Physically Motivated Soundproof Equations for Compressible Stratified Flow

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April 16, 2010

Outline

Why the fuss?

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- The speed of sound is 300 m s^{-1}.

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- A fully explicit time differencing approximation to the full compressible governing equations will require the time step to be no larger than roughly 1 second.

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- Both of the above: the Unified System

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- Constraints on stability of explicit time-differencing schemes.
  - $\Delta t < O(\Delta x / c_s)$ for the Lamb wave
  - $\Delta t < O(1/N)$ for internal gravity waves
  - Semi-implicit or similar differencing is required to stabilize systems supporting the Lamb wave when Δx is sufficiently fine.

Filtering All Acoustic Waves—I

Classical approaches

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  - Reference state is too weakly stratified for optimal numerical accuracy.

Filtering All Acoustic Waves—II

State-of-the-art anelastic systems

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    - In the thermodynamics (Bannon, 1996)

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- No additional approximations are made in the adiabatic governing equations.
- Energy conservative.
- In the presence of heating

$$\nabla \cdot \left(\overline{\rho}\overline{\theta}\mathbf{v}\right) = \frac{\rho^* H_m}{c_p \pi} \approx \frac{\rho^* H_m}{c_p \overline{\pi}}$$

where ρ^* is a pseudo-density.

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- Enables trivial derivation of many conservation relations, such as for the pseudo-incompressible vorticity $(\nabla\theta \cdot \boldsymbol{\omega})/\rho^*$.
- More useful than

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  for constructing efficient numerical algorithms.

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How do we determine a pseudo-density such that simply replacing ρ by ρ^* in the continuity yields a soundproof system?

Outline

Filtered Equation Sets

Sound Wave Propagation

Steady, Horizontally Homogeneous Reference State

Four-Dimensional Reference States

Wrap Up

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- The speed of sound is $c_s^2 = (\partial p/\partial \rho)_S$
- For isentropic motions the thermodynamic and continuity equations imply
$$\frac{1}{\rho c_s^2}\frac{Dp}{Dt} + \nabla \cdot \mathbf{v} = 0.$$

Linearized thermodynamic equation

Linearizing about a resting hydrostatically balanced basic state $\overline{\rho}(z)$ and $\overline{p}(z)$ gives,

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{v}') = -\frac{\overline{\rho}N^2 w'}{g},$$

where

$$N^2 = -g\left(\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dz} + \frac{g}{c_s^2}\right)$$

is the Brunt-Väisälä frequency.

Linearized momentum equation

Linearize the momentum equation,

$$\rho\frac{D\mathbf{v}}{Dt}+\nabla p=-\rho g\mathbf{k},$$

to obtain

$$\frac{\partial\overline{\rho}\mathbf{v}'}{\partial t}+\nabla p'=-g\rho'\mathbf{k}.$$

Linear sound waves

Eliminating $\overline{\rho}\mathbf{v}'$ between the pressure and momentum equations, one obtains

$$\frac{1}{c_s^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = g\frac{\partial \rho'}{\partial z} - \frac{\overline{\rho}N^2}{g}\frac{\partial w'}{\partial t}.$$

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- Pressure changes due to buoyancy perturbations (gravity waves) are forced through the right side.
- If the system is modified to eliminate sound waves, the pressure will satisfy a Poisson equation.

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- The true density is expressible as a function of any two distinct state variables.
- As the most general plausible alternative to the true density, let the pseudo-density have the form $\rho^*(Q, x, y, z, t)$, where Q is some thermodynamic state variable.
- We may express Q as a function of just pressure and entropy, then the convective derivative of ρ^* is

$$\frac{D\rho^*}{Dt} = \frac{\partial \rho^*}{\partial Q}\left[\left(\frac{\partial Q}{\partial p}\right)_S \frac{Dp}{Dt} + \left(\frac{\partial Q}{\partial S}\right)_p \frac{DS}{Dt}\right] + \left(\frac{\partial \rho^*}{\partial t}\right)_Q + \mathbf{v} \cdot \nabla_Q \rho^*$$

Continuity equation for this pseudo-density

Pseudo-incompressible mass continuity $D\rho^*/Dt + \rho^*\nabla \cdot \mathbf{v} = 0$ becomes

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Choosing $Q = Q(S)$ will eliminate the time tendency of the pressure and the acoustic modes.

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- The linearized thermodynamic and compressible continuity equations give

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  vary in x, y, z, t.

Efficiency in mesoscale models – II

But, the diagnostic pressure equation obtained using

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- Computationally just as simple as in the conventional anelastic system.
- Implement via generalizations of the projection method.

Pseudo-incompressible projection method – I

Approximate the pressure term in the momentum equation using the following $O\left[(\Delta t)^2\right]$ accurate expression:

$$c_p \theta \nabla \pi' \approx c_p \theta^{n+1} \nabla \left(\pi'^{n+1/2} - \pi'^{n-1/2} \right) + c_p \frac{\theta^{n+1} + \theta^n}{2} \nabla \pi'^{n-1/2}$$

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Write the momentum equation as

$$\mathbf{v}^{n+1} = \bar{\mathbf{v}} - \Delta t\, c_p\theta^{n+1}\nabla\phi$$

where $\phi = \pi'^{n+1/2} - \pi'^{n-1/2}$,

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$$\bar{\mathbf{v}} = \mathbf{v}^n + \Delta t\,\mathbf{f}(\mathbf{v}^n) + \Delta t\, c_p\frac{\theta^{n+1}+\theta^n}{2}\nabla\pi'^{n-1/2}$$

$$\mathbf{f}(\mathbf{v}) = -(\mathbf{v}\cdot\nabla)\mathbf{v} + g\frac{\theta'}{\overline{\theta}}\mathbf{k}.$$

Pseudo-incompressible projection method – II

Approximate the pseudo-incompresible continuity equation as

$$\frac{\rho^{*\,n+1} - \rho^{*\,n}}{\Delta t} + \frac{1}{2}\nabla\cdot\left(\rho^{*\,n+1}\mathbf{v}^{n+1} + \rho^{*\,n}\mathbf{v}^{n}\right) = 0.$$

Algorithm designed by Peter Blossey

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Collecting terms

$$c_p\,\nabla\cdot\left(\rho^{*\,n+1}\,\theta^{n+1}\nabla\phi\right) = \frac{2}{\Delta t}\left(\frac{\rho^{*\,n+1} - \rho^{*\,n}}{\Delta t}\right) + \frac{1}{\Delta t}\nabla\cdot\left(\rho^{*\,n+1}\bar{\mathbf{v}} + \rho^{*\,n}\mathbf{v}^n\right)$$

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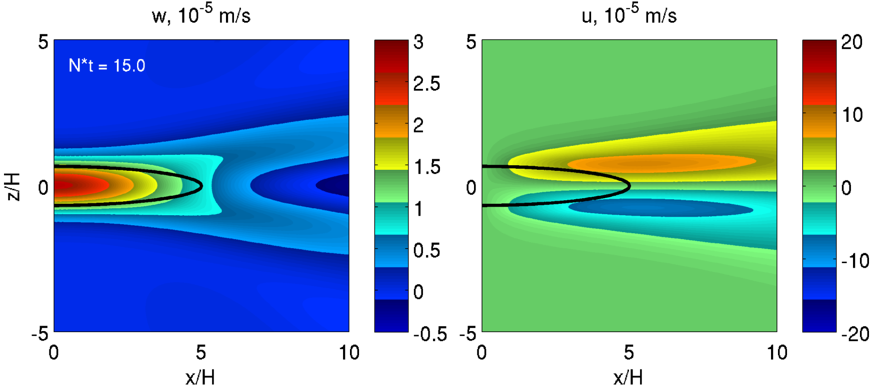
Note

- $\overline{\rho}\overline{\theta}$ depends only on z.
- $\rho^{*\,n+1} = \overline{\rho}\overline{\theta}/\theta^{n+1}$, and θ^{n+1} maybe calculated prior to this step.

Algorithm designed by Peter Blossey

Pseudo-incompressible projection method – Example

Gravity waves generated by a steady heat source.



Calculations courtesy of Peter Blossey

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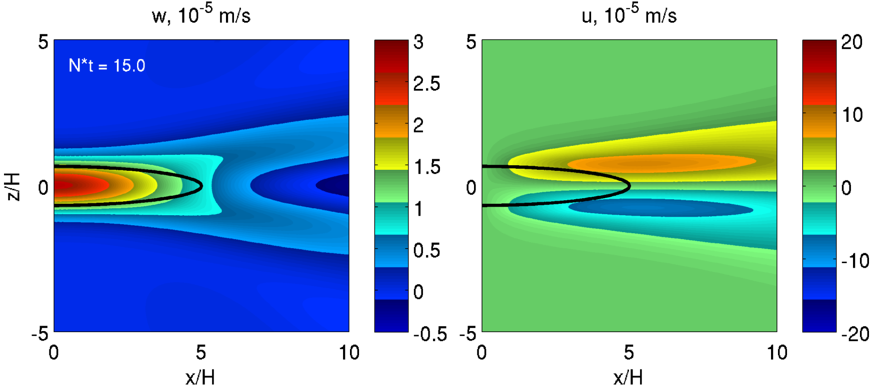

Same execution time as Lipps-Hemler-Bannon anelastic model.

Calculations courtesy of Peter Blossey

Performance relative to other equation sets – I

TABLE 3. HEIGHT-SCALE DISTORTION, ENERGY REDISTRIBUTION, AND RELOCATION OF ZEROS (SEE TEXT FOR DEFINITION OF THESE) OF INTERNAL MODES AS A FUNCTION OF EQUATION SET

Equation set	Height-scale distortion	Energy redistribution		Relocation of zeros
		Rossby	Gravity	
Fully compressible	No	No	No	No
Hydrostatic	No	No	Yes	No
Pseudo-incompressible (Durran 1989)	No	No	Yes	No
Anelastic (Wilhelmson and Ogura 1972)	Yes	Yes	Yes	Yes
Anelastic (Lipps and Hemler 1982)	No	Yes	Yes	Yes
Boussinesq	Yes	Yes	Yes	Yes

Davies, Staniforth, Wood and Thuburn (2003)

Performance relative to other equation sets –II

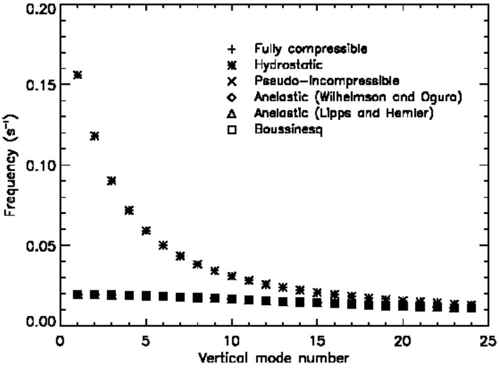

Figure 1. Frequency σ vs. vertical internal mode number m, where $k_z \equiv m\pi/z_T$ is vertical wave number, for the equation sets considered herein. Results are for a rigid lid at $z_T = 80$ km and a horizontal wavelength of $2\pi/k_x = 10$ km.

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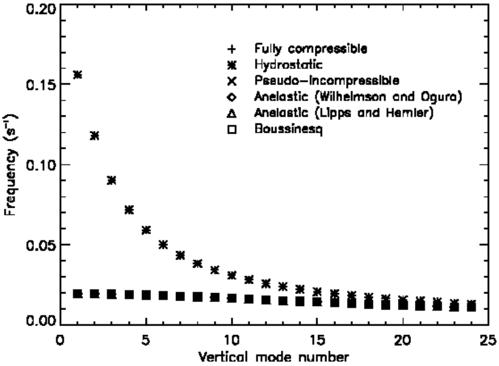

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Hydrostatic system is not appropriate for nonhydrostatic motions.

Davies, Staniforth, Wood and Thuburn (2003)

Performance relative to other equation sets – III

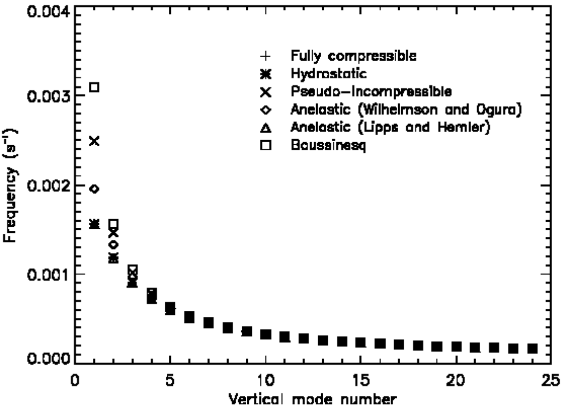
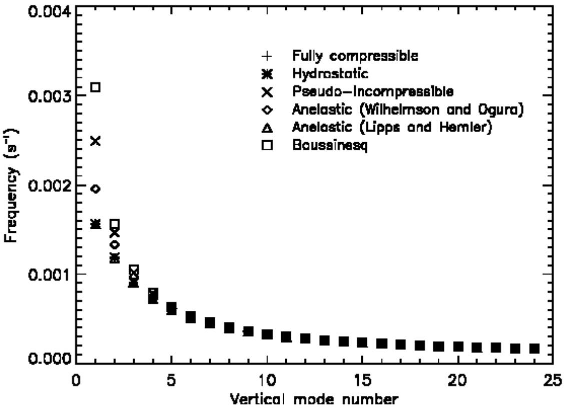


Figure 2. Same as for Fig. 1, except for a horizontal wavelength of 1000 km.

Davies, Staniforth, Wood and Thuburn (2003)

Performance relative to other equation sets – III



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Pseudo-incompressible system makes large errors in the frequency of *very deep* hydrostatic motions.

Davies, Staniforth, Wood and Thuburn (2003)

Performance relative to other equation sets – IV

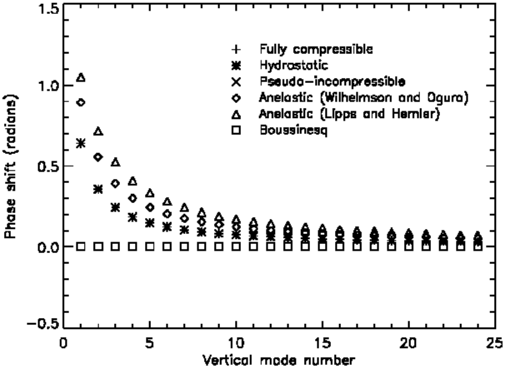

Figure 3. Phase shift Δ vs. vertical internal mode number m, where $k_z \equiv m\pi/z_T$ is vertical wave number, for the equation sets considered herein. Results are for a rigid lid at $z_T = 80$ km and are independent of horizontal wavelength $2\pi/k_x$. Note that the fully-compressible, hydrostatic and pseudo-incompressible points all coincide.

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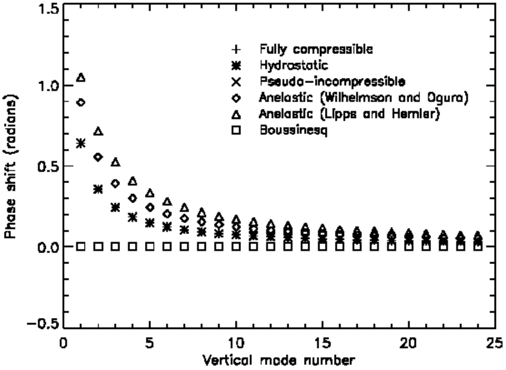
Performance relative to other equation sets – IV



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Pseudo-incompressible system makes negligible phase errors.

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Performance summary

The pseudo-incompressible system performs very well except in the very hydrostatic limit. (Davies, et al., 2003, Arakawa and Konor, 2009)

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Possible solutions:

- Use the Unified System (will require special time differencing at fine spatial resolution).
- Generalize the pseudo-incompressible system to include large-scale spatial variations and low-frequency temporal variations in the reference state.

Outline

Filtered Equation Sets

Sound Wave Propagation

Steady, Horizontally Homogeneous Reference State

Four-Dimensional Reference States

Wrap Up

Generalizing the reference state

Let $\tilde{\pi}$, $\tilde{\theta}$ and $\tilde{\rho}$ define a spatially varying reference state in hydrostatic balance

$$c_p\tilde{\theta}\frac{\partial\tilde{\pi}}{\partial z} = -g,$$

and suppose that

$$\rho^* = \frac{\tilde{\rho}(x,y,z,t)\tilde{\theta}(x,y,z,t)}{\theta}.$$

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Separate $\tilde{\pi}$ into a large horizontally uniform component $\tilde{\pi}_v(z,t)$ plus a remainder $\tilde{\pi}_h(x,y,z,t)$.

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Typically $|\tilde{\pi}_h| \ll |\tilde{\pi}_v|$.

Generalized pseudo-incompressible system

$$\frac{D\mathbf{u_h}}{Dt} + f\mathbf{k} \times \mathbf{u_h} + c_p\theta\nabla_h(\tilde{\pi}_h + \pi') = 0, \quad (2)$$

$$\frac{Dw}{Dt} + c_p\theta\frac{\partial\pi'}{\partial z} = g\frac{\theta'}{\tilde{\theta}}, \quad (3)$$

$$\frac{D\theta}{Dt} = \frac{H_m}{c_p\tilde{\pi}}, \quad (4)$$

$$\frac{D\rho^*}{Dt} + \rho^*\nabla\cdot\mathbf{v} = 0. \quad (5)$$

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- Use of ρ^* in (5).

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- U be a characteristic windspeed,
- D the vertical length scale
- L the horizontal length scale for all variations except those in the reference state fields.

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- The aspect ratio $\delta = D/L$

Conditions for validity – II

Durran (2008) showed

Case 1: if the reference state pressure gradients $\nabla_h \cdot \tilde{\pi}_h$ do not exceed $\nabla_h \cdot \pi'$, the neglected terms are small if in addition

$$\mathcal{M}^2 \ll \min(1, \mathcal{R}^2).$$

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- Strategy for extending the accuracy of the pseudo-incompressible system to accurately represent large-scale motions is to ensure the pressure signal for those motions is captured in $\tilde{\pi}$.
- How do we compute $\tilde{\pi}$ so it is effectively a large-scale compressible hydrostatic pressure?

Solution procedure –I

Starter step:

Obtain ${\rho^*}^{n+1}$ using $\mathbf{v}^n$

$$\frac{\partial \rho^*}{\partial t} = -\nabla \cdot (\rho^* \mathbf{v}).$$

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Update ${\mathbf{u_h}}^{n+1}$ using

$$\frac{\partial \mathbf{u}_h}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{u_h} - f\mathbf{k} \times \mathbf{u_h} - c_p \theta \nabla_h (\tilde{\pi}_h + \pi').$$

Solution procedure –II

Update w^{n+1} using

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Update ${\pi'}^{n+1}$ using

$$c_p \frac{\partial}{\partial x_i} \left(\rho^* \theta \frac{\partial \pi'}{\partial x_i} \right) = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\rho^* u_i u_j) + \frac{\partial^2 \rho^*}{\partial t^2} + \frac{\partial f_i}{\partial x_i}$$

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- Integrate the $\tilde{\pi}$ vertically using the hydrostatic equation

$$\tilde{\pi}^{n+1}(x,y,z) = \tilde{\pi}^{n+1}(x,y,z_s) - \int_{z_s}^{z} \frac{g}{c_p\tilde{\theta}^{n+1}}\,dz$$

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Does this actually work?

Outline

Energy conservation – Steady reference state

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Energy equation for the full compressible system

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- ρ^* replaces ρ.
- No change in internal energy produced by the energy-flux convergence.

Energy conservation – Evolving reference state

$$\frac{\pi'}{\tilde{\pi}}\frac{\partial}{\partial t}\left(\tilde{\rho}c_v\tilde{T}\right)+\frac{\partial}{\partial t}\left(\rho^* M+\tilde{\rho}c_v\tilde{T}\right)+\nabla\cdot\left[\rho^*\left(M+c_vT+\frac{p}{\rho}\right)\mathbf{v}\right]=0.$$

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- Evolving internal energy is that of the reference state.
- Global average of first term can be set to zero when solving for π'
- Local value of first term is small

$$\frac{\frac{\pi'}{\tilde{\pi}}\frac{\partial}{\partial t}\left(\tilde{\rho}c_v\tilde{T}\right)}{\frac{\partial}{\partial t}\left(\rho^*\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)}\ll O(R)$$

  - $R=10^{-2}$ on the mesoscale
  - $R=10^{-5}$ on large scales where $\tilde{\pi}$ carries the signal in pressure.

Conclusions – I

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- Using this approach, sound waves will be eliminated from the governing equations provided ρ^* is an explicit function of the space and time coordinates and of entropy.
- By construction, isentropic pressure perturbations have no influence on the pseudo-density (pseudo-incompressibility).

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It appears possible to formulate a global atmospheric model using the pseudo-incompressible system by spatially averaging the potential temperature to determine $\tilde{\theta}(\mathbf{x}, t)$ and then computing a hydrostatically balanced $\tilde{\pi}(\mathbf{x}, t)$.

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Does it work to compute the surface pressure from the equation of state?