Multiscale marine models with high order discontinuous finite elements

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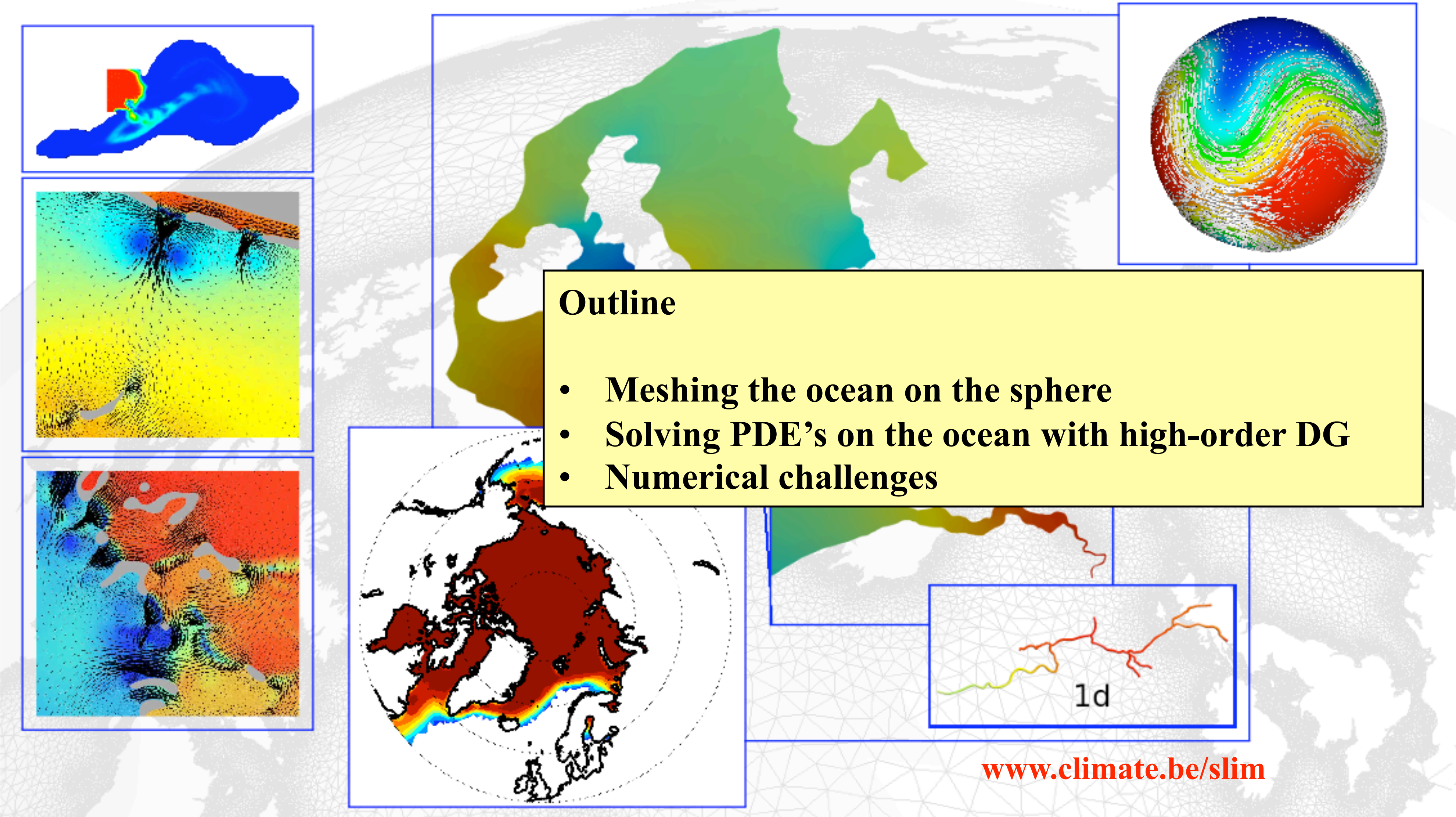
Slim : a multi-scale model for the ocean, coaslines and rivers

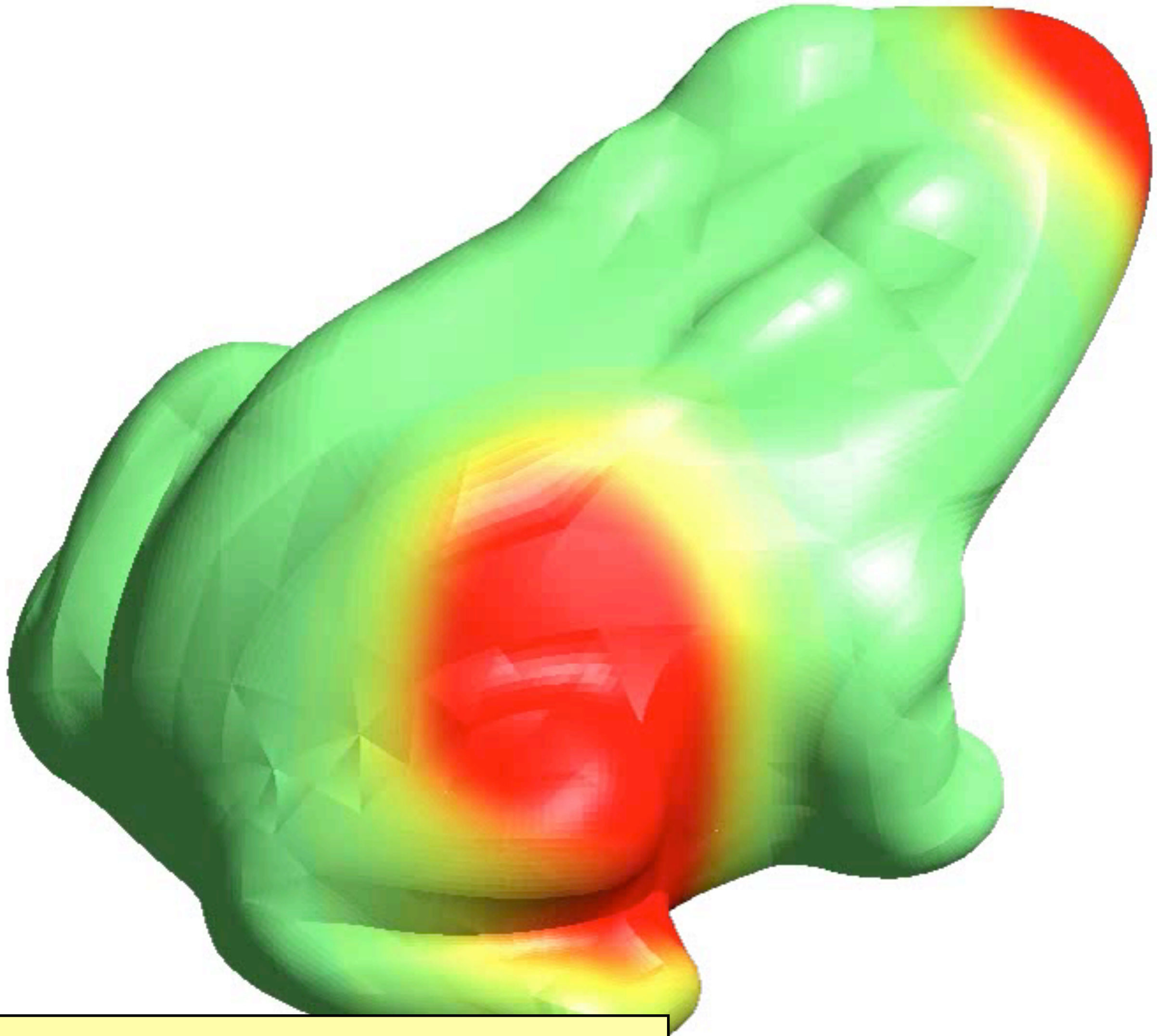


Outline

- **Meshing the ocean on the sphere**
- **Solving PDE’s on the ocean with high-order DG**
- **Numerical challenges**

www.climate.be/slim

Gravity waves on a froggy planet

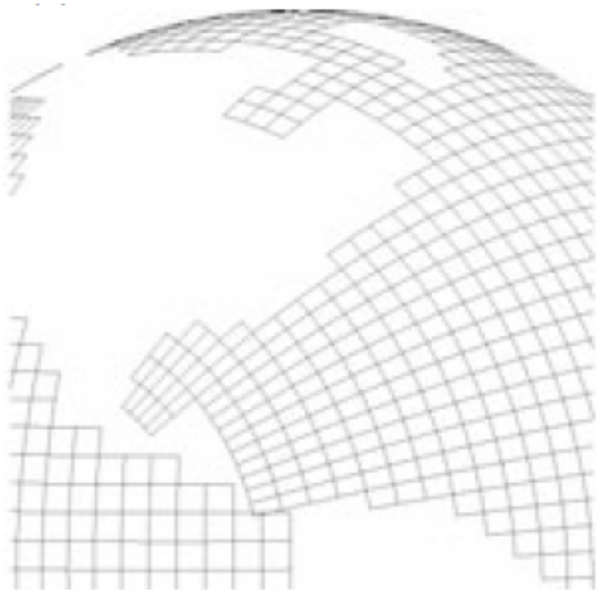


Building a general method for irregular manifolds

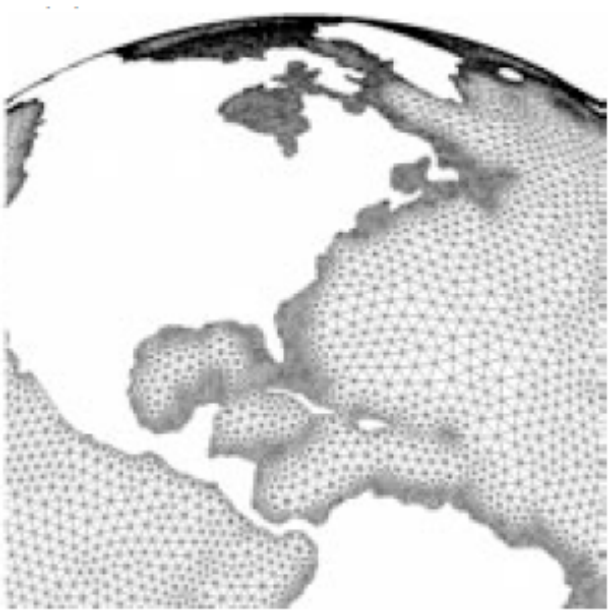
- **The method is independent of the manifold**
- **It must be easy to implement**
- **It must be robust to handle such a funny benchmark**

Structured grid …

- Finite differences are easy to implement
- Programming is easy
- Well known in the world of oceanography
- Bad representation of the coastlines
- Difficult to enhance locally the resolution
- Poles singularity

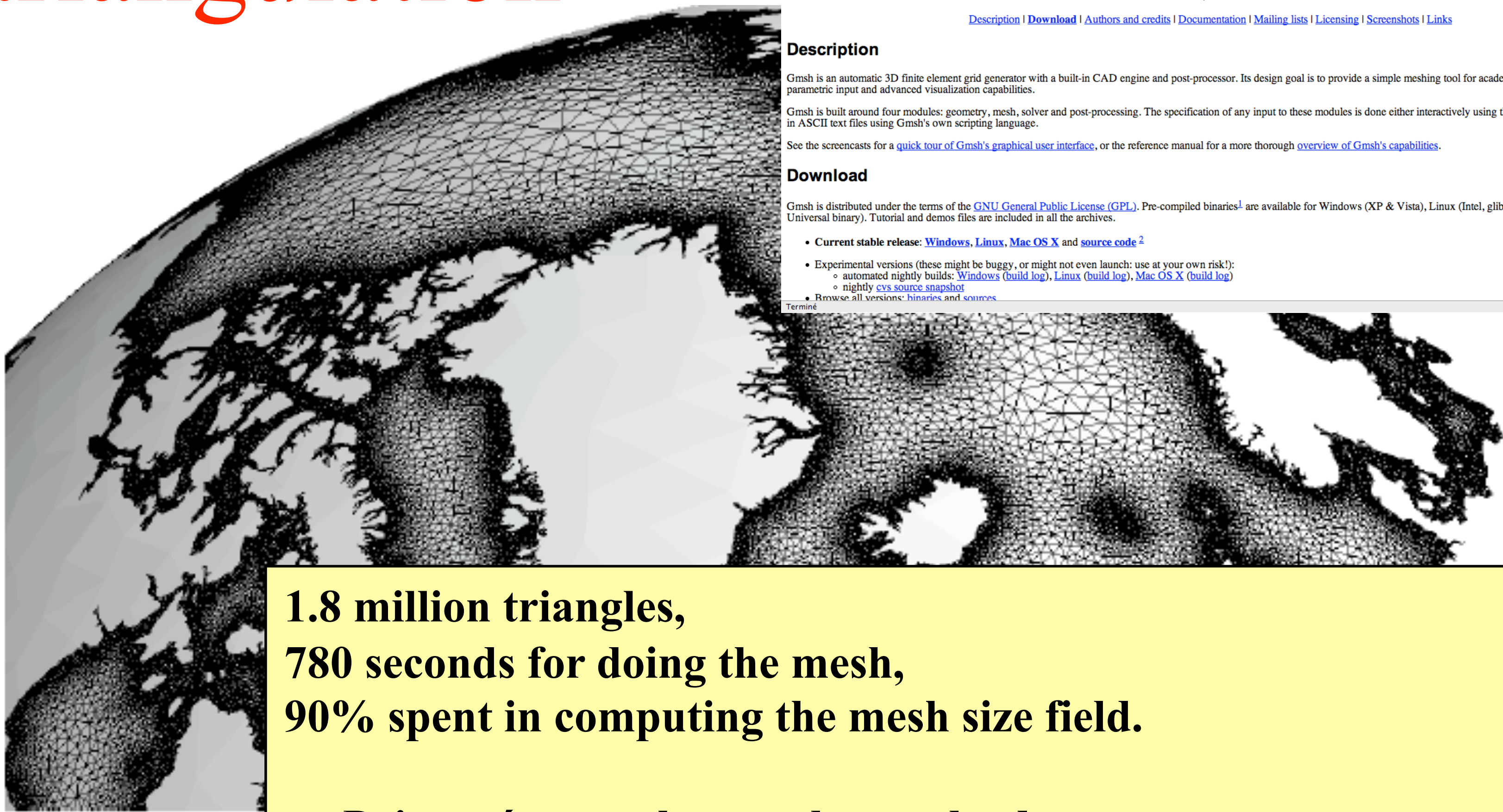


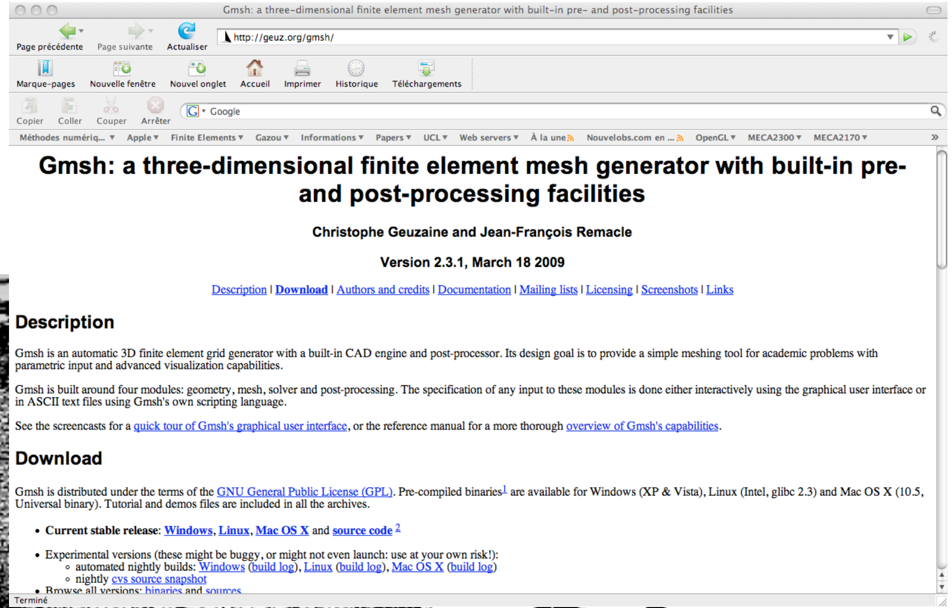
…versus unstructured grid



- Numerical methods are more complicated
- Programming is more complicated
- Not well known in the world of oceanography
- Accurate representation of the coastlines
- Enhancing the resolution is flexible
- No singular points

Delaunay based triangulation





Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities

Christophe Geuzaine and Jean-François Remacle

Version 2.3.1, March 18 2009

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Description

Gmsh is an automatic 3D finite element grid generator with a built-in CAD engine and post-processor. Its design goal is to provide a simple meshing tool for academic problems with parametric input and advanced visualization capabilities.

Gmsh is built around four modules: geometry, mesh, solver and post-processing. The specification of any input to these modules is done either interactively using the graphical user interface or in ASCII text files using Gmsh's own scripting language.

See the screencasts for a quick tour of Gmsh's graphical user interface, or the reference manual for a more thorough overview of Gmsh's capabilities.

Download

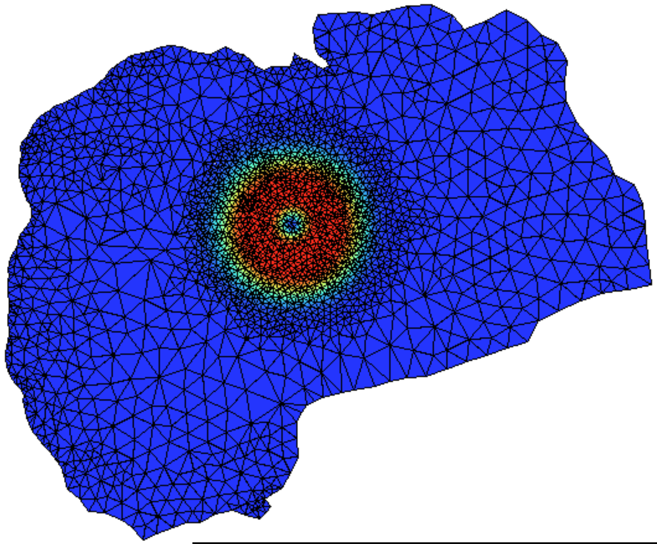
Gmsh is distributed under the terms of the GNU General Public License (GPL). Pre-compiled binaries[1] are available for Windows (XP & Vista), Linux (Intel, glibc 2.3) and Mac OS X (10.5, Universal binary). Tutorial and demos files are included in all the archives.

- **Current stable release: Windows, Linux, Mac OS X** and **source code** [2]
- Experimental versions (these might be buggy, or might not even launch: use at your own risk!):
  - automated nightly builds: Windows (build log), Linux (build log), Mac OS X (build log)
  - nightly cvs source snapshot
- Browse all versions: binaries and sources

1.8 million triangles,
780 seconds for doing the mesh,
90% spent in computing the mesh size field.

- **Poincaré waves have to be resolved**
- **Mesh size smaller along coastlines**
- **Geometry of the coastlines has to be represented**

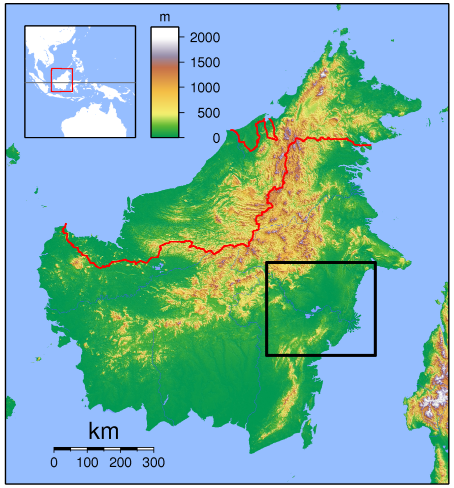
Are adaptive unstructured-grid models coming of age ?

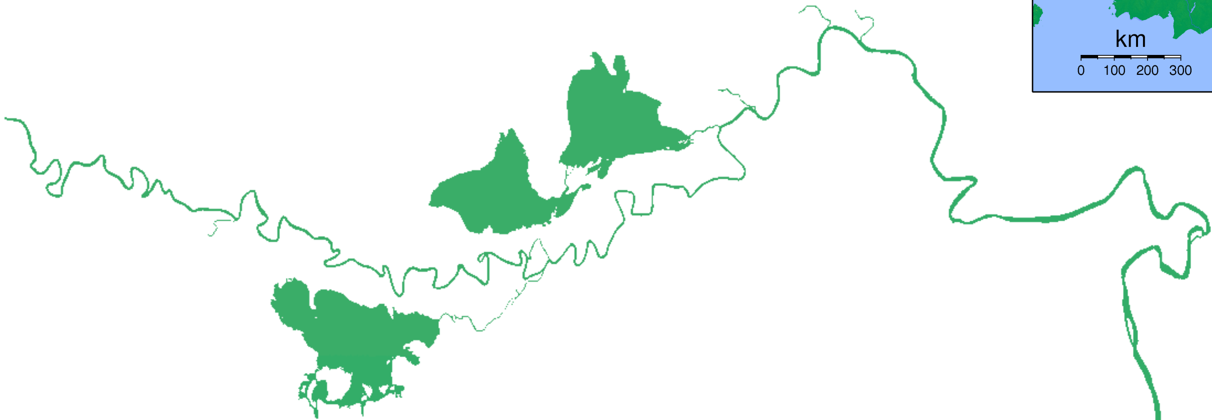


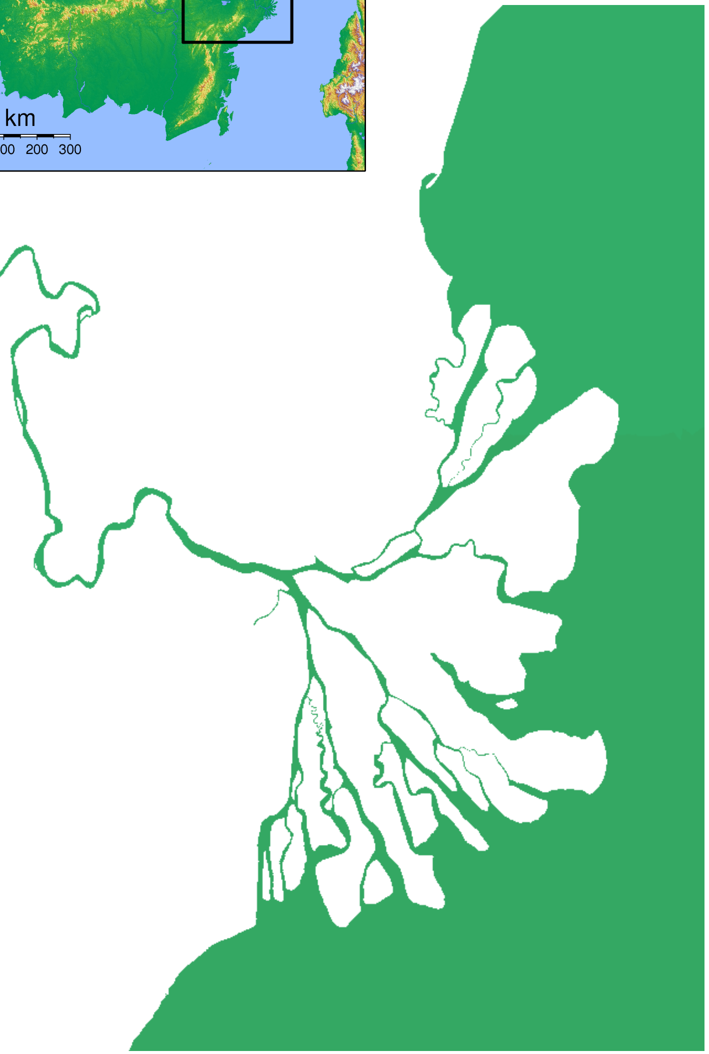
Reduced-gravity simulation of a baroclinic eddy in the Gulf of Mexico.

This simulation is several orders of magnitude cheaper than a constant resolution one of the same accuracy ! (Bernard, 2007)

- **Numerical models of marine systems should be able to explicitly represent the broadest possible range of scales.**
- **Increasing the resolution everywhere is not the best option as this often results in a very inefficient use of the computational resources.**
- **The idea is to increase the resolution where and when it is needed !**

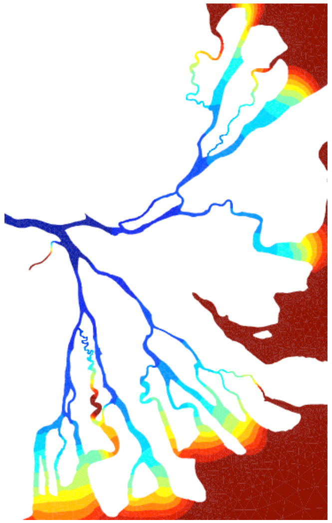


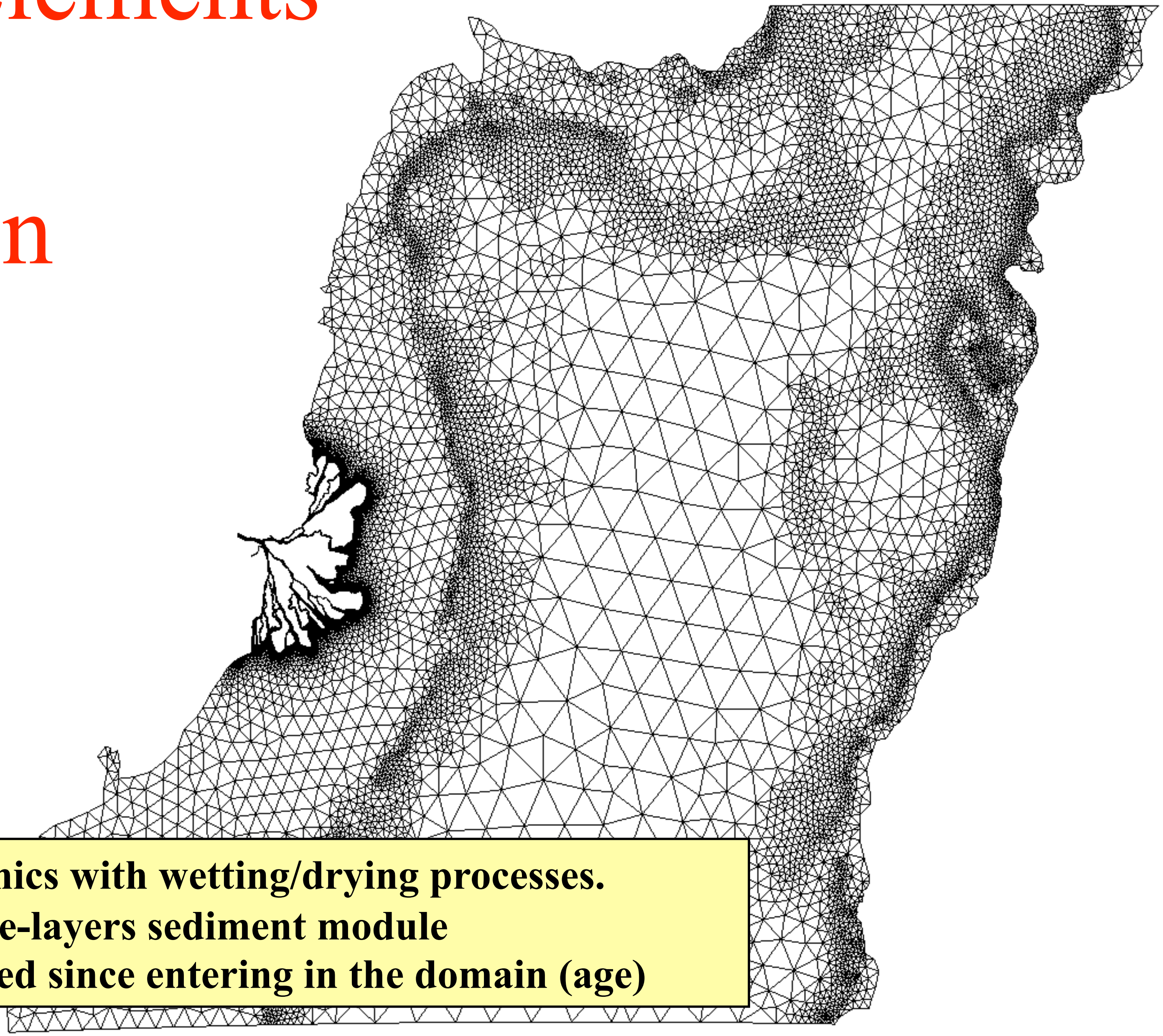




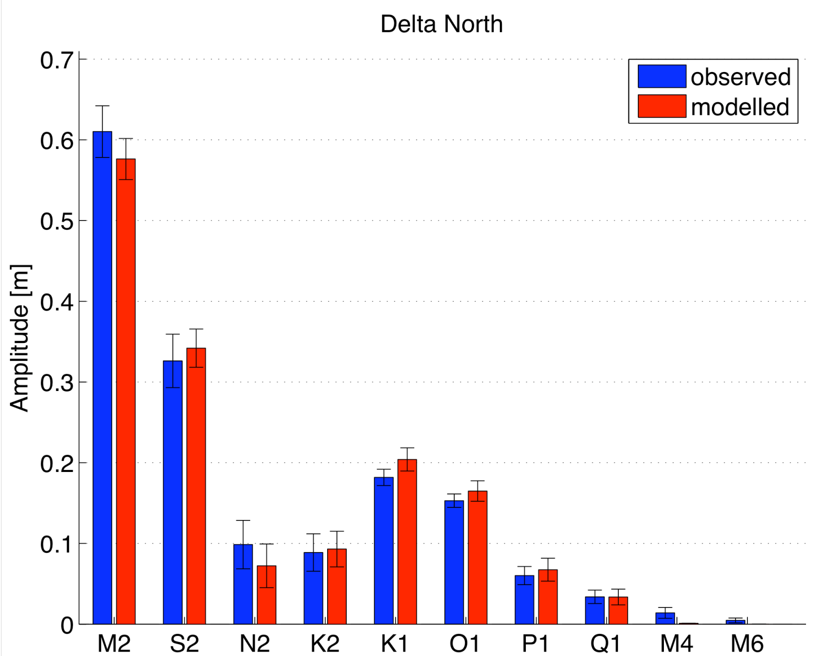
Mahakam river,
a highly multi-scaled
motivating example

72% of the elements are in 1.4% of the domain





- Validated hydrodynamics with wetting/drying processes.
- Development of a three-layers sediment module
- Computing time elapsed since entering in the domain (age)



Hydrodynamics

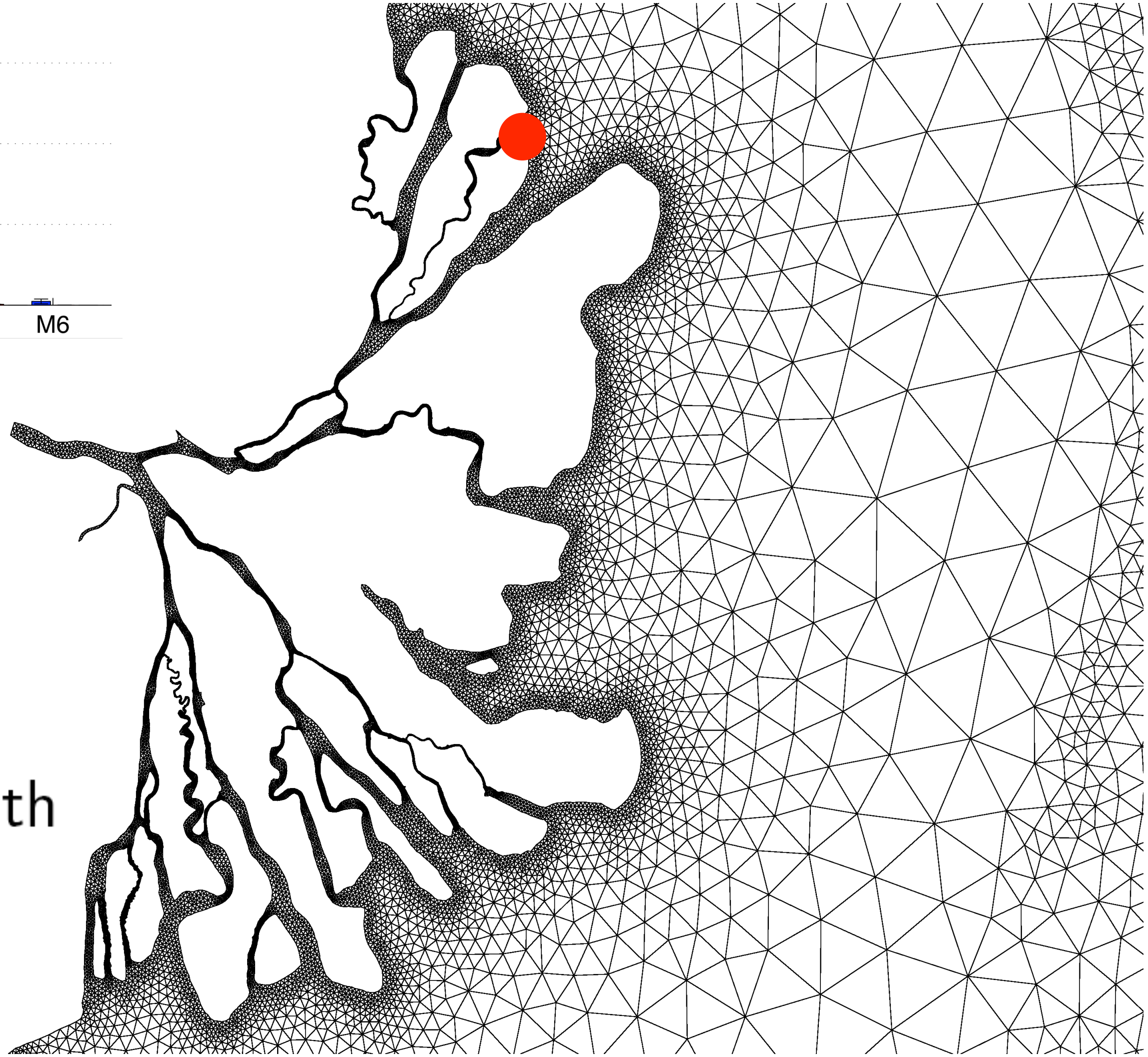
$\Delta \propto \sqrt{gH}$

$\Delta \propto$ distance to coast

$\Delta \propto ||\nabla H||^{-1}$

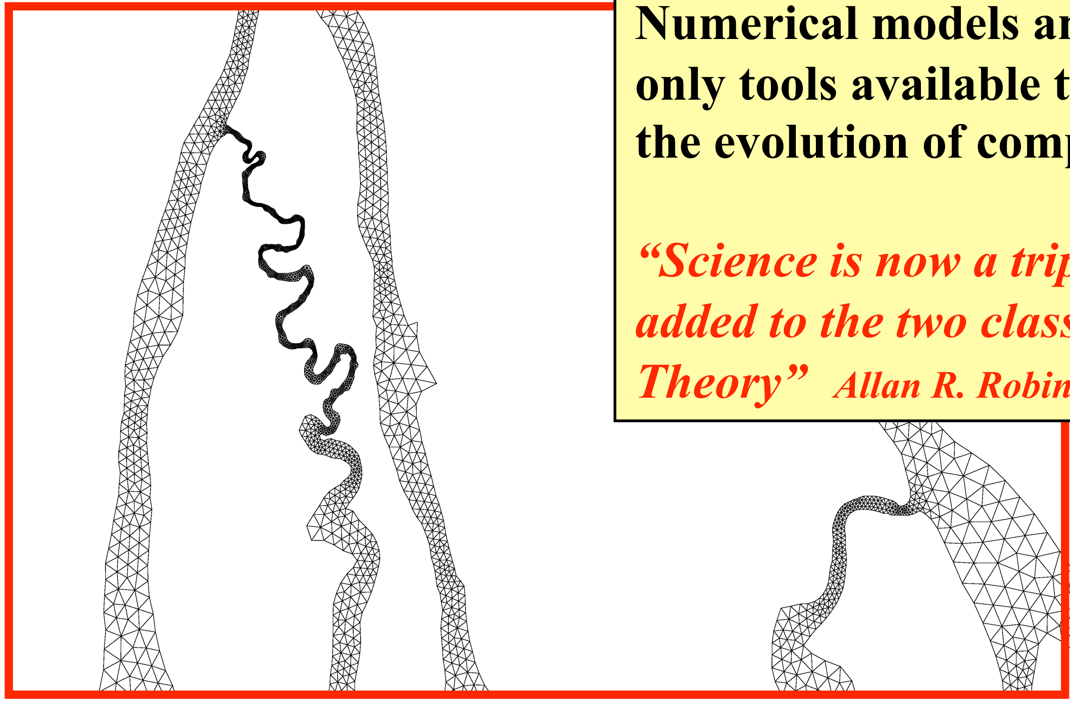
$\Delta \propto$ delta channels width

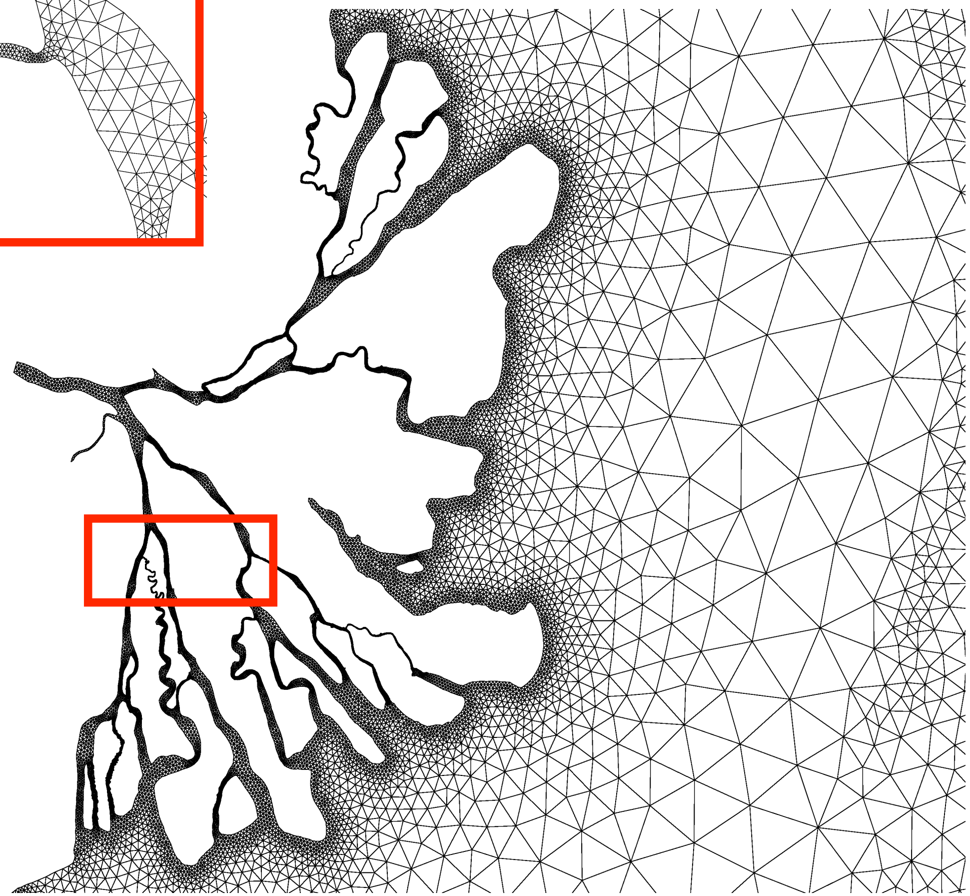
$N \approx 50\,000$ triangles



Numerical models and computer simulations are the only tools available to understand in detail and predict the evolution of complex environmental systems.

"Science is now a tripartite endeavour, with Simulation added to the two classical components, Experiment and Theory" ***Allan R. Robinson***





Size of the smallest element is 7 m

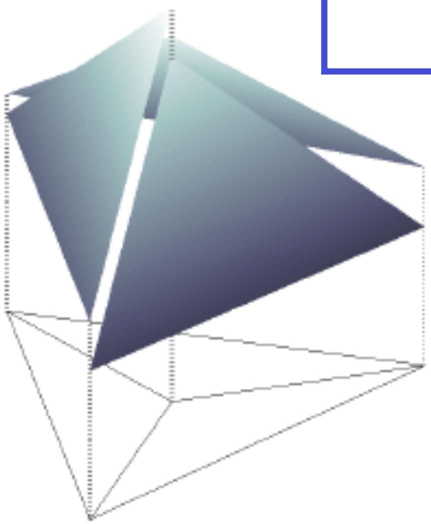
The Galerkin Discontinuous Method

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, \eta) = \cancel{\boldsymbol{f}(\boldsymbol{u}, \eta)}$$

The classical DG weak formulation reads:

$$\frac{\partial}{\partial t} < \boldsymbol{u}^h \cdot \hat{\mathbf{u}}^h >_{\Omega_e} \quad - \quad < \boldsymbol{\sigma}(\boldsymbol{u}^h, \eta^h) \cdot \nabla \hat{\mathbf{u}}^h >_{\Omega_e}$$

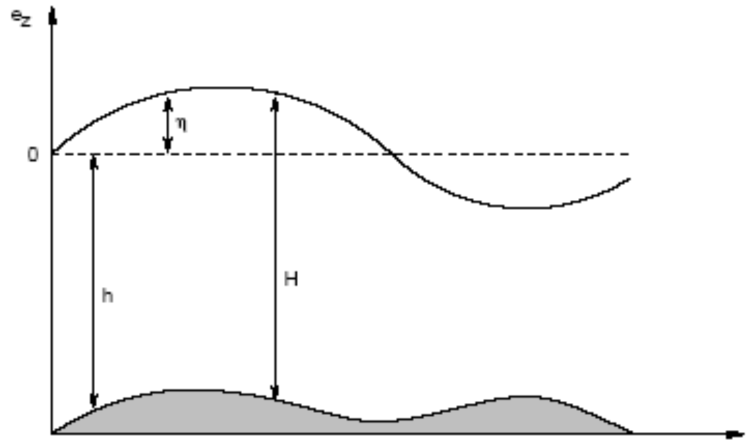
$$+ \quad \ll \boldsymbol{\sigma}^*(\boldsymbol{u}^h, \eta^h) \cdot \boldsymbol{n} \cdot \hat{\mathbf{u}}^h \gg_{\partial\Omega_e} \quad = \quad \mathbf{0}$$



- **Bloc-diagonal global matrices**
- **Transfer between elements through the flux on the edges**
- **A weak collocated formulation can be also derived**
- **Upwinding by the flux evaluation (Riemann's solver)**

The Shallow Water Equations…

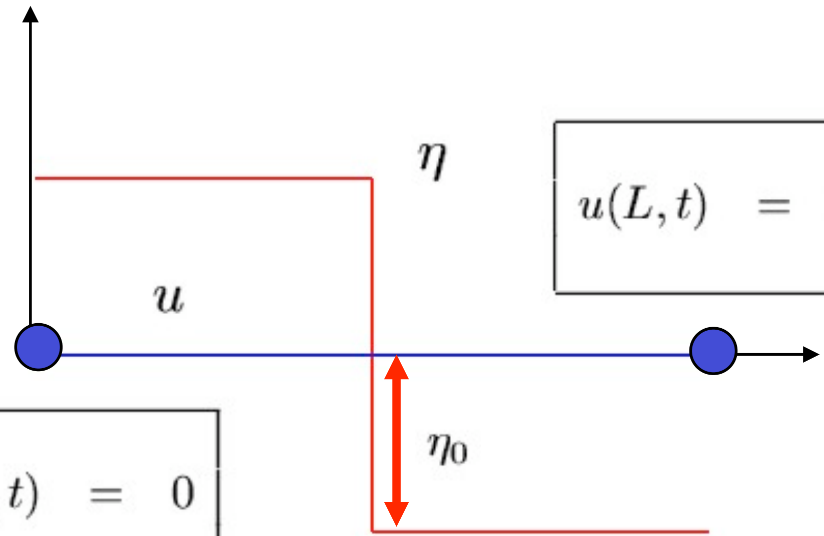
$$
\begin{aligned}
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \\
\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + fu + g\frac{\partial \eta}{\partial y} &= 0
\end{aligned}
$$



- Very crude model for geophysical flows, but allows the existence of inertia-gravity waves
- Constant depth of the ocean

A 1D sharp simplified problem in a <u>finite</u> domain

$$
\begin{aligned}
\frac{\partial \eta}{\partial t} + H\frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + fu &= 0
\end{aligned}
$$



$$u(L,t) = 0$$

$$u(0,t) = 0$$

$$
\begin{aligned}
u(x,0) &= 0 \\
v(x,0) &= 0 \\
\eta(x,0) &= \eta_0 & x \in [0, \tfrac{L}{2}[\\
\eta(x,0) &= -\eta_0 & x \in]\tfrac{L}{2}, L]
\end{aligned}
$$

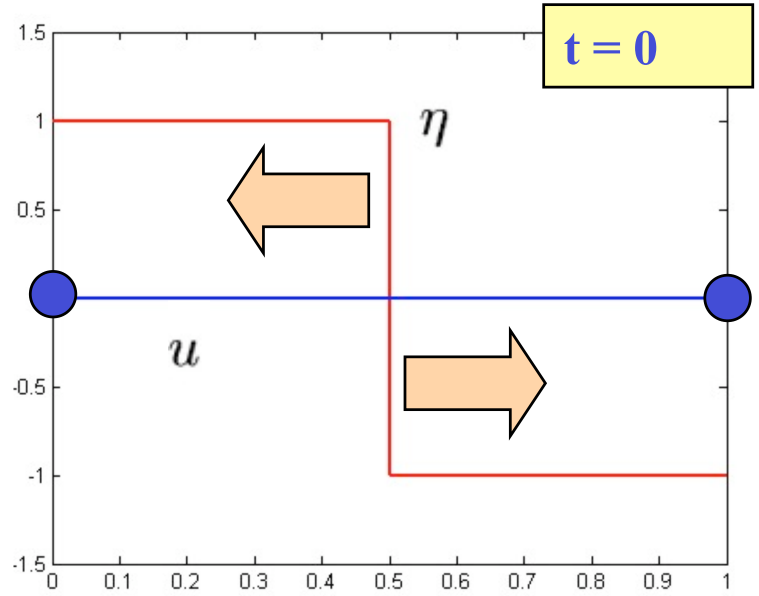
What is the solution ?

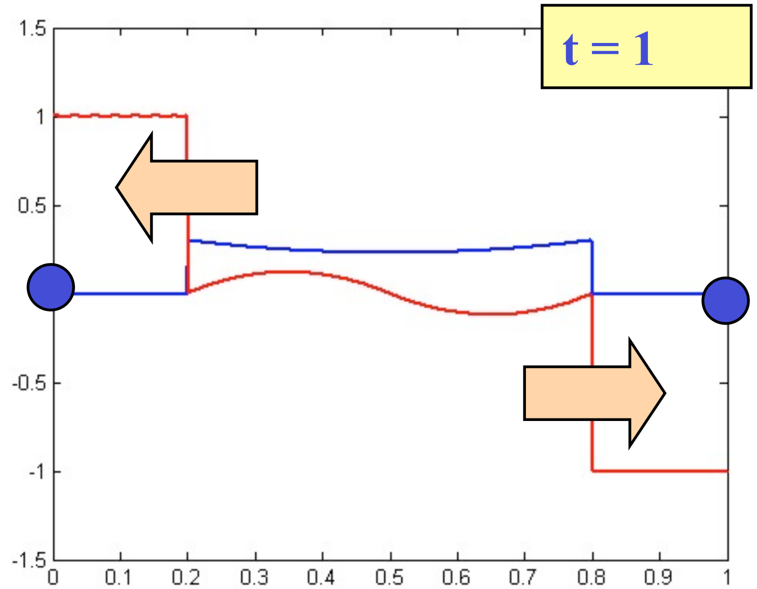
$$\begin{aligned} \frac{\partial \eta'}{\partial t'} + \frac{\partial u'}{\partial x'} &= 0 \\ \frac{\partial u'}{\partial t'} - v' + \alpha^2 \frac{\partial \eta'}{\partial x'} &= 0 \\ \frac{\partial v'}{\partial t'} + u' &= 0 \end{aligned}$$

$$x' = \frac{x}{L}, \qquad t' = f\,t, \qquad \eta' = \frac{\eta}{\eta_0}, \qquad \boldsymbol{u}' = \frac{H\boldsymbol{u}}{Lf\eta_0},$$

$$\alpha = \boxed{\frac{\sqrt{gH}}{f}}\,\frac{1}{L}$$

Rossby's radius

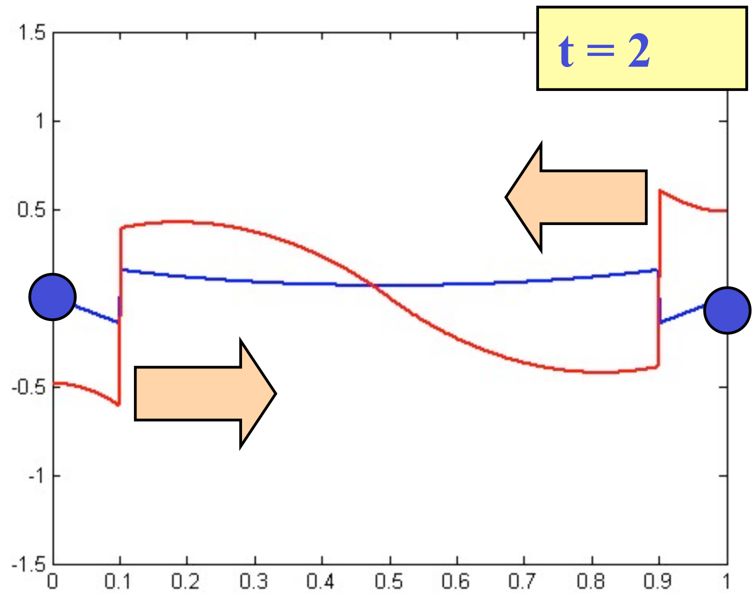


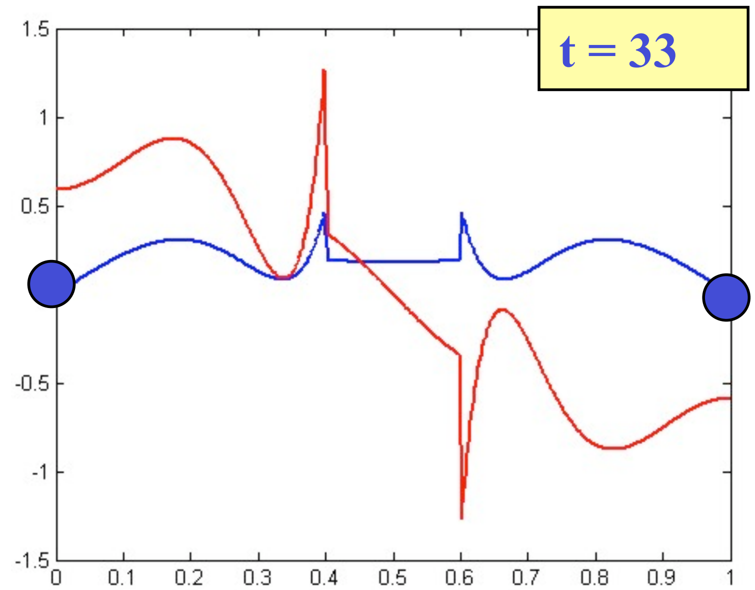


A more and more complex and interesting solution…

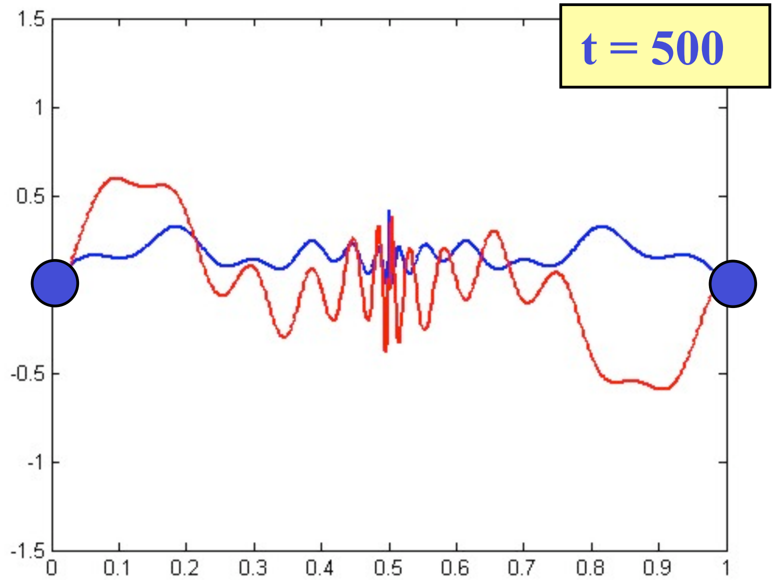
$$\alpha = \frac{\sqrt{gH}}{f}\frac{1}{L} = \frac{\sqrt{10}}{10} = 0.3162$$

$$\begin{array}{lcl} f & = & 10^{-4}\ 1/s \\ L & = & 1000\ Km \\ H & = & 100\ m \\ g & = & 10\ m/s^2 \end{array}$$





What are the equations ?



$$\begin{aligned}
\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} - v + \alpha^2 \frac{\partial \eta}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + u &= 0
\end{aligned}$$

$$\frac{\partial^2 u}{\partial t^2} - \underbrace{\frac{\partial v}{\partial t}}_{-u} + \alpha^2 \underbrace{\frac{\partial^2 \eta}{\partial t \partial x}}_{-\frac{\partial^2 u}{\partial x^2}} = 0$$

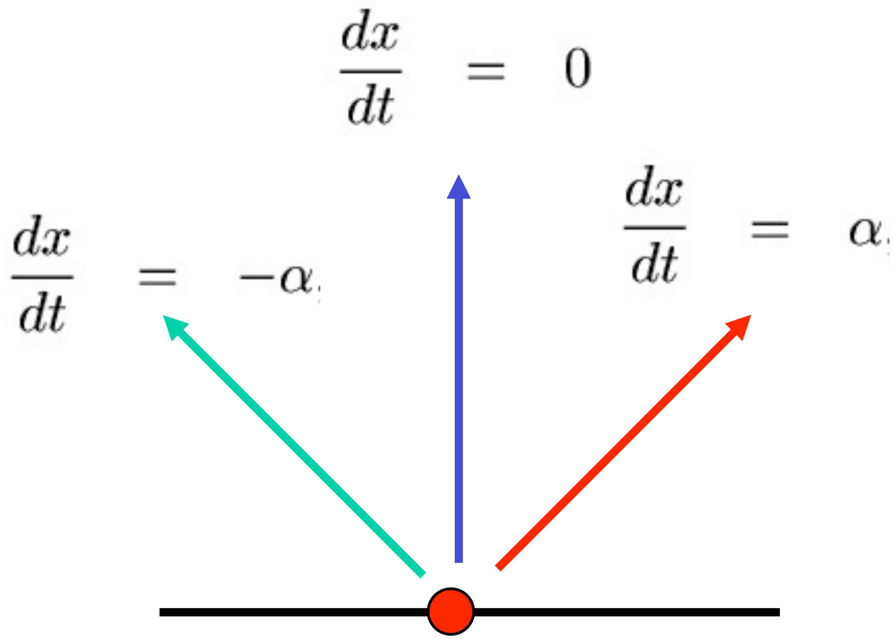
Helmholtz's Equation
Forced Wave Equation

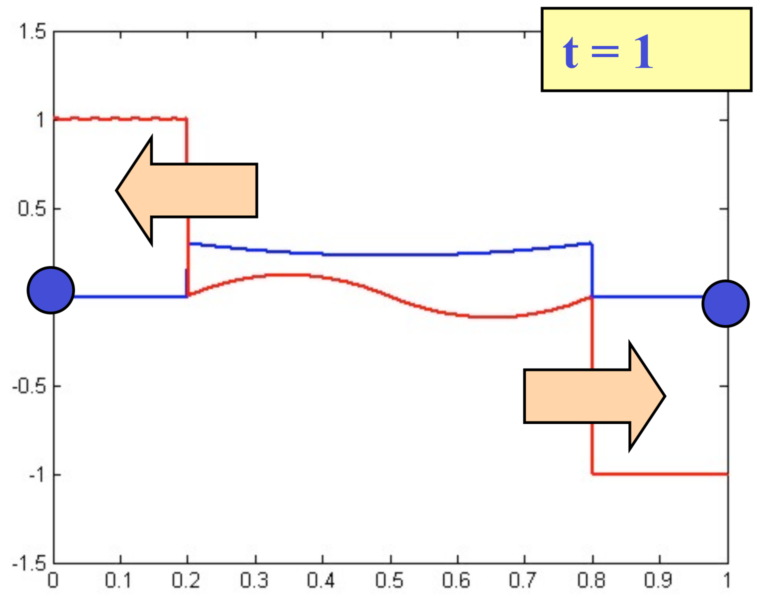
$$\frac{\partial^2 u}{\partial t^2} + u - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

How does information propagate ?

$$
\begin{aligned}
\frac{d}{dt}\boxed{(\alpha\eta - u)} &= -v && \text{on } \frac{dx}{dt} = -\alpha, \\
\frac{d}{dt}\boxed{(\alpha\eta + u)} &= v && \text{on } \frac{dx}{dt} = \alpha, \\
\frac{dv}{dt} &= -u && \text{on } \frac{dx}{dt} = 0,
\end{aligned}
$$

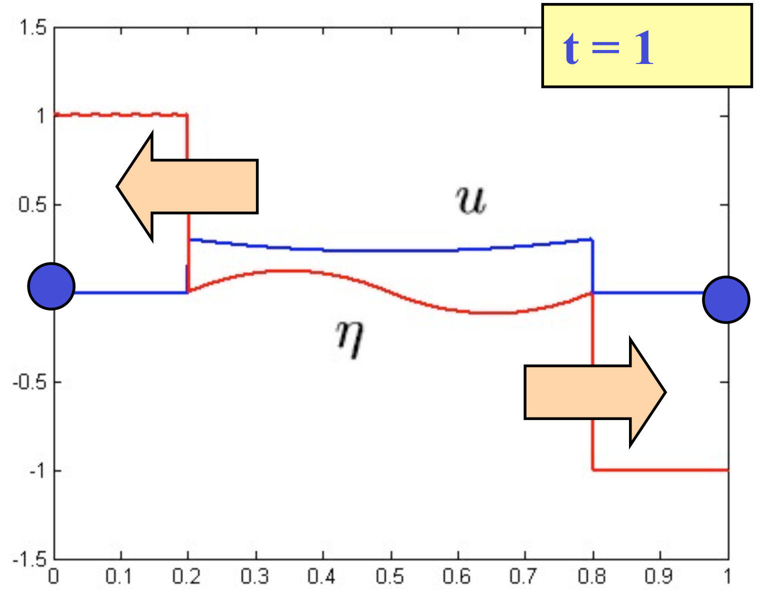
Riemann's Invariants

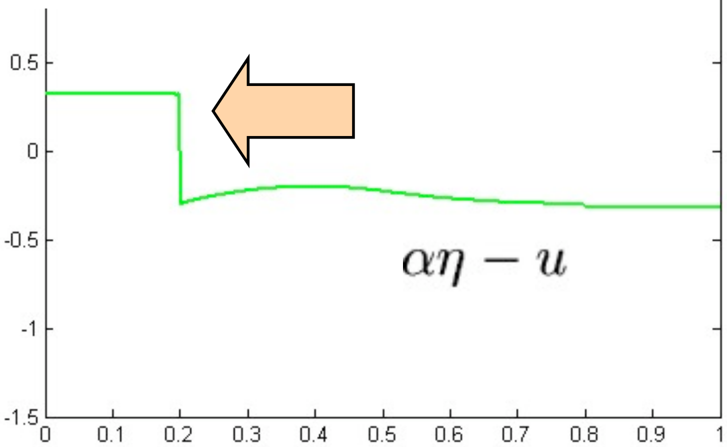


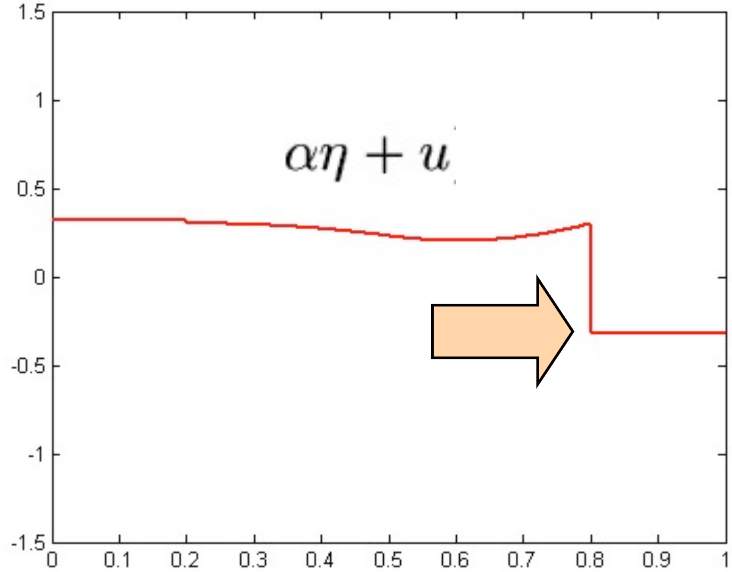


Two distinct waves…

$$
\begin{aligned}
\frac{d}{dt}\boxed{(\alpha\eta - u)} &= -v \qquad \text{on } \frac{dx}{dt} = -\alpha,\\
\frac{d}{dt}\boxed{(\alpha\eta + u)} &= v \qquad \text{on } \frac{dx}{dt} = \alpha,\\
\frac{dv}{dt} &= -u \qquad \text{on } \frac{dx}{dt} = 0,
\end{aligned}
$$

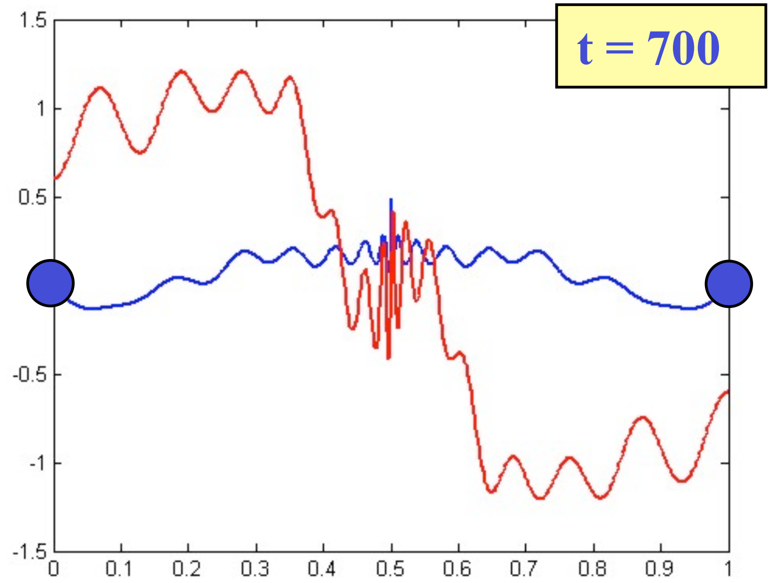






An analytical solution exists !

$$\frac{\partial^2 u}{\partial t^2} + u - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$



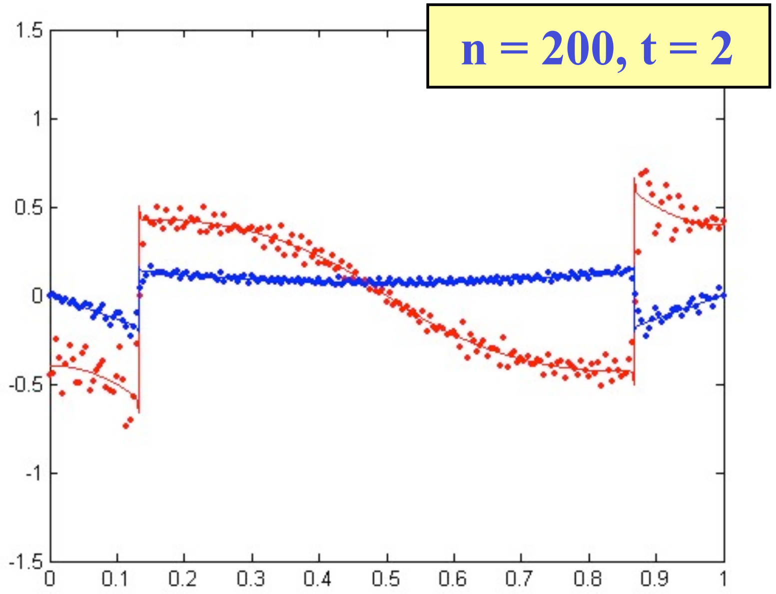
Separation of the Classical Equations with the boundary conditions

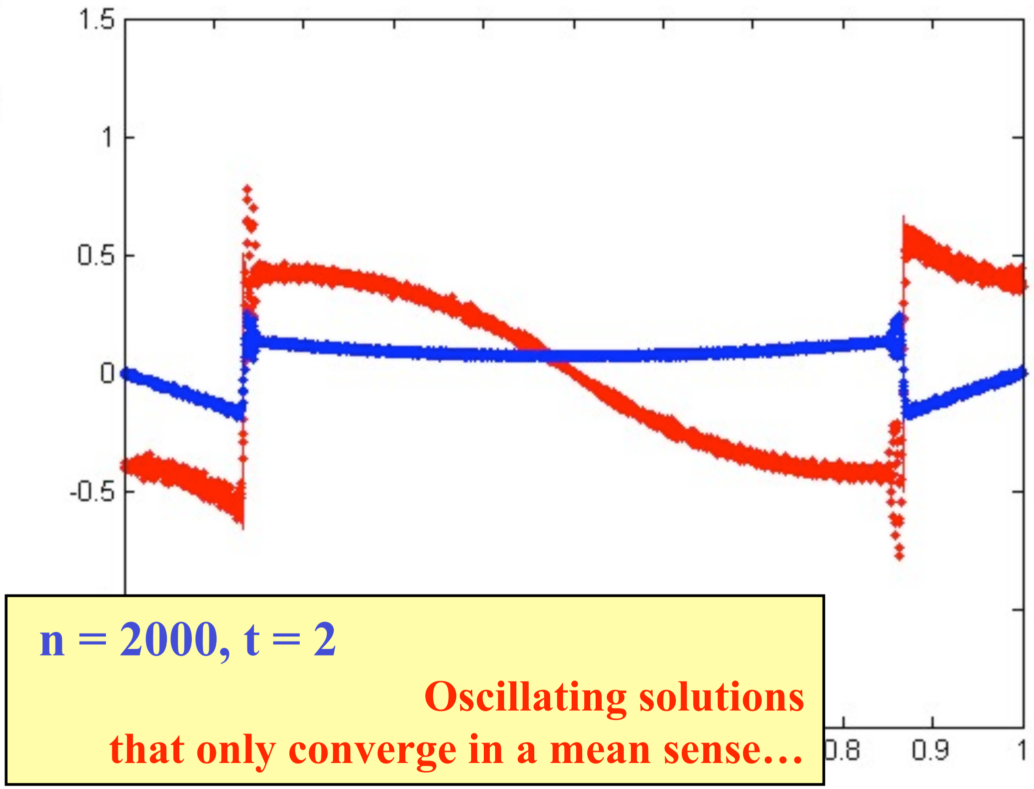
$$u(x,t) = T(t) f(x)$$

$$\frac{T''}{T} = \alpha^2 \frac{f''}{f} - 1$$

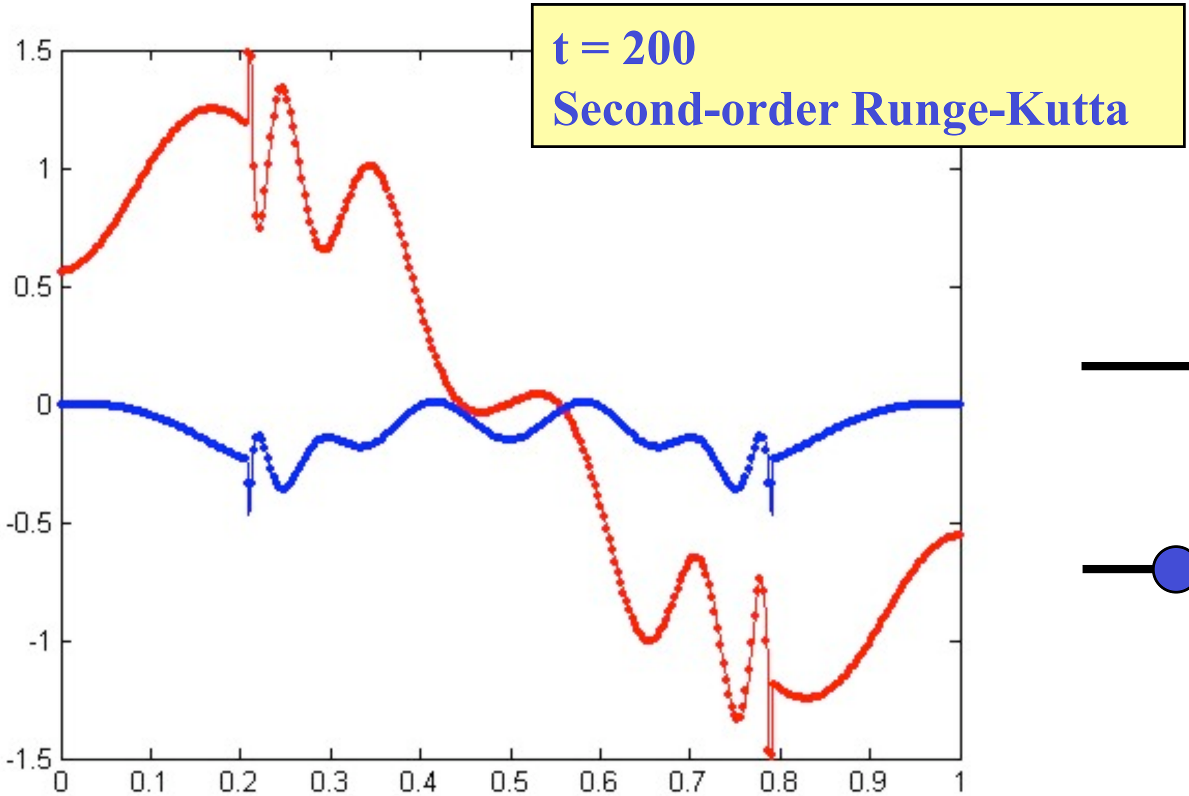
$$\begin{aligned} u(x,t) &= \sum_{i=1}^{\infty} \frac{4\alpha^2(-1)^{i+1}}{\omega_i} \sin(\omega_i t) \sin(k_i x) \\ k_i &= (2i-1)\pi \\ \omega_i &= \sqrt{1+\alpha^2 k_i^2} \end{aligned}$$

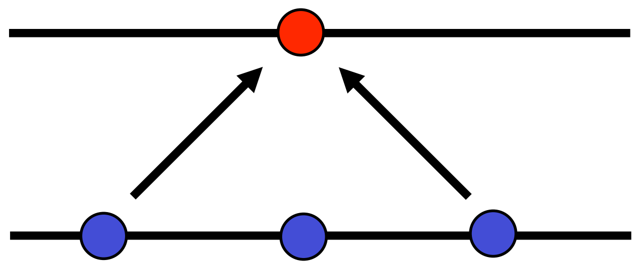
The Continuous Galerkin Method



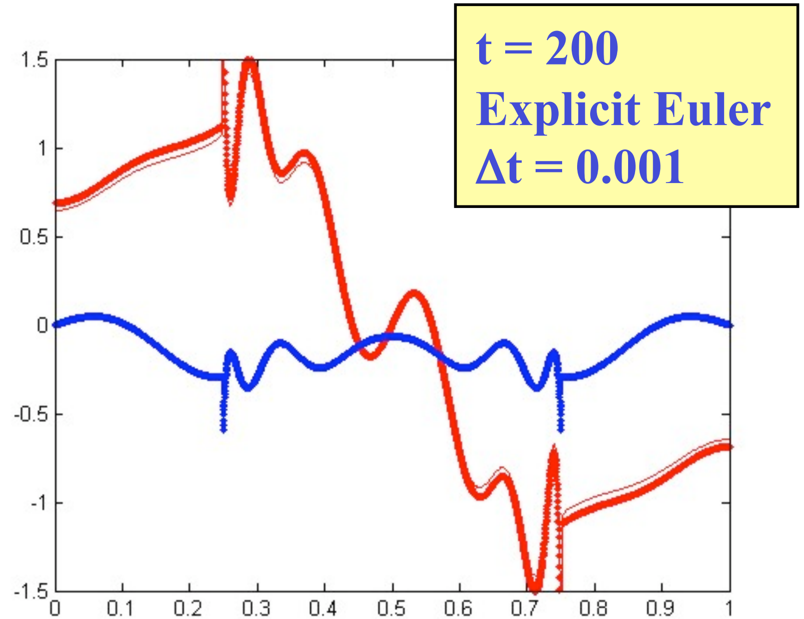


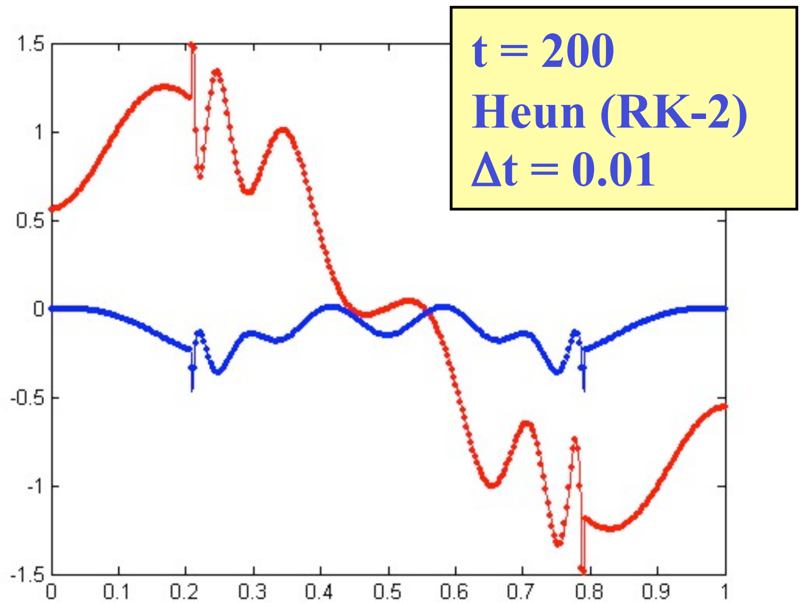
The Optimal Technique : Integrating along characteristics

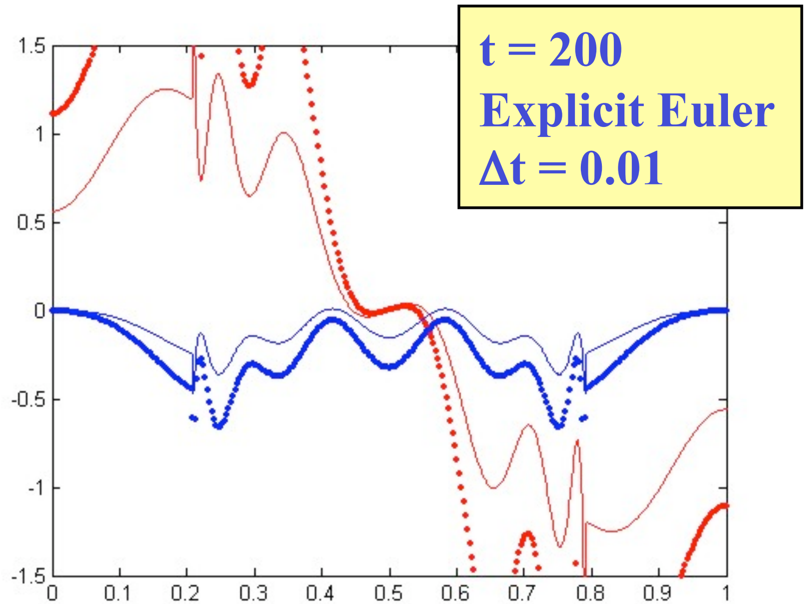




Time integration has to be accurately performed…







The Discontinuous <u>so-called</u> Galerkin Method

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

Find $u^h \in \mathcal{U}^h$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u^h}{\partial t} \hat{u}^h + \frac{\partial u^h}{\partial x} \hat{u}^h \right) dx + \boxed{\sum_{e=1}^{N_E} \left[a(\hat{u}^h)[u^h] \right]_{\Omega_e}} = 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h,$$

Penalty term to enforce weak continuity of the solution

$$a(\hat{u}^h) = \boxed{\zeta} \left(1 - \boxed{\lambda}\, \text{sign}(n) \right) \hat{u}^h$$

Penalty Factor (usually = 1)

Upwinding Factor (usually = 1)

After some tedious algebra…

Find $u^h \in \mathcal{U}^h$ such that

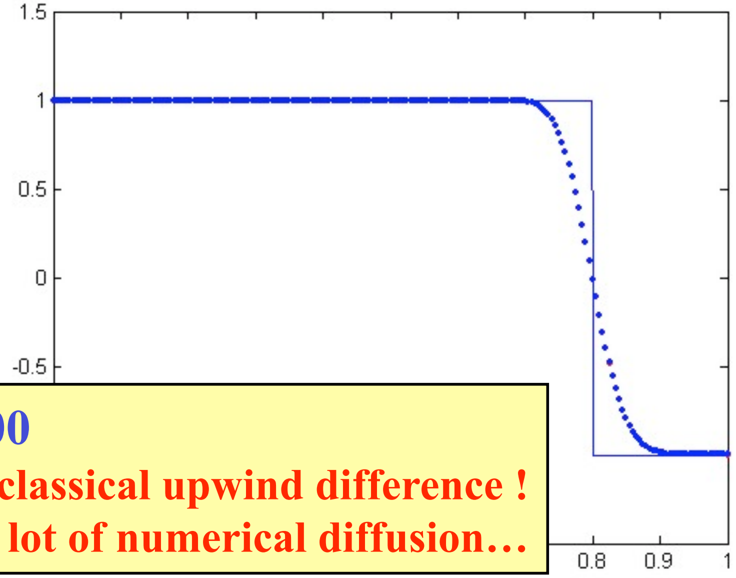
$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u^h}{\partial t} \hat{u}^h + \frac{\partial u^h}{\partial x} \hat{u}^h \right) dx + \sum_{e=1}^{N_E} \left[\, a(\hat{u}^h)[u^h] \, \right]_{\Omega_e} \quad = \quad 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h,$$

Considering only once integrals along internal segments.

Find $u^h \in \mathcal{U}^h$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u^h}{\partial t} \hat{u}^h - u^h \frac{\partial \hat{u}^h}{\partial x} \right) dx + \sum_{i=2}^{N_E} < u^h(X_i) >_\lambda [\hat{u}^h(X_i)] \quad = \quad 0 \qquad \forall \hat{u}^h \in \hat{\mathcal{U}}^h,$$

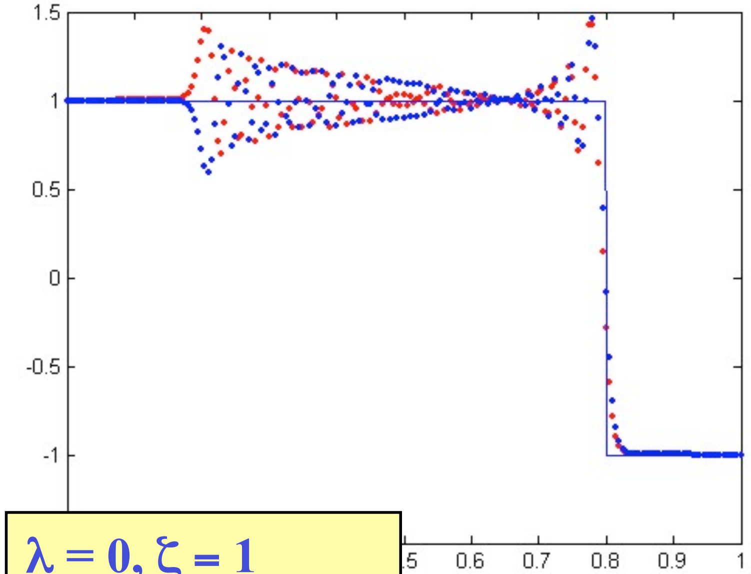
How to impose the continuity constraint ?



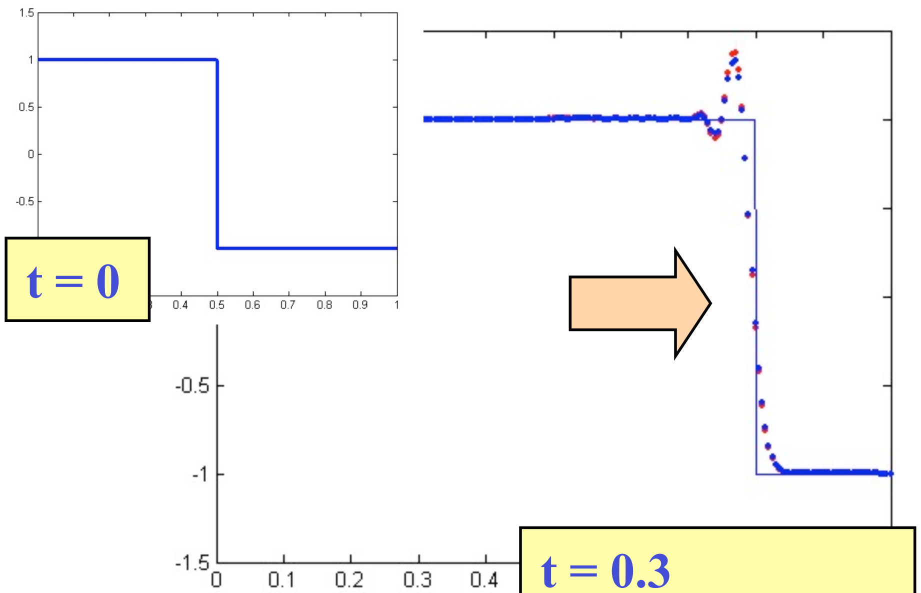
$\lambda = 1, \zeta = 100$

Almost the classical upwind difference !

Stable, but a lot of numerical diffusion…



$\lambda = 0, \zeta = 1$



$t = 0$

$t = 0.3$

$\lambda = 1, \zeta = 1$

The Discontinuous Galerkin Method

Find $\eta \in \mathcal{E}$ and $(u, v) \in \mathcal{U} \times \mathcal{U}$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial \eta}{\partial t} \hat{\eta} + \frac{\partial u}{\partial x} \hat{\eta} \right) dx + \boxed{\sum_{e=1}^{N_E} \Big[\, a(\hat{\eta})[u] \, \Big]_{\Omega_e}} = 0 \qquad \forall \hat{\eta} \in \mathcal{E},$$

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u}{\partial t} \hat{u} + v\hat{u} + \alpha^2 \frac{\partial \eta}{\partial x} \hat{u} \right) dx + \boxed{\sum_{e=1}^{N_E} \Big[\, b(\hat{u})[\alpha^2 \eta] \, \Big]_{\Omega_e}} = 0 \qquad \forall \hat{u} \in \mathcal{U},$$

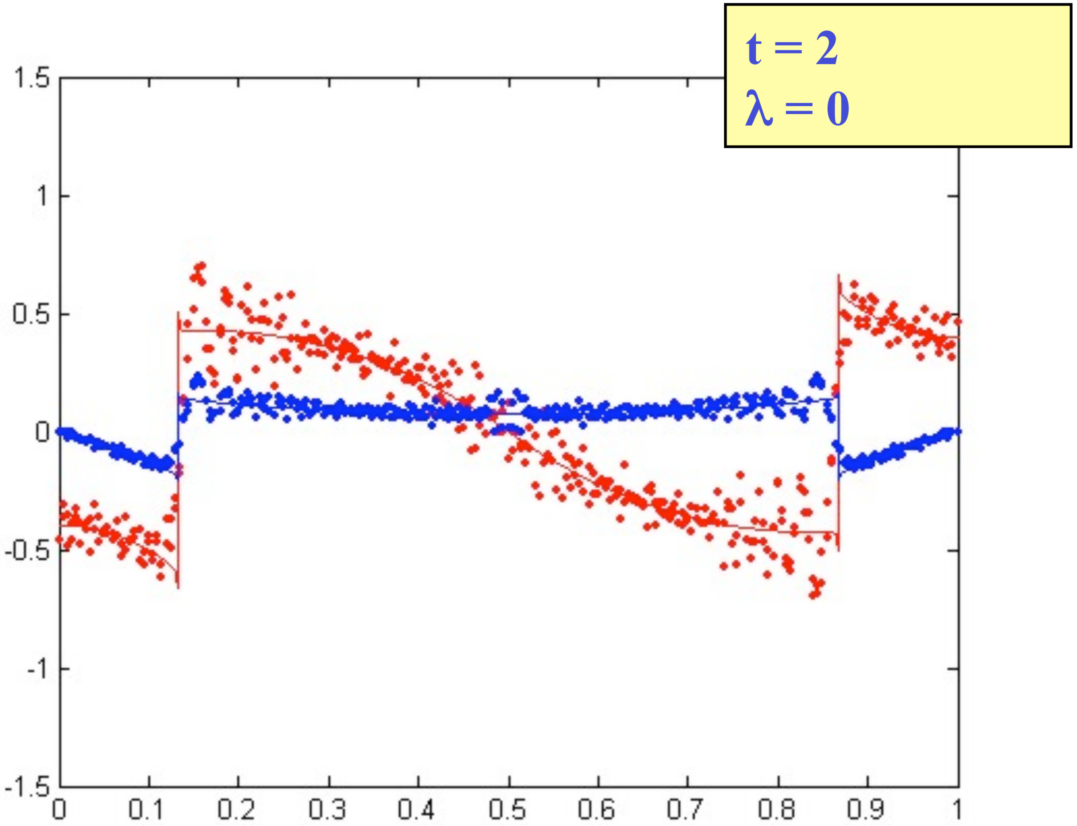
$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial v}{\partial t} \hat{v} - u\hat{v} \right) dx = 0 \qquad \forall \hat{v} \in \mathcal{U},$$

Penalty terms to enforce weak continuity of the solution

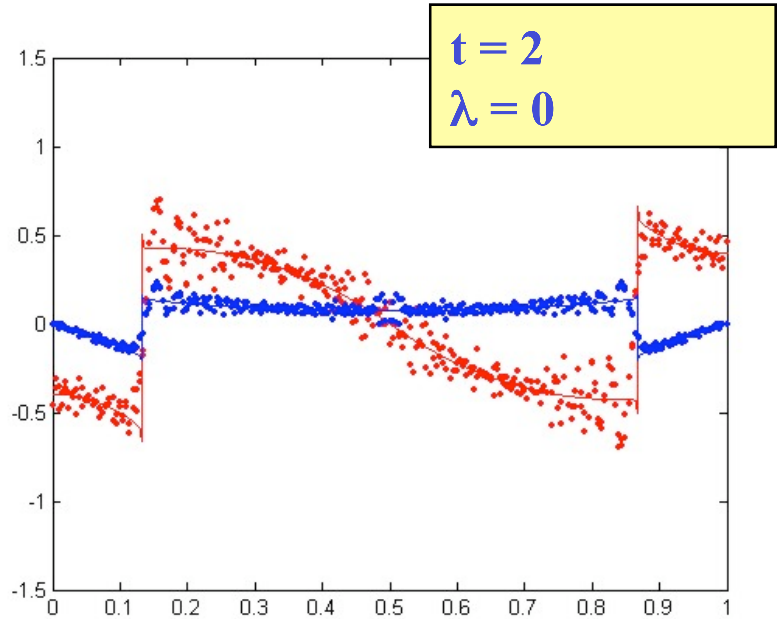
$$a(\hat{u}) = \left(1 - \boxed{\lambda} \, \mathrm{sign}(n) \right) \hat{u}$$

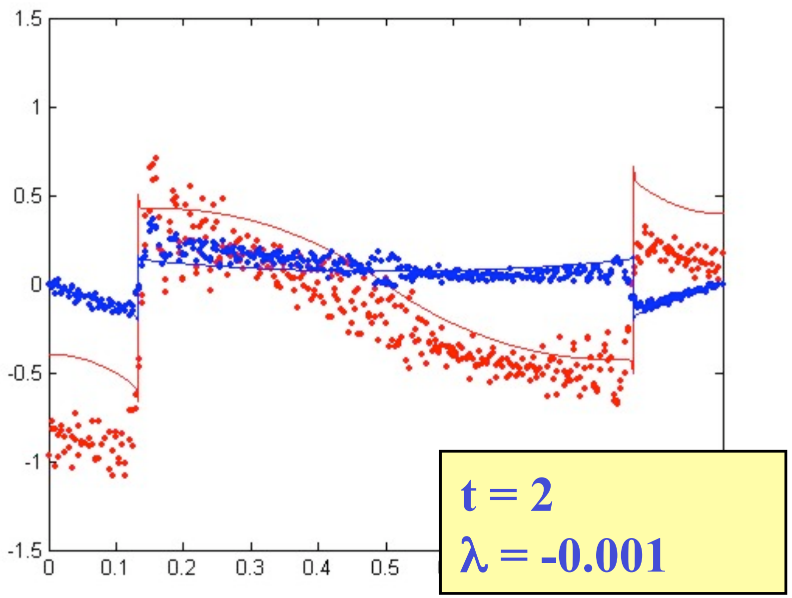
Upwinding Factor
(in fact, the best selection is = 0)

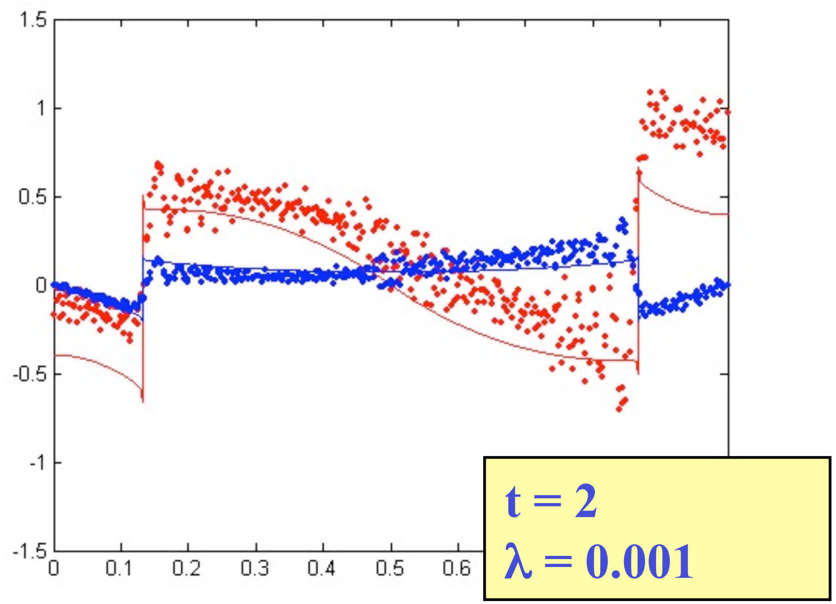
The Discontinuous Galerkin Method



How to impose continuity constraint ?







The Discontinuous Riemann-Galerkin Method

Penalty terms to enforce weak continuity of the Riemann's invariants

Find $\eta \in \mathcal{E}$ and $(u,v) \in \mathcal{U} \times \mathcal{U}$ such that

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial \eta}{\partial t} \hat{\eta} + \frac{\partial u}{\partial x} \hat{\eta} \right) dx + \boxed{\sum_{e=1}^{N_E} \Big[\, a(\hat{\eta})[u + \alpha\eta] \,\Big]_{\Omega_e}} + \boxed{\sum_{e=1}^{N_E} \Big[\, b(\hat{\eta})[u - \alpha\eta] \,\Big]_{\Omega_e}} = 0 \qquad \forall \hat{\eta} \in \mathcal{E},$$

$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial u}{\partial t} \hat{u} + v\hat{u} + \alpha^2 \frac{\partial \eta}{\partial x} \hat{u} \right) dx + \boxed{\sum_{e=1}^{N_E} \Big[\, a(\hat{u})[\alpha^2\eta + \alpha u] \,\Big]_{\Omega_e}} + \boxed{\sum_{e=1}^{N_E} \Big[\, b(\hat{u})[\alpha^2\eta - \alpha u] \,\Big]_{\Omega_e}} = 0 \qquad \forall \hat{u} \in \mathcal{U},$$

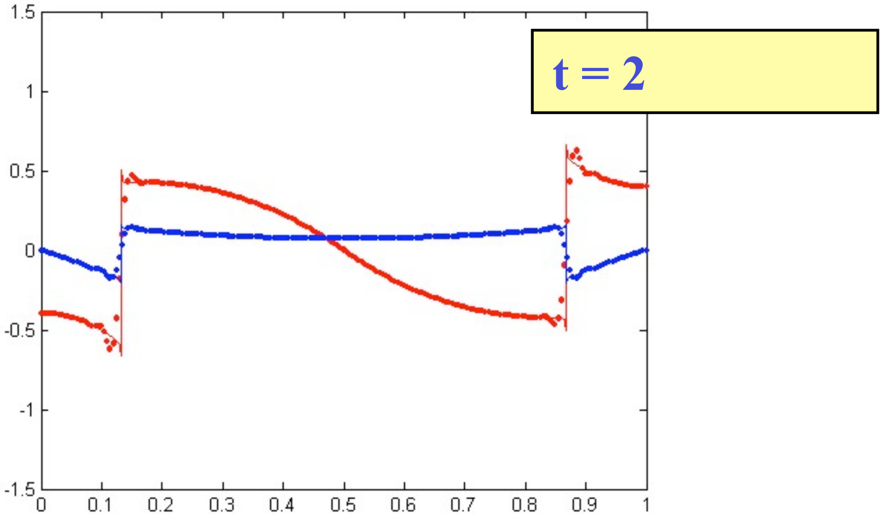
$$\sum_{e=1}^{N_E} \int_{\Omega_e} \left(\frac{\partial v}{\partial t} \hat{v} - u\hat{v} \right) dx = 0 \qquad \forall \hat{v} \in \mathcal{U},$$

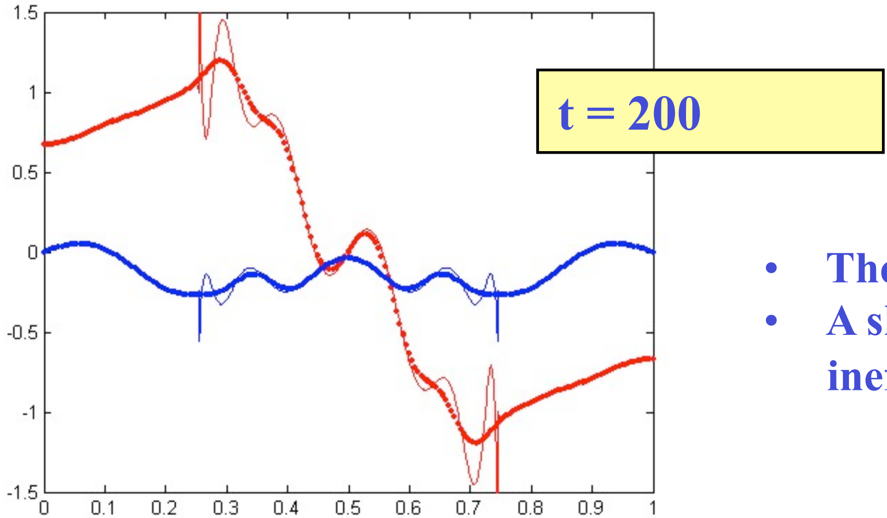
$$a(\hat{u}) = \Big(1 - \boxed{\lambda}\, \mathrm{sign}(n)\Big)\, \hat{u} \qquad\qquad b(\hat{u}) = \Big(\boxed{\lambda}\, \mathrm{sign}(n) + 1\Big)\, \hat{u}$$

Backward Upwinding

Forward Upwinding

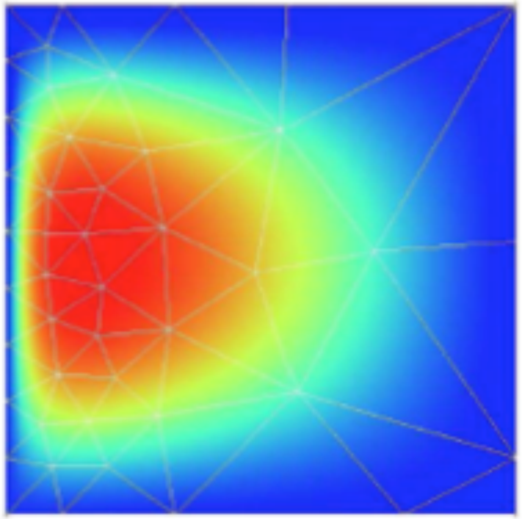
DG Method works !

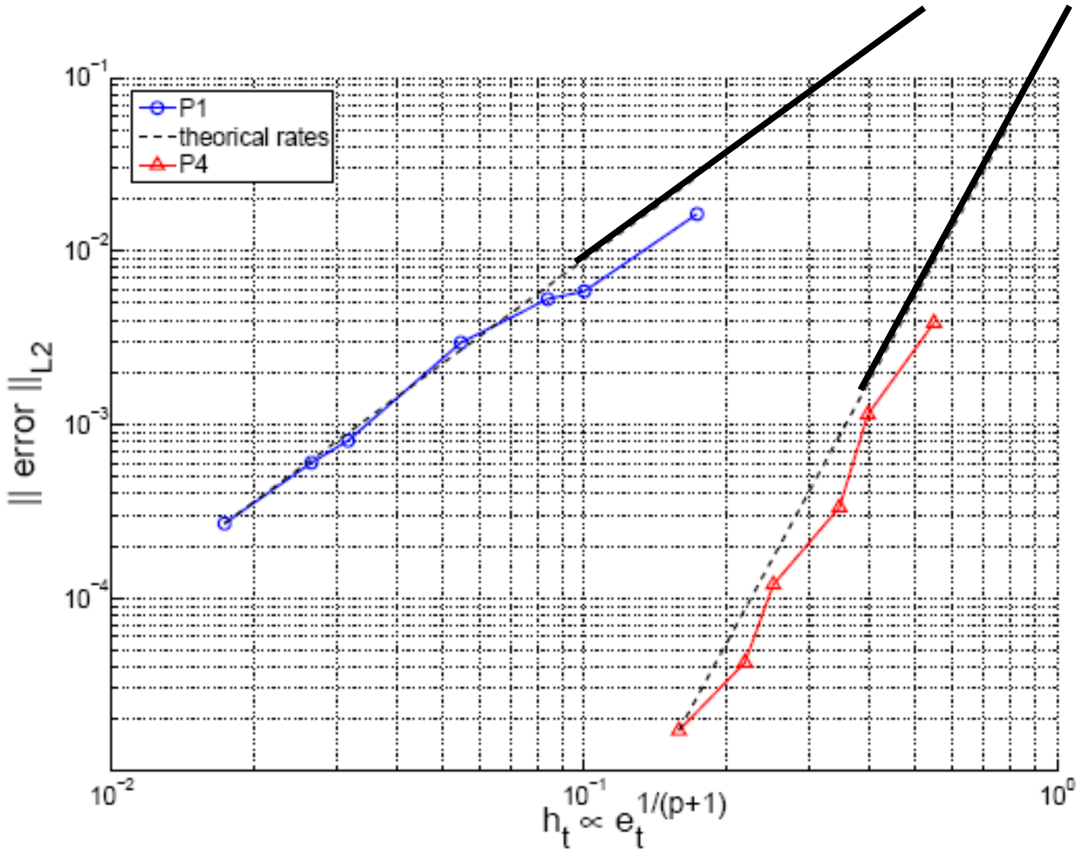




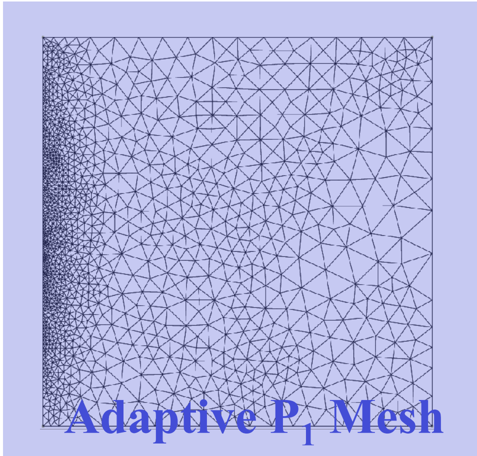
- **The use of a good Riemann solver is mandatory !**
- **A sharp problem is needed to discriminate inefficient or unstable numerical techniques**

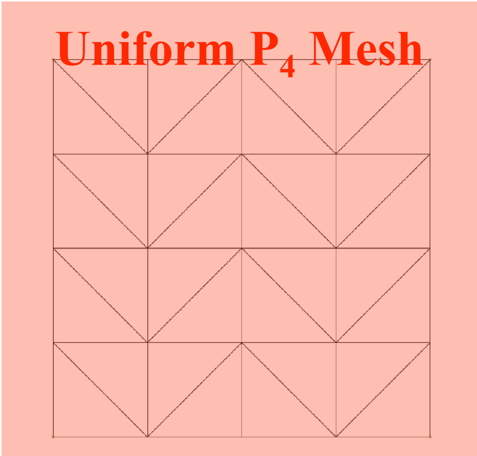
Theoretical rates of convergence are obtained for the analytical Stommel problem

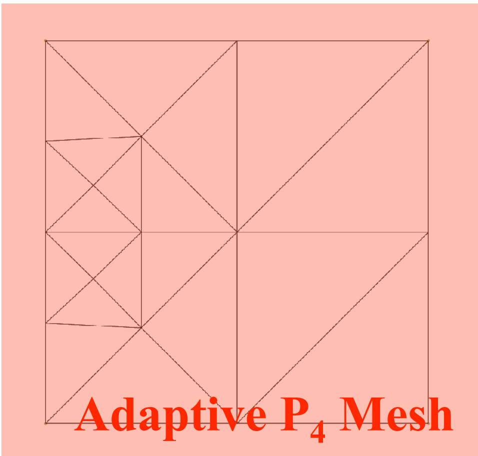


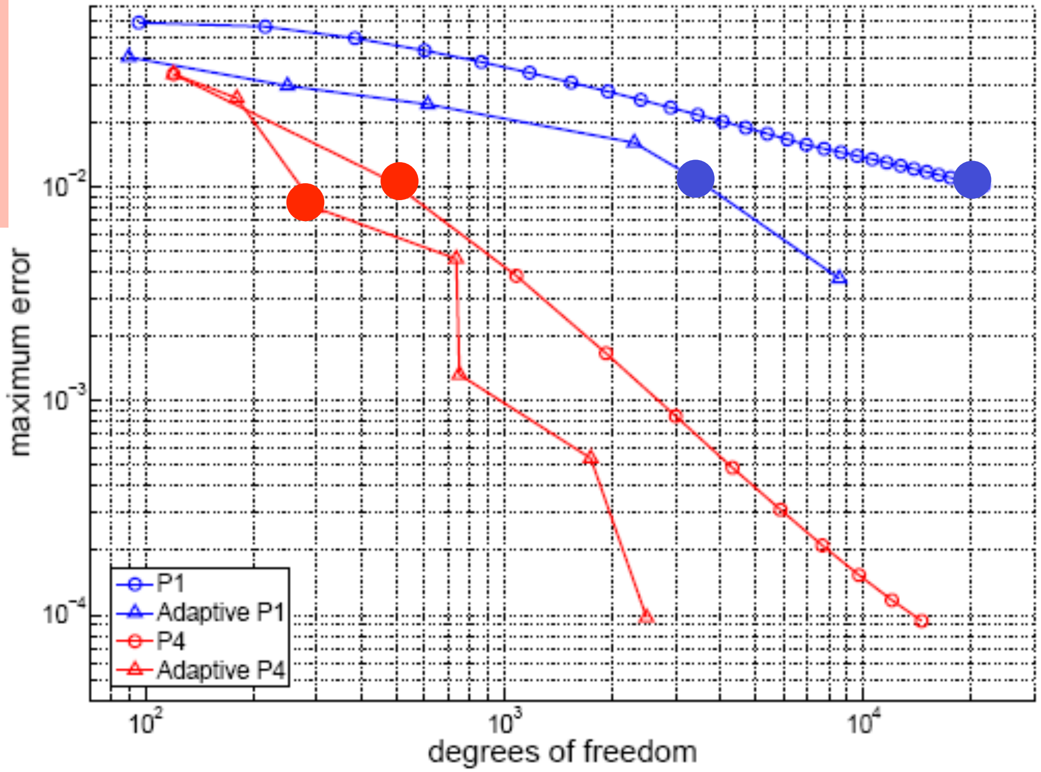


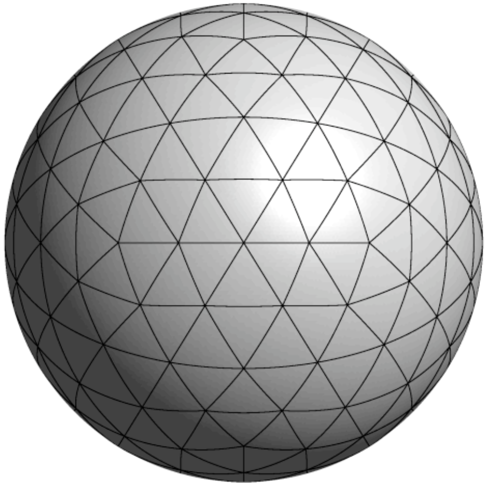
How does it converge ?



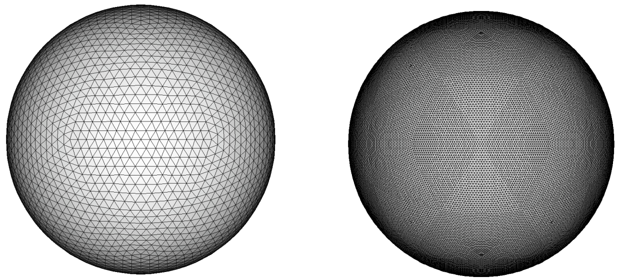




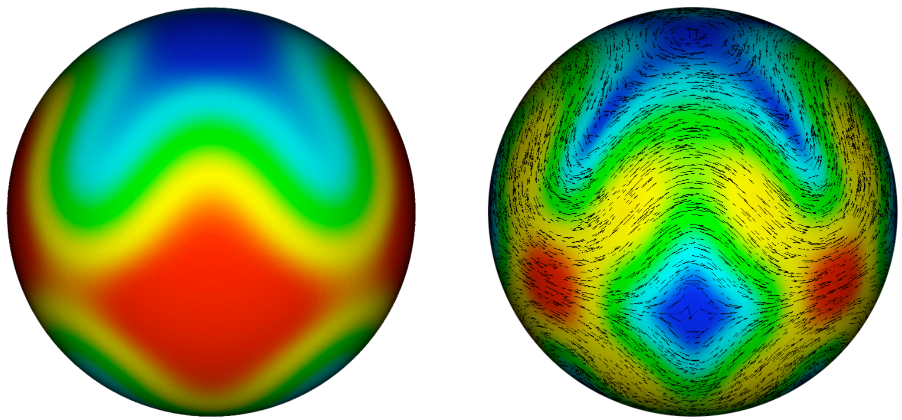




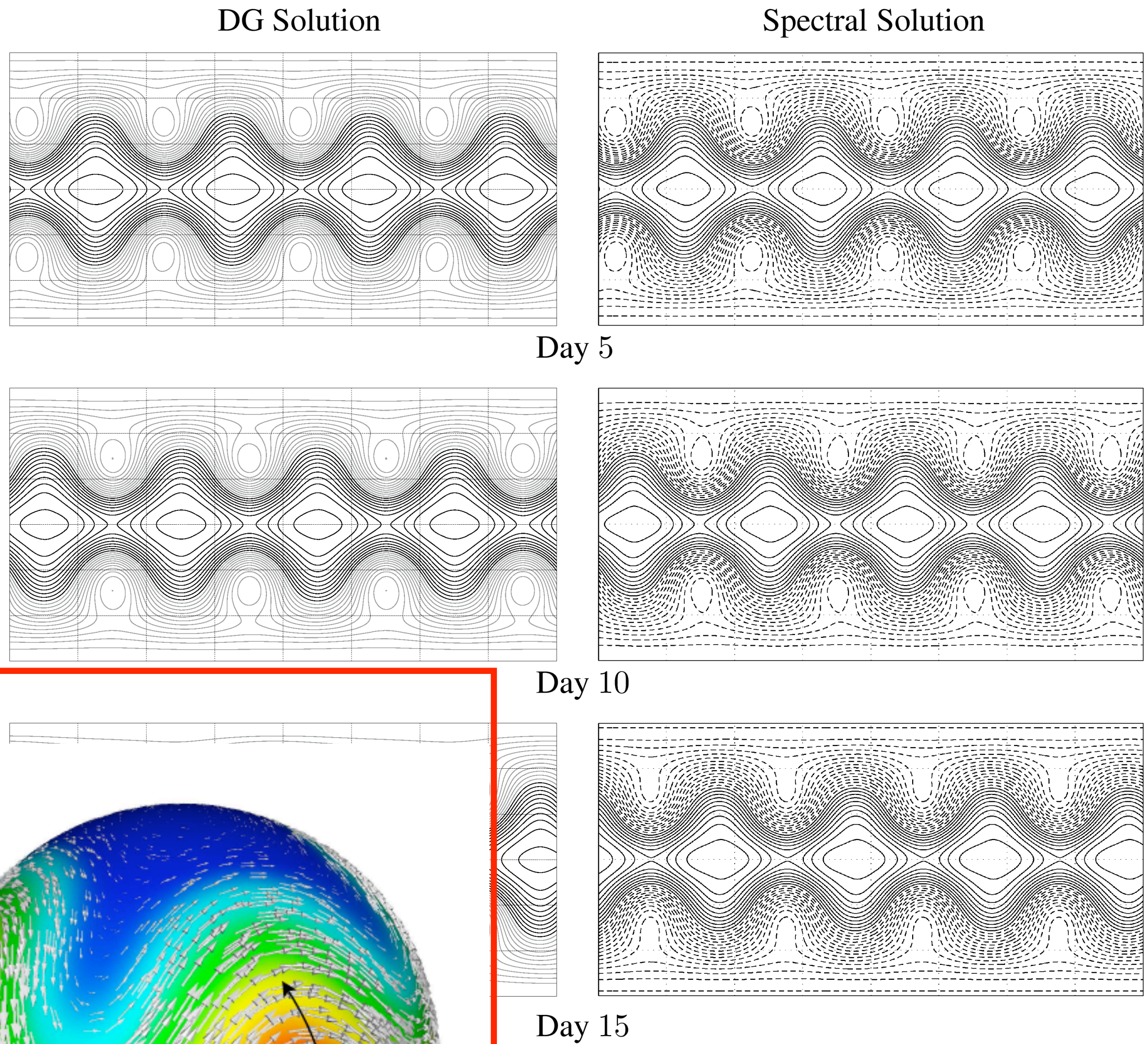
High-order versus low-order meshes



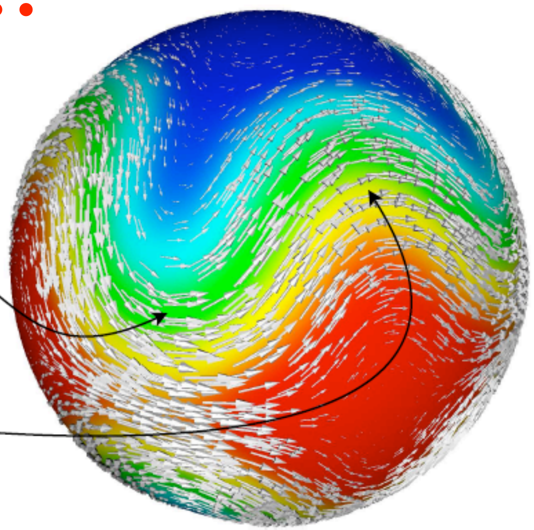
Unsteady balance between pressure term and Coriolis force



Global Rossby-Hauritz waves



Intuitively...



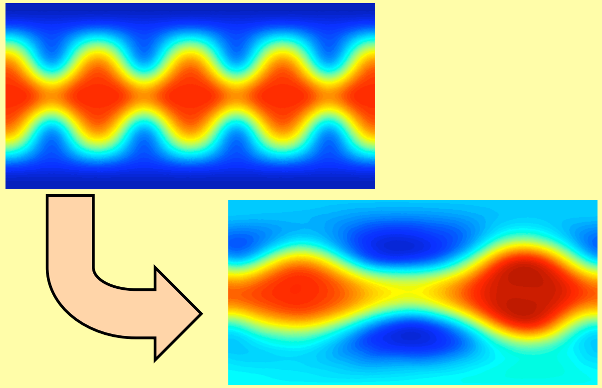
Coriolis vs elevation gradient

Elevation gradient dominating:
the flow goes leftward

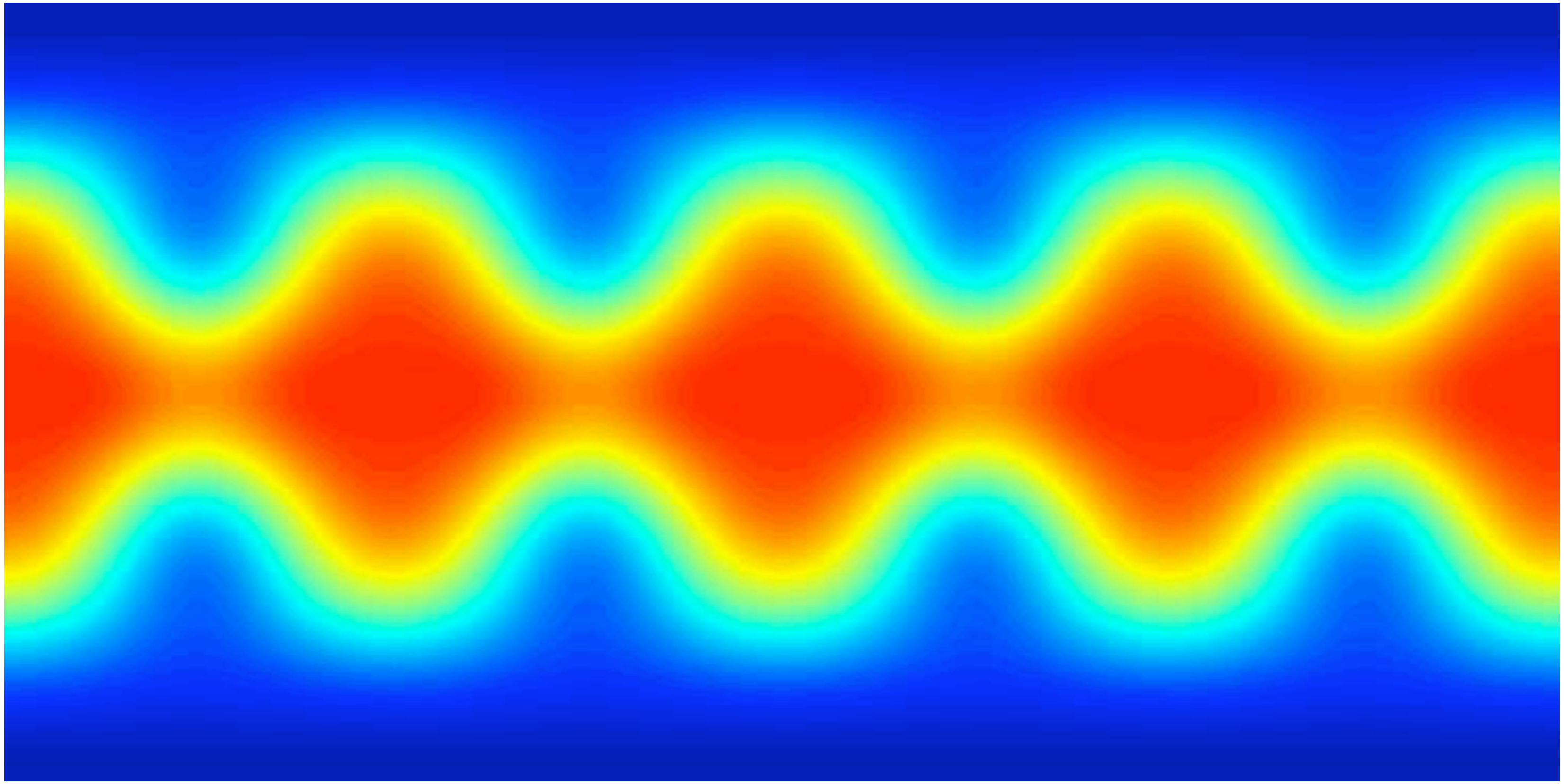
Coriolis term dominating:
the flow goes rightward

Spectral Transform Method [Jakob-Chien et al. (1995)]

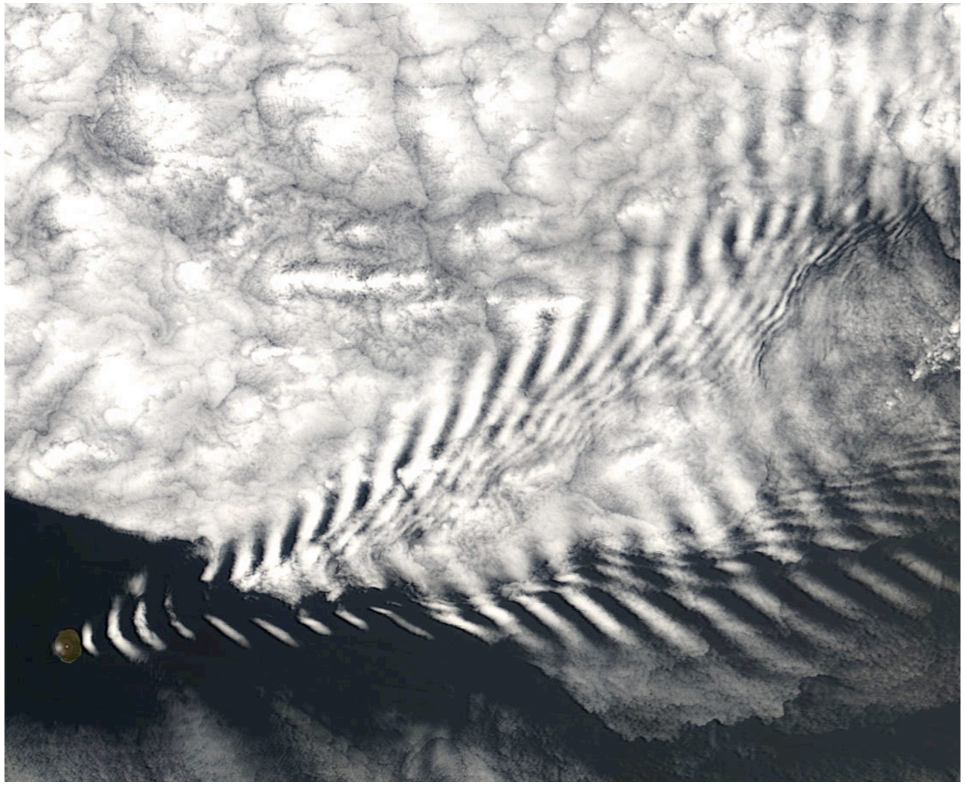
And the flow becomes instable...



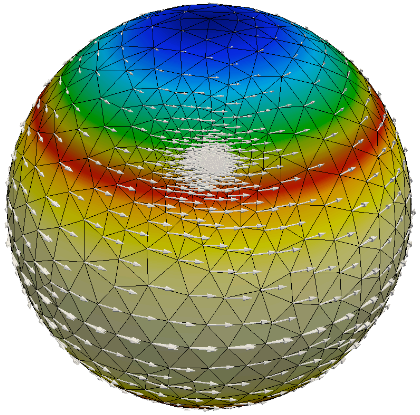
Finally, a pattern characterized by a wave number of two appears!



Internal waves in the lee of a moderately tall seamount

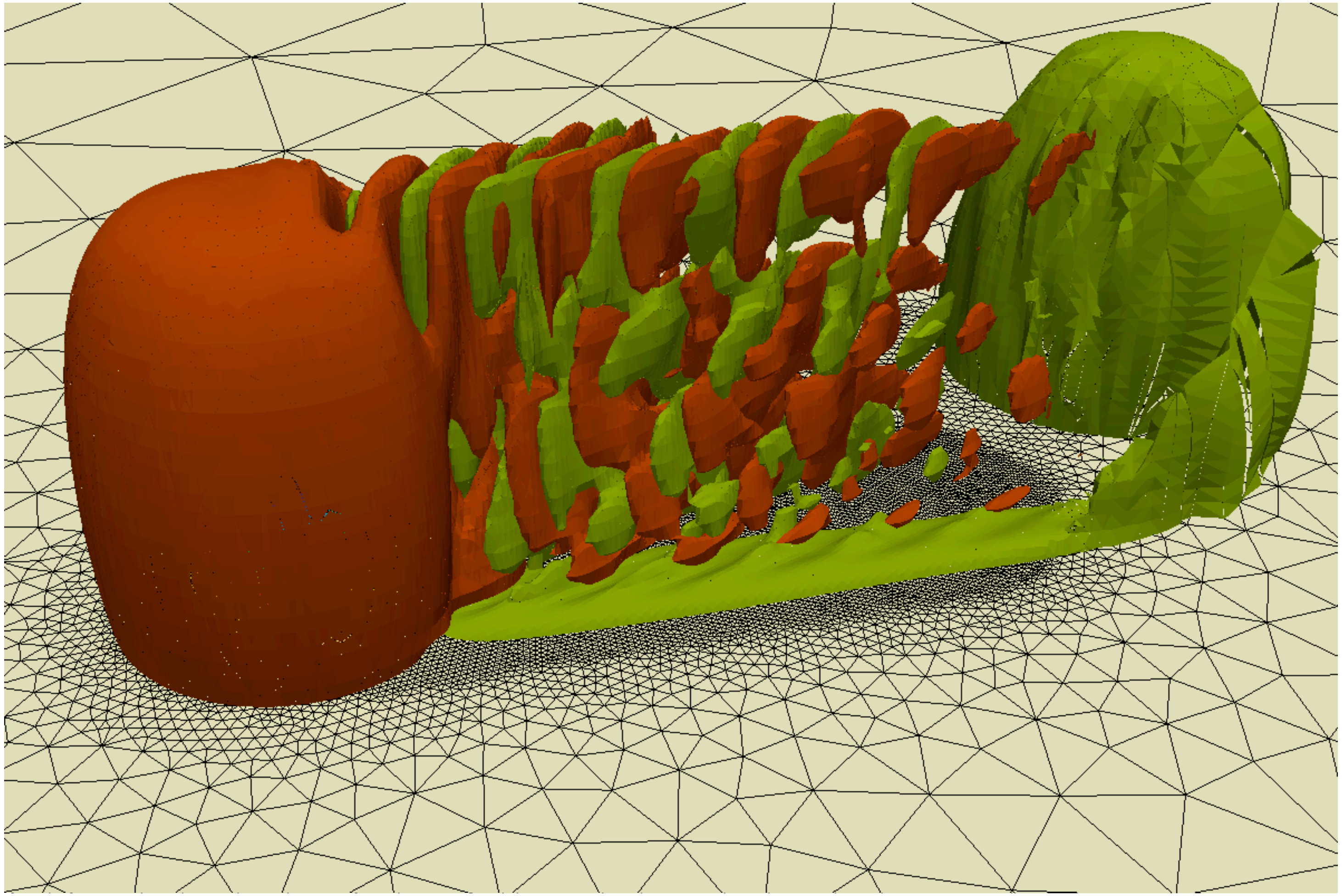


Cloud waves in the lee of Amsterdam island (NASA image from J. Schmalz)

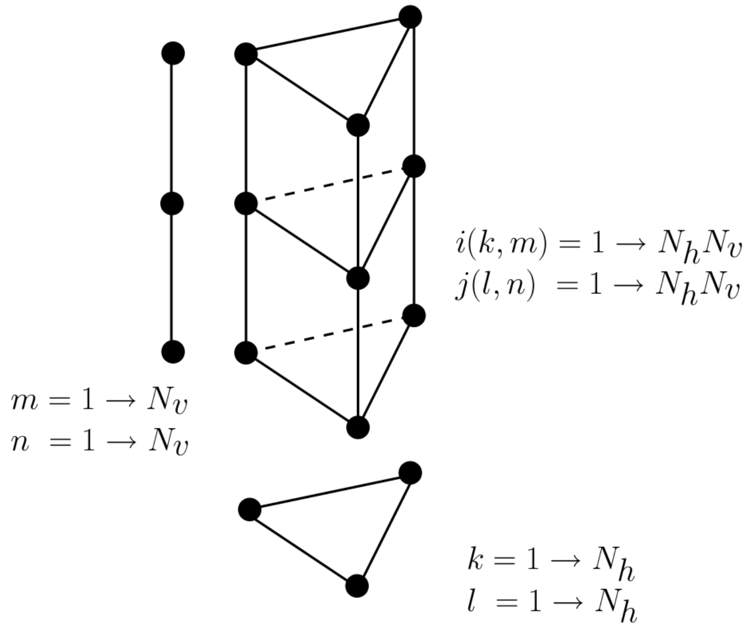


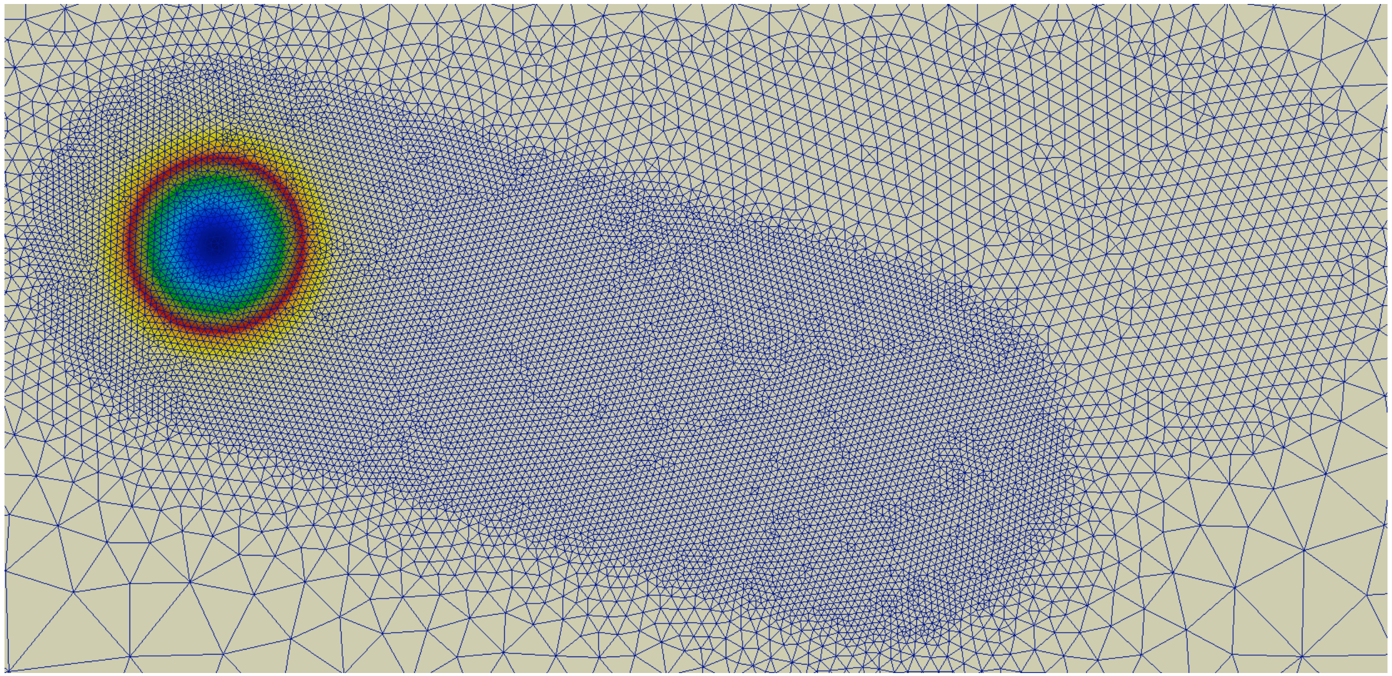
The computation starts with a global zonal geostrophic equilibrium ignoring the seamount as in Williamson testcase 5

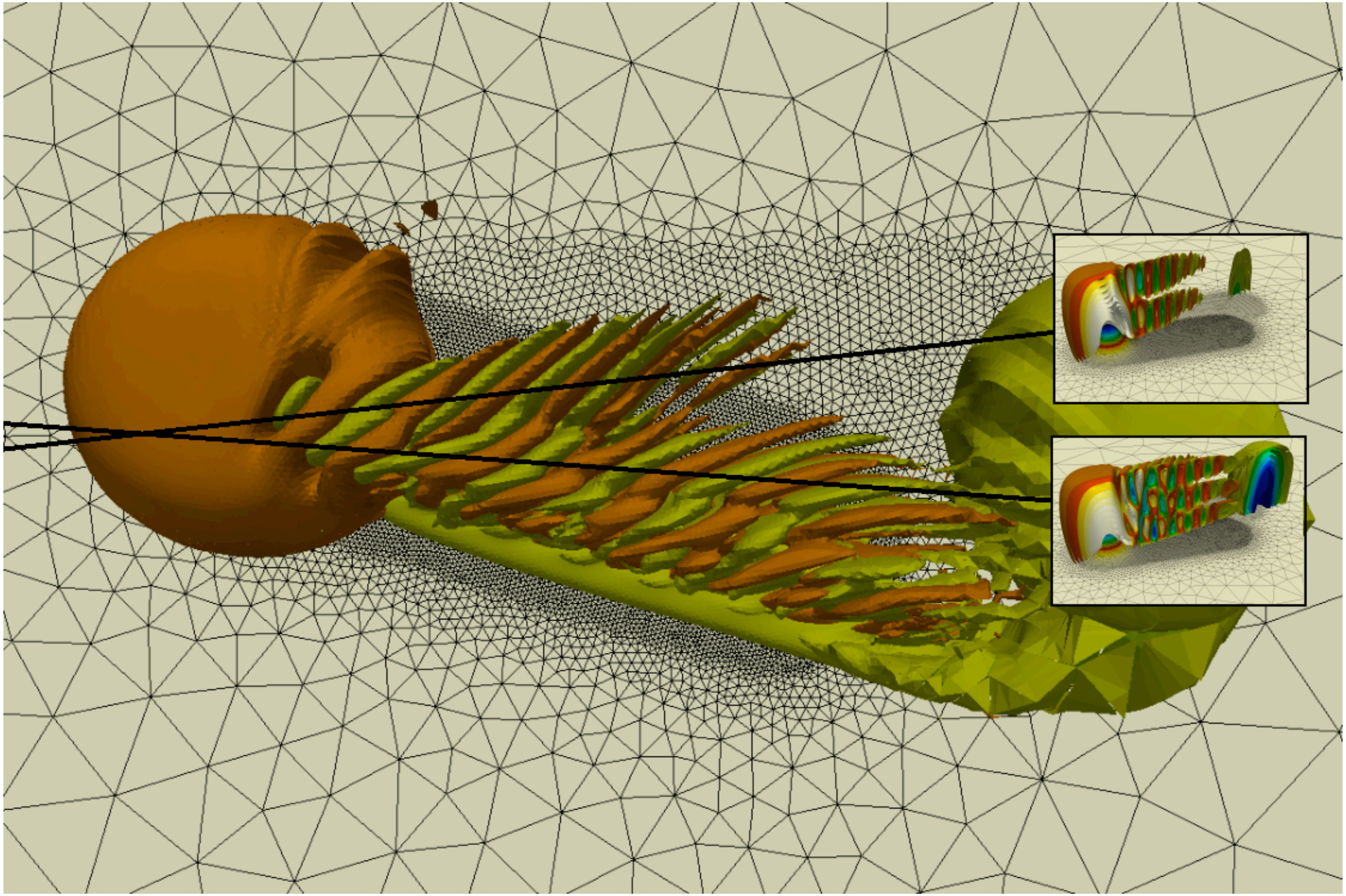
7 days evolution of density deviation field



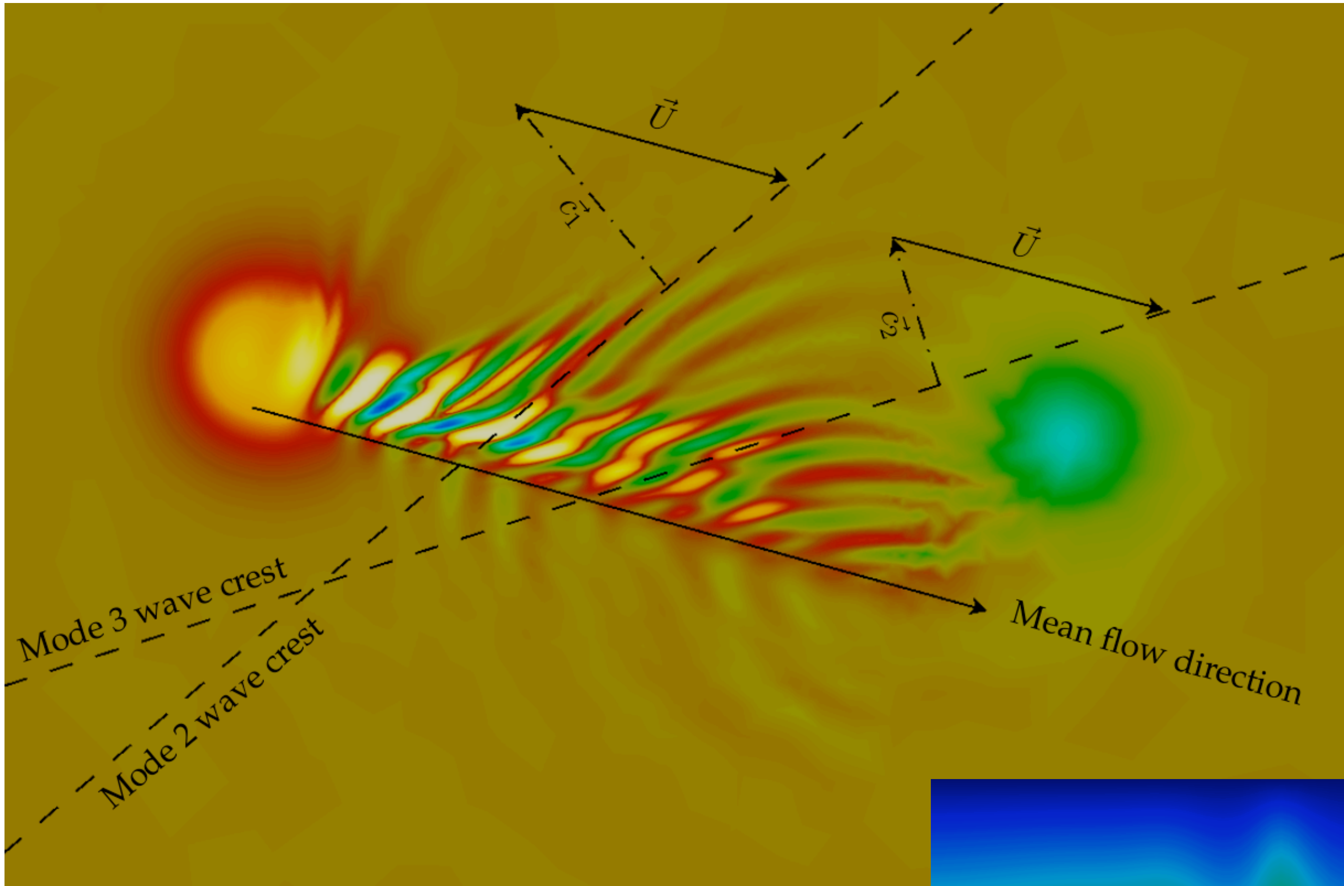
Mesh of 23562 triangles extruded into 25 σ layers

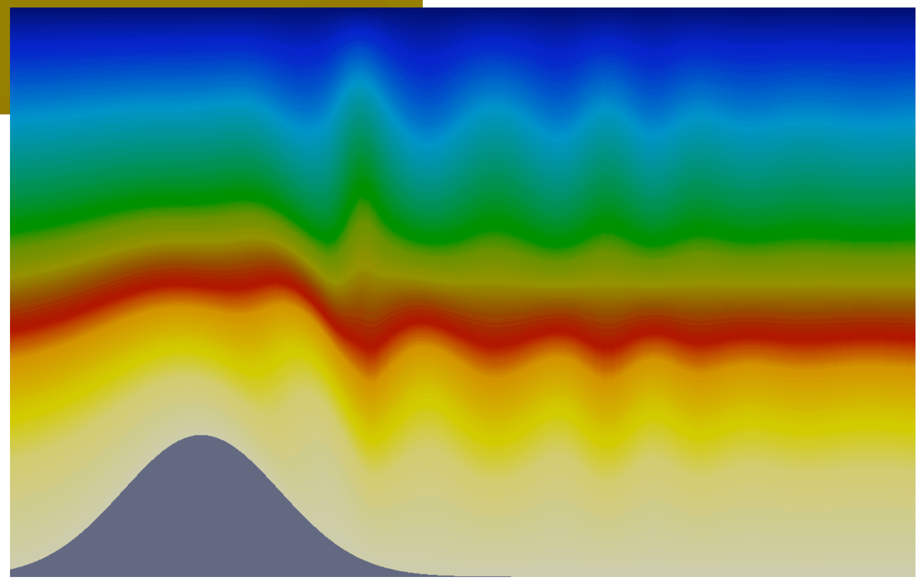






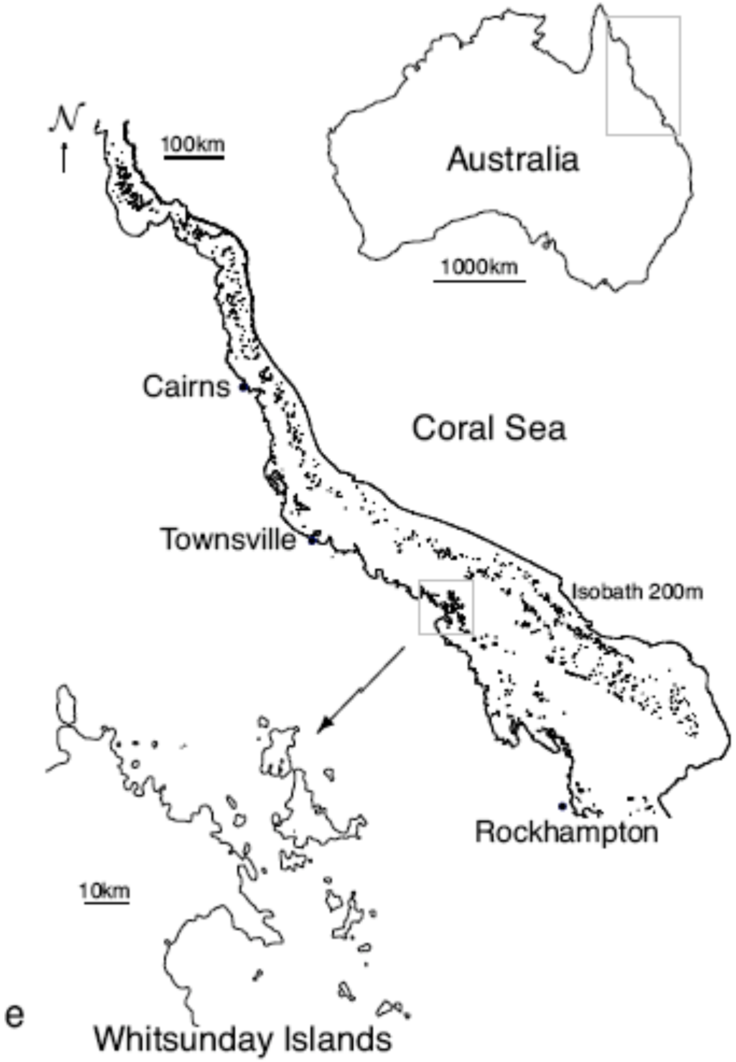
Two well separated modes at day 7

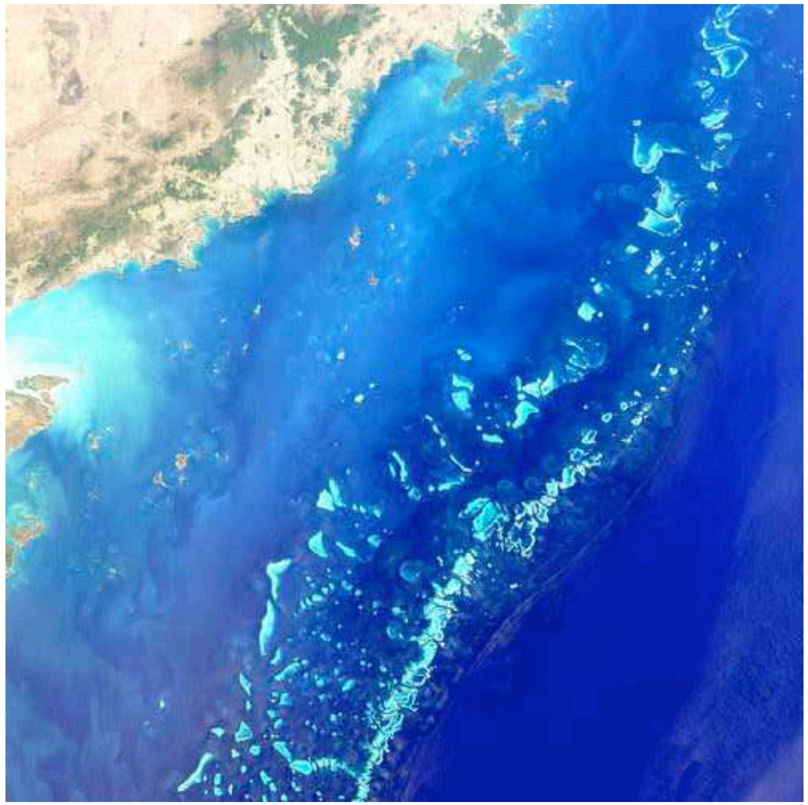




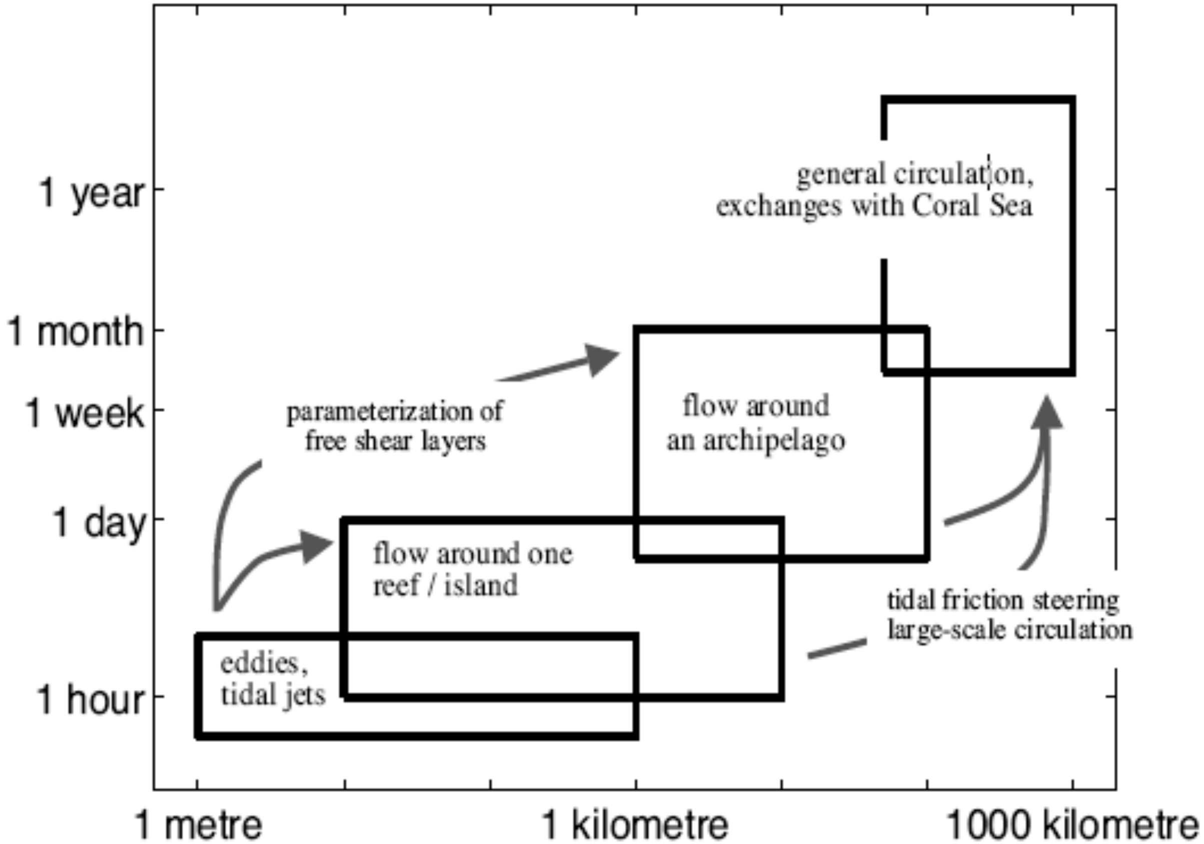
Cut in the density field at day 7

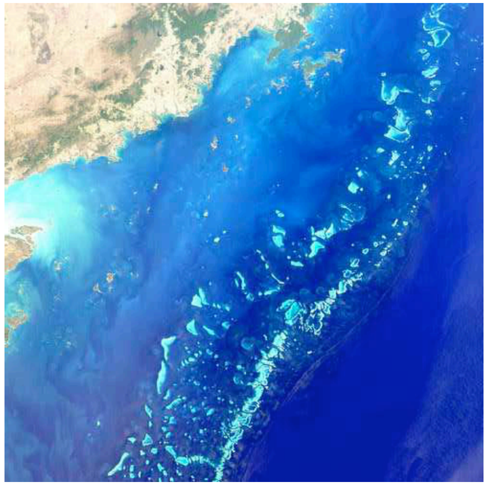
Multi-scale modelling of the Great Barreer Reef (Australia)





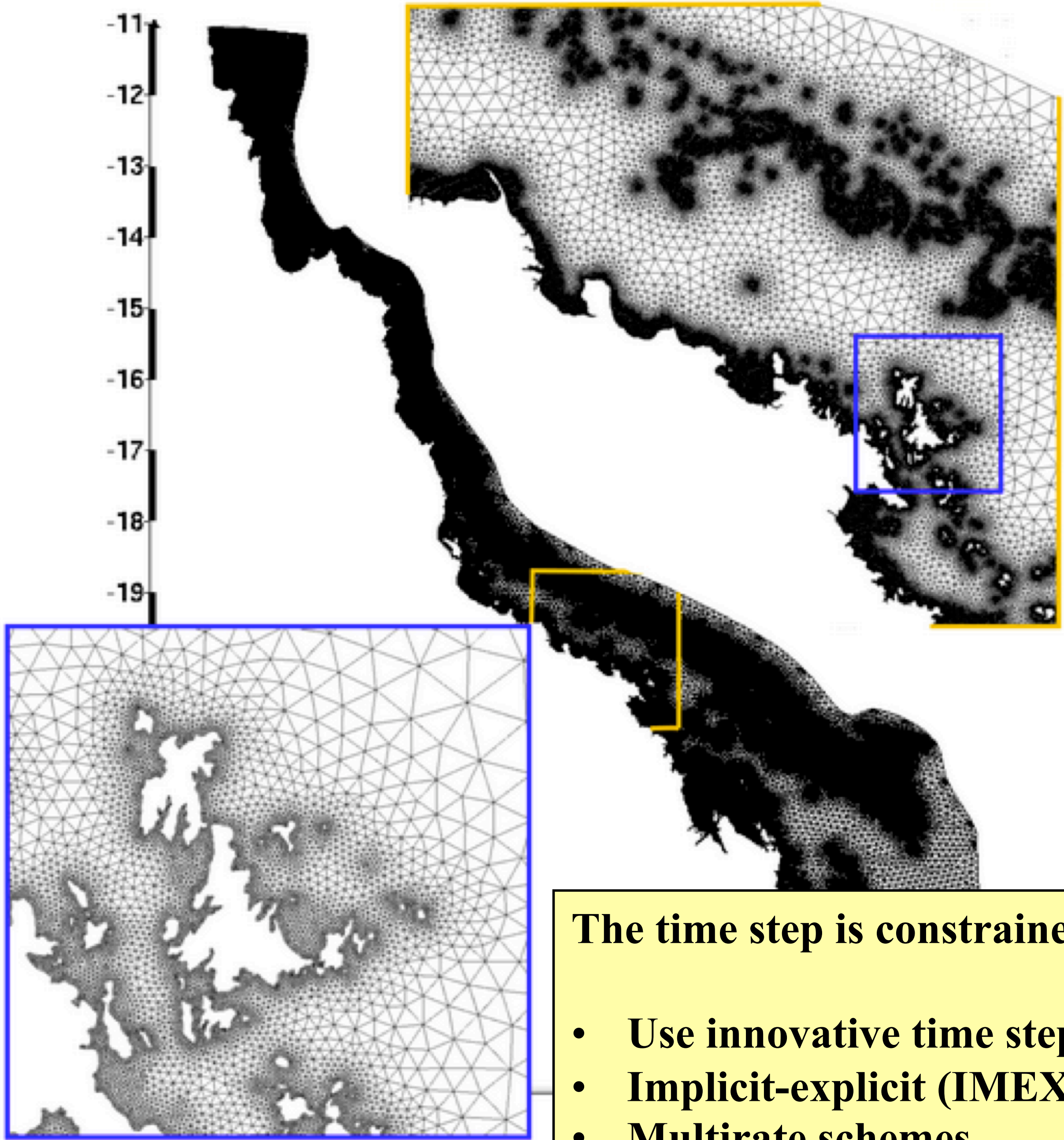
Time-space scales





- ***Forcings :***
  wind, tides, Coral Sea inflow
- ***Wide spectrum of hydrodynamics processes simulated :***
  eddies, tidal jets, sticky waters, general circulation

The time stepping issue



- *890,000 triangles*
- *Smallest element : 7 m*
- *Largest element : 3,300 m*
- *99.9 % > 60m*

The time step is constrained by the smallest element .

- **Use innovative time stepping procedures**
- **Implicit-explicit (IMEX) schemes**
- **Multirate schemes**

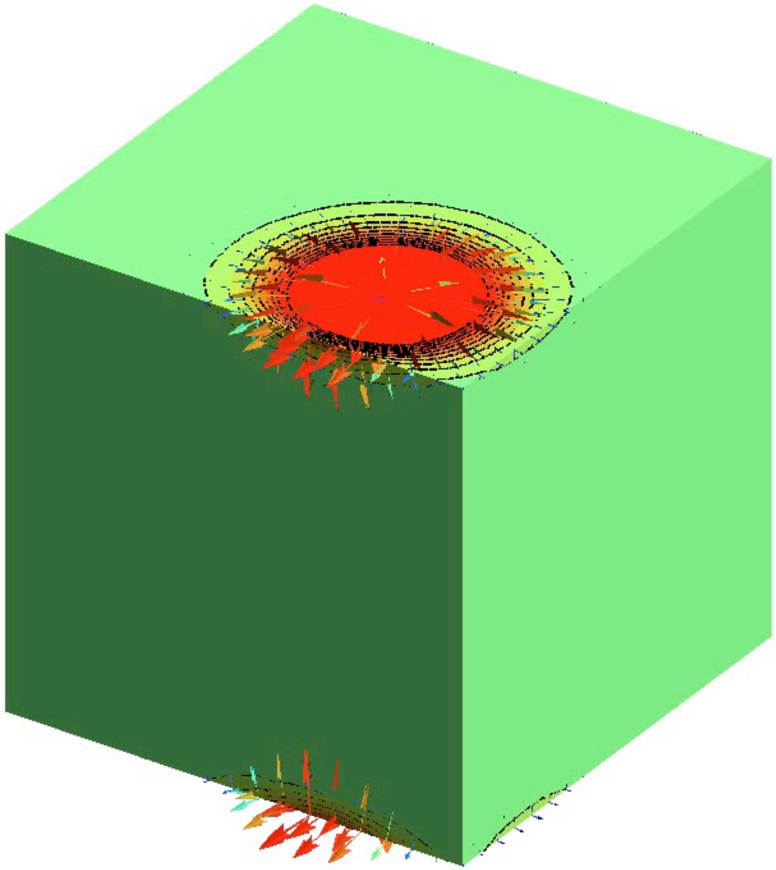
Reduce cost by 1000 !
Use high performance computers !

10 Gflops
2 processors

1.759 Pflops
224,162 processors

- **Exploit single precision BLAS/LAPACK for the efficient implementation of the explicit and implicit discontinuous Galerkin methods.**
- **Implement new time-integration procedures adapting the time step to the physical processes.**
- **Introduce multi-level methods for the implicit linear and non-linear solvers with multigrid methods as a preconditioner for stiff, non-linear and non-positive-definite systems.**

Each route could reduce the computational cost by one order of magnitude.



Conclusions

Bernard et al . (JCP, 2009)(OM, 2009) (JSC, 2008)
Comblen et al. (OM, 2009) (OD, 2010)

http://perso.uclouvain.be/vincent.legat/papers/

- **Today, the efficiency of high-order adaptative DG methods has been demonstrated in several applications.**
- **It has also been demonstrated with realistic geometry, bathymetry and forcings.**
- **An original implementation on manifolds, combining the advantages of both usual methods is given.**