv \mathbf{M}\mathbf{x}_{\mathcal{D}},$$

and

$$\mathbf{V}_{DA} \equiv \mathbf{M}\mathbf{v}_{\mathcal{D}}$$

are Departure point *coordinates* and *velocities* as seen in the Arrival-point Cartesian system

The Inverse Transformation

From $\left(X_{DA},\, Y_{DA},\, Z_{DA} \right)$ **obtain departure point as:**

$$\tan\left(\lambda_A - \lambda_D\right) = \frac{-X_{DA}}{Z_{DA}\cos\phi_A - Y_{DA}\sin\phi_A}$$

$$r_D^2 = X_{DA}^2 + Y_{DA}^2 + Z_{DA}^2$$

$$\sin\phi_D = \frac{Y_{DA}\cos\phi_A + Z_{DA}\sin\phi_A}{r_D}$$

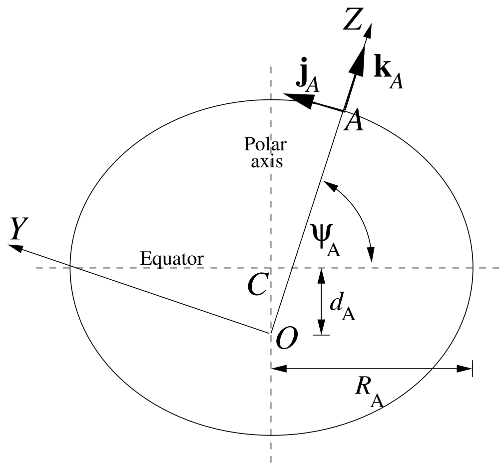
Spheroidal Case

Spheroidal case follows spherical case closely except:

$$\mathbf{x}_{\mathcal{A}} = \left(0,\ 0,\ r_A\right)^T - d_A\left(0,\ \cos\phi_A,\ \sin\phi_A\right)^T$$

and

$$\mathbf{x}_{\mathcal{D}} = \left(0,\ 0,\ r_D\right)^T - d_D\left(0,\ \cos\phi_D,\ \sin\phi_D\right)^T$$



Spheroidal Case

- **Applying M as before, but to modified position vectors, results in**

$$X_{DA} = M_{13} r_{\mathcal{D}}$$
$$Y_{DA} = M_{23} r_{\mathcal{D}} + (d_A - d_D) \cos\phi_A$$
$$X_{DA} = M_{33} r_{\mathcal{D}} + (d_A - d_D) \sin\phi_A$$

- **Inversion is a little more complicated**

Shallow-atmosphere case

Form obtained by constraining departure point to remain on arrival sphere:

$$\mathbf{x}_D = \mathbf{x}_A - \frac{\Delta t}{2}\left(\mathbf{v}_A + \mathbf{v}_D\right) + \mathcal{B}\left(\mathbf{x}_A + \mathbf{x}_D\right)$$

1. Cartesian transformation as before

$$\mathbf{X}_{DA} \equiv \mathbf{M}\mathbf{x}_{\mathcal{D}},$$

2. but velocities transform using momentum shallow-atmosphere rotation matrix

$$\mathbf{V}_{DA} \equiv \mathbf{Q}\mathbf{v}_{\mathcal{D}},$$

3. and

$$\mathbf{X}^H_{\mathcal{D}} = -\left(\frac{1 + M_{33}}{2}\right)\frac{\Delta t}{2}\left(\mathbf{V}^H_{\mathcal{A}} + \mathbf{V}^H_{\mathcal{D}}\right)$$

What does it all mean?

Single set of equations:

$$\boxed{\left(u_i - \alpha \Delta t \Psi_i\right)^{n+1} = M_{ij}\left(u_j + \beta \Delta t \Psi_j\right)_D^n}$$

with similar ones for departure point equations

Need only to specify:

1. **∇-operator in terms of the coordinate metric factors**
2. **Components of gravity vector (only vertical is non-zero)**
3. **Matrix elements M_{ij}**
4. **Components of Earth's rotation vector in chosen coordinate system**

Similar Oblate Spheroidal and Spherical Polar Coordinates

	Similar oblate spheroids (SOS) ($\varepsilon \equiv$ common eccentricity)	**Spherical polars** ($\varepsilon = 0$)
Coords	$(\lambda,\ \varphi,\ \mathtt{r})$ $\lambda =$ longitude $\varphi =$ meridional coordinate $\mathtt{r} =$ semi − major axis of spheroid	$(\lambda,\ \phi,\ r)$ $\lambda =$ longitude $\phi =$ latitude $r =$ radius of sphere
Metric factors	$h_\lambda = \left(\frac{\mathtt{r}}{h_\mathtt{r}}\right)\cos\phi$ $h_\varphi = \left(\frac{\mathtt{r}}{h_\mathtt{r}}\right)\left(\frac{\sin\phi\cos\phi}{\sin\varphi\cos\varphi}\right)$ $h_\mathtt{r} = \left(1 - \varepsilon^2\sin^2\phi\right)^{1/2}$	$h_\lambda = r\cos\phi$ $h_\phi = r$ $h_r = 1$
Gravity	$g = \frac{g_a}{\left(1-\epsilon^2\sin^2\phi\right)^{1/2}}\left(\frac{a}{\mathtt{r}}\right)^2$	$g = g_a\left(\frac{a}{r}\right)^2$
Notes	$\phi =$ geographic latitude $\varphi \rightarrow \phi$ & $\mathtt{r} \rightarrow r$ as $\varepsilon \rightarrow 0$	Geographic latitude = spherical polar latitude when $\varepsilon = 0$

Result is...

Identical numerical system encompassing hierarchy of model approximations:

1. Similar Oblate Spheroidal geopotential approximation
2. Deep-atmosphere Spherical geopotential approximation (no latitudinal variation of g)
3. Shallow-atmosphere approximation (non-Euclidean geometry)
4. Cartesian

Hydrostatic versions apply to (2)-(4)

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Where are we?

- Elements of approach form basis of deep-atmosphere, non-hydrostatic, spherical New Dynamics in MetUM

- Next version of dynamical core (ENDGame) will incorporate full functionality

- Allow clean assessment of impact of various approximations

Summary

- Can relax Spherical Geopotential Approximation
  - Retain 2/3rds latitudinal variation of gravity
- Semi-Lagrangian method can be straightforwardly applied to non-spherical coordinate systems
  - Especially when use geographic latitude
- Eulerian component equations can be derived from the rotation matrix forms

Summary

- **Matrix view gives insight into the non-Euclidean nature of the shallow-atmosphere approximation**
  - **neglect of great circle curvature**

- **Therefore desirable to avoid shallow-atmosphere approximation:**
  - **avoid loss of some (significant) Coriolis terms**
  - **and avoid non-Euclidean, space-distorting character**

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Questions?

Further details can be found in:

- Spheroidal coordinate systems for modelling global atmospheres
  White, Staniforth and Wood 2008 *Q.J.R.Meteorol.Soc.* 134 pp 261-270

- Rotation matrix treatment of vector equations in
  semi-Lagrangian models of the atmosphere I: Momentum equation
  Staniforth, White and Wood. 2010 *Q.J.R.Meteorol.Soc.* 136 pp 497-506

- Rotation matrix treatment of vector equations in
  semi-Lagrangian models of the atmosphere II: Kinematic equation
  Staniforth, White and Wood. 2010 *Q.J.R.Meteorol.Soc.* 136 pp 507-516