On the Design of Dynamical Cores for Atmospheric General Circulation Models (GCMs): Numerical and Scientific Challenges

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Overview of the 2 tutorials

Today:

• Scientific and numerical aspects in the design process
• Review different forms of the equations and variables
• Computational meshes, staggering options, the characteristics and accuracy of numerical discretizations
• Suitable time-stepping schemes and vertical coordinates
Overview of the 2 tutorials

Tomorrow:

• Physical and computational challenges: conservation, positive tracer advection, stability
• Computational aspects, computer architectures, scalability, efficiency, how it determines science decisions
• Hidden features in the design process: subgrid-scale diffusive and filtering processes
• Evaluation techniques and selected dynamical core test cases
The pursuit of the ‘perfect’ dynamical core: Design aspects

Our scientific and numerical wish list:

- Accurate
- Stable
- Simple
- Computationally efficient
- Obeys physical constraints: conservation properties (which ones?), positive-definite tracer advection
- Truthful representation of the subgrid-scale
What is a dynamical core?

- **Fluid dynamics** component of every weather or climate model
- Based on **equations of motion**, they may be approximated
- Describes the **resolved adiabatic motions** on a computational grid
- Contains **filters and diffusion** processes, mostly for numerical purposes, physical justification may be weak
- Determines the choice of the **prognostic** (forecast) variables
Components of an Atmospheric General Circulation Model (AGCM)
Modular design of NASA’s General Circulation Model GEOS-5

Dynamical core

The dynamics and physics are coupled.

Coupling is a research topic by itself (its complexities are often neglected)
Does the dynamical core matter?

• Provocative: the fluid dynamics problem is solved, physics parameterizations matter most
• Let’s take a look at 9 dynamical cores that participated in an intercomparison project during the 2008 NCAR Summer Colloquium
Mountain-triggered Rossby waves

700 hPa zonal wind at day 15 (≈1°×1°L26)
Mountain-triggered Rossby waves

Details matter, diffusion matters

700 hPa zonal wind at day 25 (≈1°×1°L26)
The choice of the equations of motion

• The governing equations are the 3D Euler equations, but we never use them in their original form
• We make simplifications (e.g. the Earth is a perfect sphere) and use scaling arguments to simplify the dynamical core design
• The Euler equations contain 6 equations:
  – Three momentum equations
  – Continuity equation (mass conservation)
  – Thermodynamic equation
  – Ideal gas laws
• 6 equations, 6 unknowns: u,v,w,T,p,ρ
Choice of the Equations: Common design decisions

• **Deep or shallow** atmosphere: is the distance ‘r’ to the center of the Earth represented as the constant radius ‘a’?

• **Hydrostatic or non-hydrostatic**: is forecast equation for \( w \) maintained?

• **Filtered** equations? Anelastic, Boussinesq, pseudo-incompressible, unified

• Which **prognostic variables** are suitable?

• Which **coordinate system** is suitable:
  - Spherical coordinates
  - Local coordinates, Cartesian coordinates
Non-hydrostatic equations of motion  
(deep atmosphere, spherical coordinates)

\[
\frac{Du}{Dt} - \frac{u vtan(\phi)}{r} + \frac{uw}{r} = -\frac{1}{\rho r \cos \phi} \frac{\partial p}{\partial \lambda} + 2\Omega v \sin(\phi) - 2\Omega w \cos(\phi) + \nu \nabla^2 (u)
\]

\[
\frac{Dv}{Dt} + \frac{u^2 \tan(\phi)}{r} + \frac{vw}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} - 2\Omega u \sin(\phi) + \nu \nabla^2 (v)
\]

\[
\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} - g + 2\Omega u \cos(\phi) + \nu \nabla^2 (w)
\]

with variable \( g = \frac{d\Phi}{dr} = G \frac{a^2}{r^2} \)

\[
\frac{D\rho}{Dt} + \frac{\rho}{r \cos \phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right] + \frac{\rho}{r^2} \frac{\partial (r^2 w)}{\partial r} = 0
\]

\[
c_v \frac{DT}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) = J
\]

\( p = \rho RT \)

\[
\frac{D\left( \begin{array}c u \\ v \\ w \end{array} \right)}{Dt} = \frac{\partial \left( \begin{array}c u \\ v \\ w \end{array} \right)}{\partial t} + u \frac{\partial \left( \begin{array}c u \\ v \\ w \end{array} \right)}{r \cos \phi \partial \lambda} + \frac{v}{r} \frac{\partial \left( \begin{array}c u \\ v \\ w \end{array} \right)}{\partial \phi} + w \frac{\partial \left( \begin{array}c u \\ v \\ w \end{array} \right)}{\partial r}
\]

Only approximation:  
Earth is a perfect sphere
Quasi-hydrostatic equations of motion (deep atmosphere, spherical coordinates)

\[
\begin{align*}
\frac{Du}{Dt} &- \frac{uvtan(\phi)}{r} + \frac{uw}{r} = -\frac{1}{\rho \left(r \cos\phi\right)} \frac{\partial p}{\partial \lambda} + 2\Omega v \sin(\phi) - 2\Omega w \cos(\phi) + \nu \nabla^2(u) \\
\frac{Dv}{Dt} &+ \frac{u^2 \tan(\phi)}{r} + \frac{vw}{r} = -\frac{1}{\rho \left(r \cos\phi\right)} \frac{\partial p}{\partial \phi} - 2\Omega u \sin(\phi) + \nu \nabla^2(v) \\
\frac{Dw}{Dt} &- \frac{u^2 + v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} - g + 2\Omega u \cos(\phi) + \nu \nabla^2(w) \quad \text{with variable } g = \frac{d\Phi}{dr} = G \frac{a^2}{r^2} \\
\frac{D\rho}{Dt} &+ \frac{\rho}{r \cos\phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos\phi)}{\partial \phi} \right] + \frac{\rho}{r^2} \frac{\partial (r^2 w)}{\partial r} = 0 \\
c_v \frac{DT}{Dt} + p \frac{D}{Dt} \left(\frac{1}{\rho}\right) = J
\end{align*}
\]

- neglect of \(Dw/Dt\) has the effect of removing vertically propagating acoustic modes
- puts the vertical momentum balance into a diagnostic form
Shallow atmosphere approximation

- Approximate distance $r = a + z$ to the center of the Earth with the constant mean radius of the Earth $a$
- Replace $r$ by $a$ and $\partial/\partial r$ by $\partial/\partial z$, where $z$ is height above mean sea level
- Omit all the metric terms not involving $\tan \varphi$
- Omit those Coriolis terms that vary as the cosine of the latitude
- Neglect all variations of the gravity $g$ (constant)
- Neglect the vertical component of the diffusion
- All is necessary to guarantee energy and absolute momentum conservation on a shallow Earth
Non-hydrostatic equations of motion (shallow atmosphere)

\[
\begin{aligned}
\frac{Du}{Dt} - \frac{uvtan(\phi)}{a} + \frac{uw}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial \lambda} + 2\Omega v \sin(\phi) - 2\Omega w \cos(\phi) + \nu \nabla^2(u) \\
\frac{Dv}{Dt} + \frac{u^2 \tan(\phi)}{a} + \frac{vw}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial \phi} - 2\Omega u \sin(\phi) + \nu \nabla^2(v) \\
\frac{ Dw}{Dt} - \frac{u^2 + v^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2\Omega u \cos(\phi) + \nu \nabla^2(w) \\
\frac{D\rho}{Dt} + \frac{\rho}{a \cos \phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right] + \frac{\rho}{r^2} \frac{\partial (r^2 w)}{\partial z} &= 0 \\
c_v \frac{DT}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) &= J \\
p = \rho RT \\
\frac{D( )}{Dt} = \frac{\partial ( )}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial ( )}{\partial \lambda} + \frac{v}{a} \frac{\partial ( )}{\partial \phi} + w \frac{\partial ( )}{\partial z}
\end{aligned}
\]

with variable \( g = \frac{d\Phi}{dr} = C \frac{\mu^2}{r^2} \)

Omit terms
Replace \( r \) with \( a \)
Non-hydrostatic equations of motion (shallow atmosphere)

\[
\begin{align*}
\frac{Du}{Dt} - \frac{uvtan(\phi)}{a} &= -\frac{1}{\rho \ a \ cos \ \phi} \ \frac{\partial p}{\partial \lambda} + 2\Omega v \ sin(\phi) + \nu \nabla^2 (u) \\
\frac{Dv}{Dt} + \frac{u^2tan(\phi)}{a} &= -\frac{1}{\rho \ a \ \phi} \ \frac{\partial p}{\partial \phi} - 2\Omega u \ sin(\phi) + \nu \nabla^2 (v) \\
\frac{Dw}{Dt} &= -\frac{1}{\rho \ \phi} \ \frac{\partial p}{\partial \phi} - g \quad \text{with constant } \ g \\
\frac{Dp}{Dt} + \frac{\rho}{a \ cos \ \phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \ cos \ \phi)}{\partial \phi} \right] + \frac{\partial w}{\partial \phi} &= 0 \\
C_v \frac{DT}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) &= J \\
p &= \rho RT \\
\frac{D( )}{Dt} &= \frac{\partial( )}{\partial t} + \frac{u}{a \ cos \ \phi} \frac{\partial( )}{\partial \lambda} + \frac{v}{a} \frac{\partial( )}{\partial \phi} + w \frac{\partial( )}{\partial \phi}
\end{align*}
\]
Hydrostatic equations of motion (shallow atmosphere): **Primitive Equations**

\[
\frac{Du}{Dt} - \frac{uv\tan(\phi)}{a} = -\frac{1}{\rho a \cos \phi} \frac{\partial p}{\partial \lambda} + 2\Omega v \sin(\phi) + \nu \nabla^2(u)
\]

\[
\frac{Dv}{Dt} + \frac{u^2\tan(\phi)}{a} = -\frac{1}{\rho a \cos \phi} \frac{\partial p}{\partial \phi} - 2\Omega u \sin(\phi) + \nu \nabla^2(v)
\]

\[
\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \text{with constant } g
\]

\[
\frac{D\rho}{Dt} + \rho \frac{\partial}{\partial \phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right] + \frac{\partial w}{\partial z} = 0
\]

\[
c_v \frac{DT}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) = J
\]

\[
p = \rho RT
\]

\[
\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u \frac{\partial(\ )}{a \cos \phi \partial \lambda} + \frac{\partial(\ )}{\partial \phi} + w \frac{\partial(\ )}{\partial z}
\]
Design choices

- The hydrostatic shallow atmosphere equations are called **Primitive Equations (PE)**
- **PE most popular choice** in today’s GCMs
- Vertically propagating sound waves are removed
- But note: acoustic modes can also be removed by selecting filtered equation sets
- Filtered equations are sometimes used for special purposes like cloud models, meso-scale models
- Word of caution: filtered equations are not a good choice for global GCMs
Filtered equations: Getting rid of sound waves

- Vertically propagating sound waves are a nuisance in weather and climate models (not important)
- They propagate at high speed and require small time steps in numerical schemes (stability constraints)
- The hydrostatic approximation filters vertically propagating sound waves
- If non-hydrostatic equations need to be used, a filtered equation set might be a choice (just be careful and know the limitations: e.g. shallow flows, static stability requirements)
Filtered equations: Boussinesq

- Boussinesq approximation: set the density to a constant unless it is multiplied with the gravity term
- Continuity equation becomes:

\[
\frac{\partial \rho}{\partial t} + \frac{1}{a \cos \phi} \left[ \frac{\partial (\rho u)}{\partial \lambda} + \frac{\partial (\rho v \cos \phi)}{\partial \phi} \right] + \frac{\partial (\rho w)}{\partial z} = 0
\]

\[\iff \nabla \cdot \vec{v} = 0\]

- Flow is non-divergent
- Limited to shallow flows in the boundary layer
Filtered equations: Anelastic

• Anelastic approximation: the density varies according to a prescribed vertical profile $\bar{\rho}(z)$

• Continuity equation becomes:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{1}{a \cos \phi} \left[ \frac{\partial (\bar{\rho} u)}{\partial \lambda} + \frac{\partial (\bar{\rho} v \cos \phi)}{\partial \phi} \right] + \frac{\partial (\bar{\rho} w)}{\partial z} = 0$$

$$\Leftrightarrow \nabla \cdot (\bar{\rho}(z) \vec{v}) = 0$$

• Justification from scale analysis: density variations in the vertical direction are bigger than horizontal variations

• Problem: Specifying a generic profile is difficult, sometimes $\bar{\rho}(z)$ assumes isentropic conditions

Ogura and Phillips (1962)
Filtered equations: Pseudo-incompressible

- Neglects the influence of perturbation $p'$ on $\rho'$
- The continuity equation includes the steady reference fields $\overline{\theta}(x,y,z)$ and $\overline{\rho}(x,y,z)$ that need to obey the equation of state

$$\overline{p} = p_0 \left( \frac{R_d}{p_0} \overline{\rho} \overline{\theta} \right)^{\frac{c_p}{c_v}}$$

- Continuity equation becomes:

$$\nabla \cdot \left( \overline{\rho} \overline{\theta} \overline{v} \right) = 0$$

- Less severe restriction, wider application range
- Not used in GCMs

Durran (JAS, 1989)
More design decisions:
The form of the equations

- Lagrangian versus Eulerian form
- Advective form versus flux form
- Model variables
- Vertical coordinate transformations
Lagrangian versus Eulerian framework

• Lagrangian form: The variations are observed following a moving particle, requires the total derivative, e.g. the continuity equation is:

\[
\frac{D\rho}{Dt} + \frac{\rho}{a \cos \phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right] + \frac{\partial w}{\partial z} = 0
\]

• Eulerian form: The variations are observed at a fixed location and snapshot in time, requires partial derivatives, e.g. the continuity equation is:

\[
\frac{\partial \rho}{\partial t} + \frac{1}{a \cos \phi} \left[ \frac{\partial (\rho u)}{\partial \lambda} + \frac{\partial (\rho v \cos \phi)}{\partial \phi} \right] + \frac{\partial (\rho w)}{\partial z} = 0
\]
Advective form versus the flux form

• Consider a tracer advection equation for tracer q:
  \[ \frac{Dq}{Dt} = 0 \]
  \[ \Leftrightarrow \frac{\partial q}{\partial t} + \vec{v} \cdot \nabla q = 0 \]

• This is the so-called **advective form**

• The **flux form** can be formed by incorporating the continuity equation:
  \[ \frac{\partial (\rho q)}{\partial t} + \nabla \cdot (\rho q \vec{v}) = 0 \]

• The flux form has great advantages concerning mass conservation, especially in finite-volume models
Mass conservation in flux form

• The continuity equation in the Eulerian framework is also an equation in flux form

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

• For simplicity, let us assume the equation is 1D:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \]

• The finite-difference discretization (numerical scheme) for this PDE may be (n time index):

\[ \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{1}{\Delta x} \left( (\rho u)^n_{i+1/2} - (\rho u)^n_{i-1/2} \right) = 0 \]

• Mass-conserving by design
Mass conservation in flux form

• Rewrite the equation with numerical fluxes $F$ in the $x$ direction:

$$\rho_{i}^{n+1} = \rho_{i}^{n} - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{n} - F_{i-1/2}^{n} \right)$$

• The density variations at the next time step $n+1$ are determined by the **balance of incoming and outgoing fluxes** at the grid interfaces with indices $i+1/2, i-1/2$

• Picture this: $F_{i-1/2}^{n}$ and $F_{i+1/2}^{n}$

  assume $u > 0$
Choice of the model variables

• We can choose (within limits) the model variables
• Hydrostatic models lose the ability to forecast the vertical velocity (becomes diagnostic)
• The choices are also determined by the numerical schemes (e.g. vertical coordinate system)
• A common set is $u, v, T, p_s, \rho$
• Another common set is $\zeta, \delta, T, p_s, \rho$ where $\zeta, \delta$ are the relative vorticity and horizontal divergence
• The thermodynamic variable is sometimes the potential temperature $\theta$ instead of $T$
• Advantage: built-in conservation $\frac{\partial (\rho \theta)}{\partial t} + \nabla \cdot (\rho \theta \vec{v}) = 0$
Choice of the vertical coordinate

• First decision to make: Orography-intersecting model levels or orography-following coordinate?

Most common choice: orography following, e.g.

• Pressure-based, so-called σ-coordinate:
  \[
  \sigma = \frac{(p-p_t)}{(p_s-p_t)}
  \]
  with \( p_t \) (p at the model top), \( p_s \) is surface pressure

• Hybrid σ-ρ coordinate called η-coordinate:
  \[
  \eta = A p_0 + B p_s
  \]
  with prescribed coefficients A and B (dependent on vertical position), constant \( p_0 = 1000 \) hPa, used in many GCMs
Hybrid ($\eta$) vertical coordinates

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<th>Interface index</th>
<th>Interface pressure</th>
<th>Interface coordinate</th>
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$p_{\text{lev}} = \sigma \frac{p}{p_s}$

$\sigma = \frac{p}{p_s}$
Floating Lagrangian vertical coordinate

- 2D transport calculations, let layers expand
- Layers are material surfaces, no vertical advection
- Periodic re-mapping of the Lagrangian layers onto reference grid
Choice of the vertical coordinate

Other choices might be:
- Height based coordinates
- Floating Lagrangian coordinate (Lin (2004))
- Isentropic based (hybrid $\theta$-$p$)
- Shaved cells, step coordinate

• **Requirement:** the new vertical coordinate needs to be **monotonic**

• Whatever we choose it requires a coordinate transformation and modifies the equations of motion, e.g. see the example on the next slide for a pressure-based coordinate $s=p$
Vertical coordinate transformations

Pressure gradient along ‘s’:

\[
\left( \frac{\partial p}{\partial x} \right)_s = \frac{\partial p}{\partial z} \left( \frac{\partial z}{\partial x} \right)_s + \left( \frac{\partial p}{\partial x} \right)_z
\]

Example:
For \( s = p \):

\[
\left( \frac{\partial p}{\partial x} \right)_z = \left( \frac{\partial p}{\partial x} \right)_p - \frac{\partial p}{\partial z} \left( \frac{\partial z}{\partial x} \right)_p
\]

\[
= -(-\rho g) \left( \frac{\partial z}{\partial x} \right)_p = \rho \left( \frac{\partial g}{\partial x} \right)_p = \rho \left( \frac{\partial \Phi}{\partial x} \right)_p
\]
Vertical coordinate transformations

• New vertical coordinates introduce new vertical velocities

• Example: in a pressure-based system the vertical velocity becomes

\[ \omega = \frac{dp}{dt} = \dot{p} \]

• In a hybrid \( \sigma-\rho (\eta) \) system the vertical velocity becomes

\[ \dot{\eta} = \frac{d\eta}{dt} \]

• The new vertical velocity enters the equations of motion, e.g.

\[ \frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial(\ )}{\partial \lambda} + \frac{v}{a} \frac{\partial(\ )}{\partial \phi} + \dot{\eta} \frac{\partial(\ )}{\partial \eta} \]
The pursuit of the ‘perfect’ model grid

• How to distribute grid points over the sphere: yet to be solved

• Possible design criteria:
  – Highly uniform coverage
  – Orthogonal
  – Structured versus unstructured
  – Adaptive mesh
Platonic solids - Regular grid structures

- Platonic solids can be enclosed in a sphere
Computational grids (horizontal)

Hexagonal grid

Cubed sphere

Icosahedral grid
Latitude-Longitude Grid

- Popular choice in the past
- Meridians converge: requires polar filters or/and small time steps
- Orthogonal
Adaptive Mesh Refinements (AMR)

AMR on a cubed-sphere mesh

AMR on a latitude-longitude grid

St-Cyr, Jablonowski, et.al (MWR, 2008)
Example of an AMR simulation
Other non-uniform (nested) grids

Model ICON

Icosahedral grid with nested high-resolution regions under development at the German Weather Service (DWD) and MPI, Hamburg, Germany

Source: DWD
Why do we might want AMR grids in GCMs?

GEOS-5 Modeled Clouds at 7 km Global Resolution for Aug 17, 2009 21z through Aug 26, 2009 21z

NASA/GSFC:
Dr. Bill Putman
Dr. Max Suarez
Greg Shirah
The pursuit of the ‘perfect’ positions of variables in the discrete system

- Having decided on the basic distribution of grid points, a choice has to be made as to how to arrange the different prognostic variables on the grid
- Most obvious choice of representing all variables at the same point has disadvantages
- There are many choices, called: A, B, C, D, E, Z or ZM grid (the first five are based on a classification by Akio Arakawa (UCLA))
Example: Grid staggerings

- Many choices how to place scalars and vector winds in the computational grid
- Examples are

  - Staggerings determine properties of the numerical schemes: dispersion and diffusion properties
  - Additional staggering options in the vertical
Vertical grid staggerings: Lorenz

Level index | Interface index
---|---
1 | 1
2 | 2
k-1 | k-1
k | k
plev | plev
plev + 1 |

- \( \dot{\eta}, p \) (interface)
- \( u, v, \theta \) (full level)

Good conservation properties.
But contains a computational mode
Vertical grid staggerings: Charney-Phillips

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\[
\dot{\eta}, p, \theta \quad \text{(interface)}
\]

\[
u, v \quad \text{(full level)}
\]
The pursuit of the ‘perfect’ numerical scheme

• We want: high order of accuracy, but computationally cheap method
• We need: discretizations in space and time
• Many space discretization philosophies:
  – Finite difference methods (FD)
  – Finite volume methods (FV)
  – Finite (spectral) element methods (FE, SE)
  – Spectral methods
• Sometimes different spatial methods are used in the horizontal and vertical directions
The pursuit of the ‘perfect’ numerical scheme

• Phase errors and damping should be small (often a compromise)
• Explicit scheme is ‘easy’ to program, but it will only be conditionally stable and so the choice of time step is limited
• Implicit schemes are absolutely stable; however at every time step a system of simultaneous equations has to be solved
• More than two time levels: additional computational modes and possibly separation of the solution at odd and even time steps. Higher storage (memory) requirements.
Discretizations in time: explicit versus implicit

• The earlier example used a so-called explicit time stepping scheme with two time levels \( n+1 \) and \( n \)

\[
\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^n - F_{i-1/2}^n \right)
\]

• How about rewriting this equation as an implicit scheme

\[
\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1} \right)
\]

• It uses the (unknown) fluxes at the future time step, requires sophisticated numerical methods to solve (more expensive, e.g. iterative methods)

• Big advantage: increased numerical stability, allows longer time steps
Multi-level time discretizations

• We can increase the order of accuracy of a time stepping scheme by using **multiple time levels**, e.g. three time levels n+1, n, n-1

• Picture an equation like (describes Rayleigh friction)

\[ \frac{\partial u}{\partial t} = -\tau u \]

• Possible time discretizations are:

\[ \frac{u^{n+1} - u^n}{\Delta t} = -\tau u^n \quad \text{or} \quad \frac{u^{n+1} - u^n}{\Delta t} = -\tau u^{n+1} \quad \text{(explicit / implicit)} \]

\[ \frac{u^{n+1} - u^{n-1}}{2\Delta t} = -\tau u^n \quad \text{(3-step method, called Leapfrog method)} \]
Multi-level time discretizations

- Leapfrog is a popular choice in today’s GCMs
- Uses 3 time levels and computes the forcing (here friction) at the center time with index $n$, forcing is applied over a time span of $2\Delta t$

$$u^{n+1} = u^{n-1} - 2\Delta t\tau u^n$$

- 2$^{nd}$-order accurate
- Unfortunately, the Leapfrog method can be numerically unstable (has a computational mode, separates odd and even time steps)
- But it then can be stabilized by applying a time filter that mimics diffusion in time (Asselin filter)
Time-splitting

- Sometimes, equations are separated into two parts (e.g., horizontal and vertical part) and solved independently (in a time-split fashion).

- Example: Consider the tracer conservation equation

\[
\frac{\partial q}{\partial t} + \vec{v} \cdot \nabla q = 0
\]

- Can be split like (using Leapfrog)

\[
q^* = q^{n-1} + 2\Delta t \left( \vec{v}_h \cdot \nabla_h q \right)^n
\]

\[
q^{n+1} = q^* + 2\Delta t \left( w \frac{\partial q}{\partial z} \right)^n
\]

- Allows to use different techniques for the horizontal (h) and vertical advection, introduces a splitting error.
The flavor of FV spatial discretizations

• Finite volume discretization are based on an integrated version of the equations of motion

• Consider 2D example with $h$ (=height of shallow water system), this is a conservation law:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \vec{v}) = 0$$

• Conservation equation can be integrated over spatial domain $\Omega$ with “volume” (here an area) $A_\Omega$ and time $t$

$$\int_{t_n}^{t_{n+1}} \int_\Omega \left( \frac{\partial}{\partial t} h \right) d\Omega \, dt + \int_\Omega \int_{t_n}^{t_{n+1}} \nabla \cdot (h \vec{v}) \, dt \, d\Omega = 0$$
The flavor of FV spatial discretizations

- Integration over spatial domain $\Omega$ with area/volume $A_\Omega$ and time $t$, can be rearranged:

$$\int_{t_n}^{t_{n+1}} \left( \frac{\partial h}{\partial t} \right) dt + \frac{\Delta t}{A_\Omega} \int_{\Omega} \nabla \cdot \vec{F} \, d\Omega = 0$$

- Overbar denotes the spatial mean, $\vec{F}$ denotes time averaged fluxed across the interface of a volume
- Apply Gauss’ divergence theorem to second term:

$$\int_{t_n}^{t_{n+1}} \left( \frac{\partial h}{\partial t} \right) dt + \frac{\Delta t}{A_\Omega} \oint_{\partial\Omega} \vec{F} \cdot d\vec{n} = 0$$

Introduces line integral, $\vec{n}$ is the line segment vector normal to the boundary
The flavor of FV spatial discretizations

• Leads to discretized forecast equation:

\[ \bar{h}^{n+1} = \bar{h}^n - \frac{\Delta t}{A \Omega} \sum_{i=1}^{4} l_i \vec{F}_i \cdot \vec{n}_i \]

• Where \( l_i \) denotes the length of a line segment, \( \vec{n}_i \) is the unit vector normal to the boundary ‘i’

• The future time step is determined by the sum of all fluxes across the boundaries of a finite volume

• Order of accuracy determined by the fluxes \( \vec{F} \), rely on subgrid-scale representations (constant, linear, quadratic, cubic) of transported variable (here \( h \))

• Idea: express subgrid structure of \( h \) with polynomials
The flavor of spectral transform models

• Spectral transform methods on latitude-longitude grids have been very popular in the past
• Some GCM (ECHAM, ECMWF’s weather model IFS) still use it
• Idea:
  – Use a model formulation with vorticity and divergence as prognostic variables
  – Use Fourier and Legendre transformation to transform/represent the flow in spectral space
  – Solve the linear parts of the Eqs. in spectral space (exact)
  – Solve the nonlinear parts in grid point space
• Highly accurate, but suffers from Gibb’s ringing, non-local discretization
The flavor of spectral transform models

Structure of the global basis functions:

Spectral representations of variable $q$:

$$q(\lambda,\varphi,t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} q_n^m(t)Y_n^m(\lambda,\varphi)$$

Spherical harmonics

$$Y_n^m(\lambda,\varphi) = P_n^m(\sin\varphi)\exp(im\lambda)$$

associated Legendre functions

Fourier modes

Triangular wave number range $m, n$

(m=0, n=6)  (m=3, n=6)  (m=6, n=6)
The flavor of spectral transform models

- Triangular truncation $T$... with $N(m) = M$ is unique, provides uniform spatial resolution over the entire surface of the sphere, e.g. $M=42, 85, 170$
- Eliminates pole problem in latitude-longitude grids
- Allows reduced grids with fewer points towards poles
Summary

Today we reviewed:

- Scientific and numerical aspects in the design process
- Different forms of the equations and variables
- Computational meshes, staggering options
- Characteristics (and accuracy) of numerical discretizations
- Suitable time-stepping schemes and vertical coordinates