On the Design of Dynamical Cores for Atmospheric General Circulation Models (GCMs): Numerical and Scientific Challenges

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Overview of the 2 tutorials

Today:

- Scientific and numerical aspects in the design process
- Review different forms of the equations and variables
- Computational meshes, staggering options, the characteristics and accuracy of numerical discretizations
- Suitable time-stepping schemes and vertical coordinates

Overview of the 2 tutorials

Tomorrow:

- Physical and computational challenges: conservation, positive tracer advection, stability
- Computational aspects, computer architectures, scalability, efficiency, how it determines science decisions
- Hidden features in the design process: subgrid-scale diffusive and filtering processes
- Evaluation techniques and selected dynamical core test cases

The pursuit of the ‘perfect’ dynamical core: Design aspects

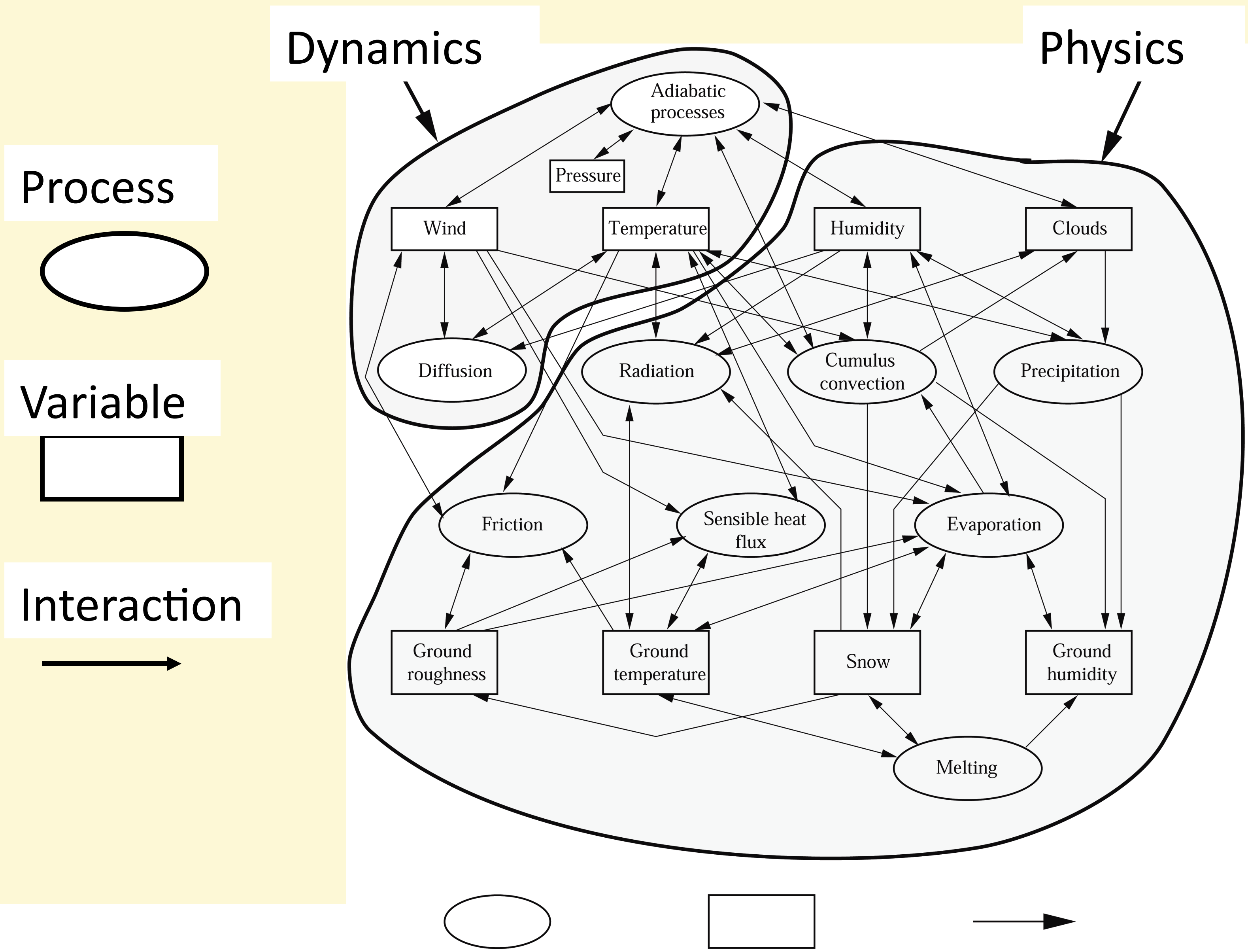
Our scientific and numerical wish list:

- Accurate
- Stable
- Simple
- Computationally efficient
- Obeys physical constraints: conservation properties (which ones?), positive-definite tracer advection
- Truthful representation of the subgrid-scale

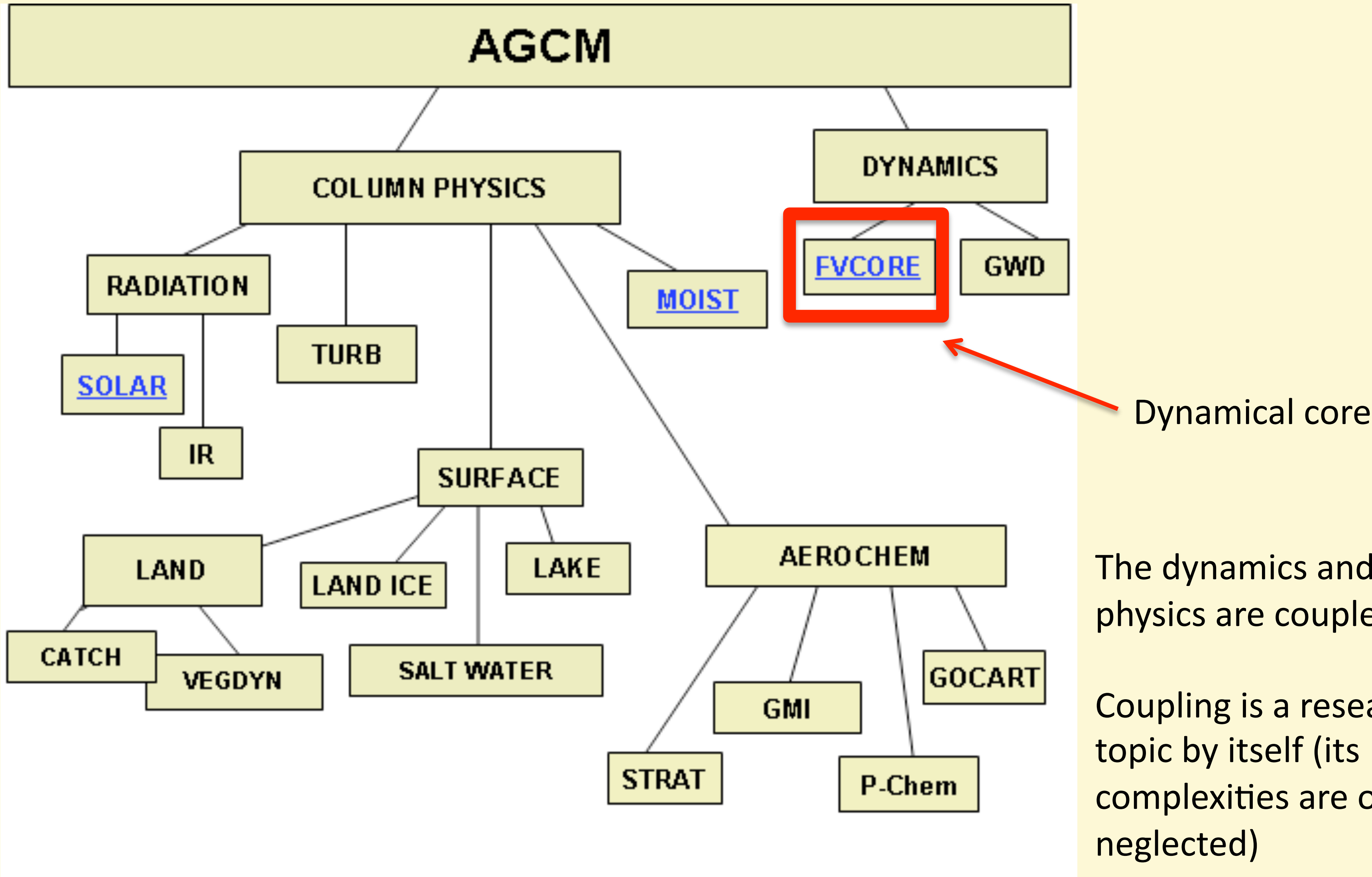
What is a dynamical core?

- **Fluid dynamics** component of every weather or climate model
- Based on **equations of motion**, they may be approximated
- Describes the **resolved adiabatic motions** on a computational grid
- Contains **filters and diffusion** processes, mostly for numerical purposes, physical justification may be weak
- Determines the choice of the **prognostic** (forecast) variables

Components of an Atmospheric General Circulation Model (AGCM)



Modular design of NASA's General Circulation Model GEOS-5



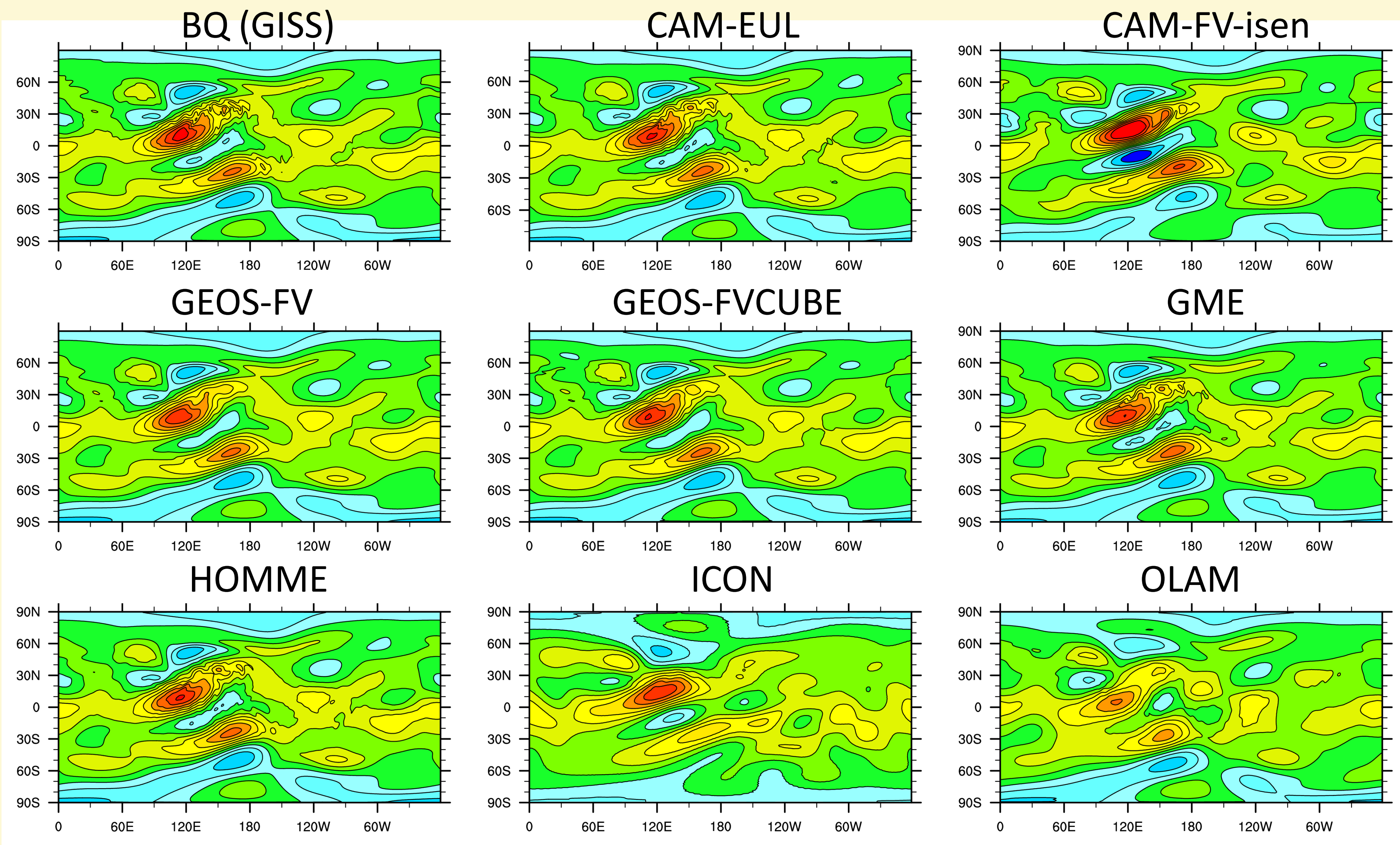
The dynamics and physics are coupled.

Coupling is a research topic by itself (its complexities are often neglected)

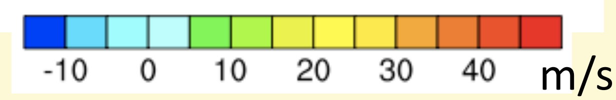
Does the dynamical core matter?

- Provocative: the fluid dynamics problem is solved, physics parameterizations matter most
- Let's take a look at 9 dynamical cores that participated in an intercomparison project during the 2008 NCAR Summer Colloquium

Mountain-triggered Rossby waves

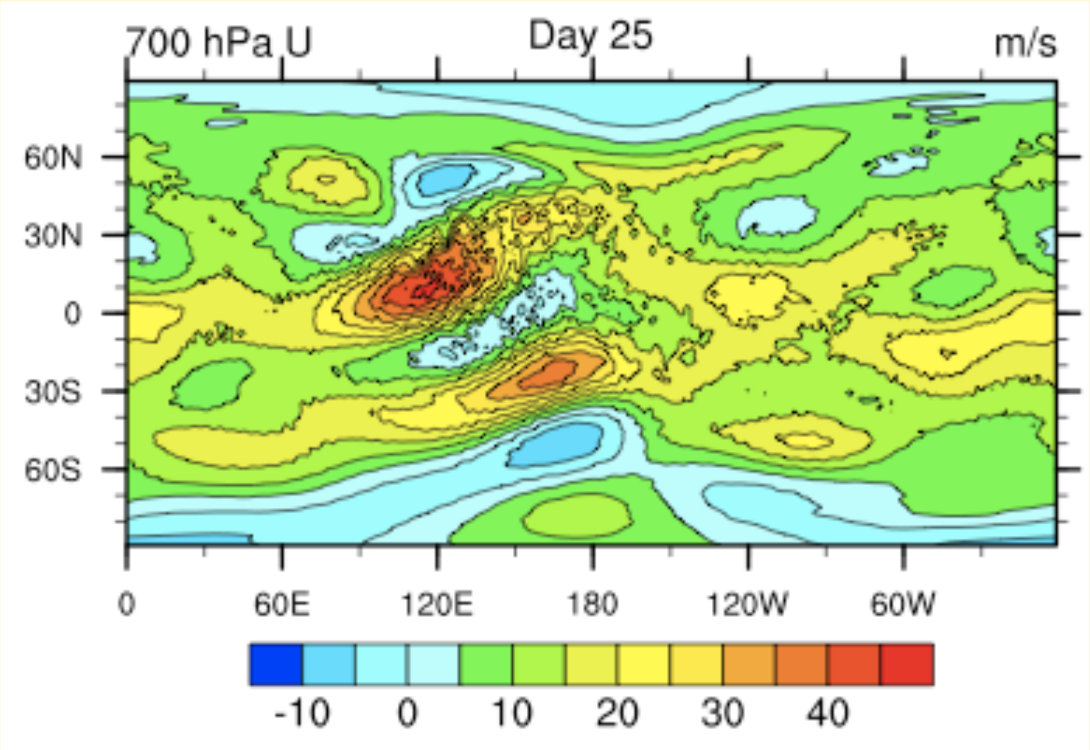


700 hPa zonal wind at day 15 (≈1°×1°L26)



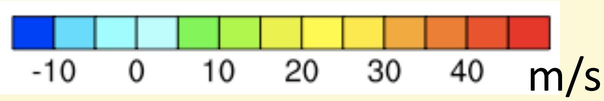
Mountain-triggered Rossby waves

CAM-EUL



Details matter, diffusion matters

700 hPa zonal wind at day 25 (≈1°×1°L26)



The choice of the equations of motion

- The governing equations are the 3D Euler equations, but we never use them in their original form
- We make **simplifications** (e.g. the Earth is a perfect sphere) and use scaling arguments to simplify the dynamical core design
- The Euler equations contain 6 equations:
  - Three momentum equations
  - Continuity equation (mass conservation)
  - Thermodynamic equation
  - Ideal gas las
- **6 equations, 6 unknowns**: u, v, w, T, p, ρ

Choice of the Equations: Common design decisions

- **Deep or shallow** atmosphere:
  is the distance ‘r’ to the center of the Earth represented as the constant radius ‘a’?
- **Hydrostatic or non-hydrostatic**:
  is forecast equation for w maintained?
- **Filtered** equations? Anelastic, Boussinesq, pseudo-incompressible, unified
- Which **prognostic variables** are suitable?
- Which **coordinate system** is suitable:
  - Spherical coordinates
  - Local coordinates, Cartesian coordinates

Non-hydrostatic equations of motion (deep atmosphere, spherical coordinates)

$$\frac{Du}{Dt} - \frac{u\mathrm{v}\tan(\phi)}{r} + \frac{uw}{r} = -\frac{1}{\rho\, r\cos\phi}\frac{\partial p}{\partial \lambda} + 2\Omega \mathrm{v}\sin(\phi) - 2\Omega w\cos(\phi) + \nu\nabla^2(u)$$

$$\frac{D\mathrm{v}}{Dt} + \frac{u^2\tan(\phi)}{r} + \frac{\mathrm{v}w}{r} = -\frac{1}{\rho\, r}\frac{\partial p}{\partial \phi} - 2\Omega u\sin(\phi) + \nu\nabla^2(\mathrm{v})$$

$$\frac{Dw}{Dt} - \frac{u^2 + \mathrm{v}^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega u\cos(\phi) + \nu\nabla^2(w)$$

with variable $g = \frac{d\Phi}{dr} = G\frac{a^2}{r^2}$

$$\frac{D\rho}{Dt} + \frac{\rho}{r\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial(v\cos\phi)}{\partial \phi}\right] + \frac{\rho}{r^2}\frac{\partial(r^2 w)}{\partial r} = 0$$

$$c_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = J$$

$$p = \rho R T$$

$$\frac{D(\;)}{Dt} = \frac{\partial(\;)}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial(\;)}{\partial \lambda} + \frac{v}{r}\frac{\partial(\;)}{\partial \phi} + w\frac{\partial(\;)}{\partial r}$$

Only approximation: Earth is a perfect sphere

Quasi-hydrostatic equations of motion (deep atmosphere, spherical coordinates)

$$\frac{Du}{Dt} - \frac{u\mathrm{v}\tan(\phi)}{r} + \frac{uw}{r} = -\frac{1}{\rho\, r\cos\phi}\frac{\partial p}{\partial \lambda} + 2\Omega \mathrm{v}\sin(\phi) - 2\Omega w\cos(\phi) + \nu\nabla^2(u)$$

$$\frac{D\mathrm{v}}{Dt} + \frac{u^2\tan(\phi)}{r} + \frac{\mathrm{v}w}{r} = -\frac{1}{\rho\, r}\frac{\partial p}{\partial \phi} - 2\Omega u\sin(\phi) + \nu\nabla^2(\mathrm{v})$$

$$\cancel{\frac{Dw}{Dt}} - \frac{u^2+\mathrm{v}^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + 2\Omega u\cos(\phi) + \cancel{\nu\nabla^2(w)} \qquad \text{with variable } g = \frac{d\Phi}{dr} = G\frac{a^2}{r^2}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{r\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right] + \frac{\rho}{r^2}\frac{\partial(r^2 w)}{\partial r} = 0$$

$$c_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = J$$

$$p = \rho R T$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial(\ )}{\partial\lambda} + \frac{v}{r}\frac{\partial(\ )}{\partial\phi} + w\frac{\partial(\ )}{\partial r}$$

- neglect of *Dw/Dt* has the effect of removing vertically propagating acoustic modes
- puts the vertical momentum balance into a diagnostic form

Shallow atmosphere approximation

- Approximate distance $r = a+z$ to the center of the Earth with the constant mean radius of the Earth a
- Replace r by a and $\partial/\partial r$ by $\partial/\partial z$, where z is height above mean sea level
- Omit all the metric terms not involving tan φ
- Omit those Coriolis terms that vary as the cosine of the latitude
- Neglect all variations of the gravity g *(constant)*
- Neglect the vertical component of the diffusion
- All is necessary to guarantee **energy** and **absolute momentum conservation** on a shallow Earth

Non-hydrostatic equations of motion (shallow atmosphere)

$$\frac{Du}{Dt} - \frac{u\mathrm{v}\tan(\phi)}{a} + \cancel{\frac{uw}{r}} = -\frac{1}{\rho\, a\cos\phi}\frac{\partial p}{\partial\lambda} + 2\Omega\mathrm{v}\sin(\phi) - \cancel{2\Omega w\cos(\phi)} + \nu\nabla^2(u)$$

$$\frac{D\mathrm{v}}{Dt} + \frac{u^2\tan(\phi)}{a} + \cancel{\frac{\mathrm{v}w}{r}} = -\frac{1}{\rho\, a}\frac{\partial p}{\partial\phi} - 2\Omega u\sin(\phi) + \nu\nabla^2(\mathrm{v})$$

$$\frac{Dw}{Dt} - \cancel{\frac{u^2+\mathrm{v}^2}{r}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \cancel{2\Omega u\cos(\phi)} + \cancel{\nu\nabla^2(w)} \qquad \text{with } \cancel{\text{variable}}\ g = \cancel{\frac{d\Phi}{dr}} = \cancel{G\frac{a^2}{r^2}}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{a\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right] + \frac{\rho}{\cancel{r^2}}\frac{\partial(\cancel{r^2}w)}{\partial z} = 0$$

$$c_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = J$$

$$p = \rho RT$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial(\ )}{\partial\lambda} + \frac{v}{a}\frac{\partial(\ )}{\partial\phi} + w\frac{\partial(\ )}{\partial z}$$

Omit terms

Replace r with a

Non-hydrostatic equations of motion (shallow atmosphere)

$$\frac{Du}{Dt} - \frac{u\mathrm{v}\tan(\phi)}{a} = -\frac{1}{\rho\, a\cos\phi}\frac{\partial p}{\partial \lambda} + 2\Omega \mathrm{v}\sin(\phi) + \nu\nabla^2(u)$$

$$\frac{D\mathrm{v}}{Dt} + \frac{u^2\tan(\phi)}{a} = -\frac{1}{\rho\,\mathrm{a}}\frac{\partial p}{\partial \phi} - 2\Omega u\sin(\phi) + \nu\nabla^2(\mathrm{v})$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \text{with constant g}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{a\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial\left(v\cos\phi\right)}{\partial \phi}\right] + \frac{\partial w}{\partial z} = 0$$

$$c_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = J$$

$$p = \rho R T$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial(\ )}{\partial \lambda} + \frac{v}{a}\frac{\partial(\ )}{\partial \phi} + w\frac{\partial(\ )}{\partial z}$$

Hydrostatic equations of motion (shallow atmosphere): **Primitive Equations**

$$\frac{Du}{Dt} - \frac{u\mathrm{v}\tan(\phi)}{a} = -\frac{1}{\rho\, a\cos\phi}\frac{\partial p}{\partial\lambda} + 2\Omega \mathrm{v}\sin(\phi) + \nu\nabla^2(u)$$

$$\frac{D\mathrm{v}}{Dt} + \frac{u^2\tan(\phi)}{a} = -\frac{1}{\rho\,\mathrm{a}}\frac{\partial p}{\partial\phi} - 2\Omega u\sin(\phi) + \nu\nabla^2(\mathrm{v})$$

$$\cancel{\frac{Dw}{Dt}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \text{with constant g}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{a\cos\phi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(v\cos\phi)}{\partial\phi}\right] + \frac{\partial w}{\partial z} = 0$$

$$c_v\frac{DT}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = J$$

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$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial(\ )}{\partial\lambda} + \frac{v}{a}\frac{\partial(\ )}{\partial\phi} + w\frac{\partial(\ )}{\partial z}$$

Design choices

- The hydrostatic shallow atmosphere equations are called **Primitive Equations (PE)**
- **PE most popular choice** in today's GCMs
- Vertically propagating sound waves are removed
- But note: acoustic modes can also be removed by selecting filtered equation sets
- Filtered equations are sometimes used for special purposes like cloud models, meso-scale models
- Word of caution: filtered equations are not a good choice for global GCMs

Filtered equations: Getting rid of sound waves

- Vertically propagating sound waves are a nuisance in weather and climate models (not important)
- They propagate at high speed and require small time steps in numerical schemes (stability constraints)
- The hydrostatic approximation filters vertically propagating sound waves
- If non-hydrostatic equations need to be used, a filtered equation set might be a choice (just be careful and know the limitations: e.g. shallow flows, static stability requirements)

Filtered equations: Boussinesq

- Boussinesq approximation: set the density to a constant unless it is multiplied with the gravity term
- Continuity equation becomes:

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{a\cos\phi}\left[\frac{\partial(\cancel{\rho} u)}{\partial \lambda} + \frac{\partial(\cancel{\rho} v\cos\phi)}{\partial \phi}\right] + \frac{\partial(\cancel{\rho} w)}{\partial z} = 0$$

$$\Leftrightarrow \nabla \bullet \vec{v} = 0$$

- Flow is non-divergent
- Limited to shallow flows in the boundary layer

Filtered equations: Anelastic

- Anelastic approximation: the density varies according to a prescribed vertical profile $\overline{\rho}(z)$
- Continuity equation becomes:

$$\cancel{\frac{\partial\overline{\rho}}{\partial t}} + \frac{1}{a\cos\phi}\left[\frac{\partial(\overline{\rho}u)}{\partial\lambda} + \frac{\partial(\overline{\rho}v\cos\phi)}{\partial\phi}\right] + \frac{\partial(\overline{\rho}w)}{\partial z} = 0$$

$$\Leftrightarrow \nabla \bullet \left(\overline{\rho}(z)\ \vec{v}\right) = 0$$

- Justification from scale analysis: density variations in the vertical direction are bigger than horizontal variations
- Problem: Specifying a generic profile is difficult, sometimes $\overline{\rho}(z)$ assumes isentropic conditions

Ogura and Phillips (1962)

Filtered equations: Pseudo-incompressible

- Neglects the influence of perturbation p' on ρ'
- The continuity equation includes the steady reference fields $\overline{\theta}(x,y,z)$ and $\overline{\rho}(x,y,z)$ that need to obey the equation of state

$$\overline{p} = p_0 \left(\frac{R_d}{p_0} \overline{\rho}\ \overline{\theta} \right)^{c_p / c_v}$$

- Continuity equation becomes:

$$\nabla \bullet \left(\overline{\rho}\ \overline{\theta}\ \vec{v} \right) = 0$$

- Less severe restriction, wider application range
- Not used in GCMs

Durran (JAS, 1989)

More design decisions: The form of the equations

- Lagrangian versus Eulerian form
- Advective form versus flux form
- Model variables
- Vertical coordinate transformations

Lagrangian versus Eulerian framework

- Lagrangian form: The variations are observed following a moving particle, requires the total derivative, e.g. the continuity equation is:

$$\frac{D\rho}{Dt} + \frac{\rho}{a\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial (v\cos\phi)}{\partial \phi}\right] + \frac{\partial w}{\partial z} = 0$$

- Eulerian form: The variations are observed at a fixed location and snapshot in time, requires partial derivatives, e.g. the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \frac{1}{a\cos\phi}\left[\frac{\partial (\rho u)}{\partial \lambda} + \frac{\partial (\rho v\cos\phi)}{\partial \phi}\right] + \frac{\partial (\rho w)}{\partial z} = 0$$

Advective form versus the flux form

- Consider a tracer advection equation for tracer q:

$$\frac{Dq}{Dt} = 0$$

$$\Leftrightarrow \frac{\partial q}{\partial t} + \vec{v} \bullet \nabla q = 0$$

- This is the so-called **advective form**
- The **flux form** can be formed by incorporating the continuity equation:

$$\frac{\partial(\rho q)}{\partial t} + \nabla \bullet (\rho q \vec{v}) = 0$$

- The flux form has great advantages concerning mass conservation, especially in finite-volume models

Mass conservation in flux form

- The continuity equation in the Eulerian framework is also an equation in flux form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho \vec{v} \right) = 0$$

- For simplicity, let us assume the equation is 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u \right) = 0$$

- The finite-difference discretization (numerical scheme) for this PDE may be (n time index):

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \frac{1}{\Delta x}\left(\left(\rho u \right)_{i+1/2}^n - \left(\rho u \right)_{i-1/2}^n \right) = 0$$

- Mass-conserving by design

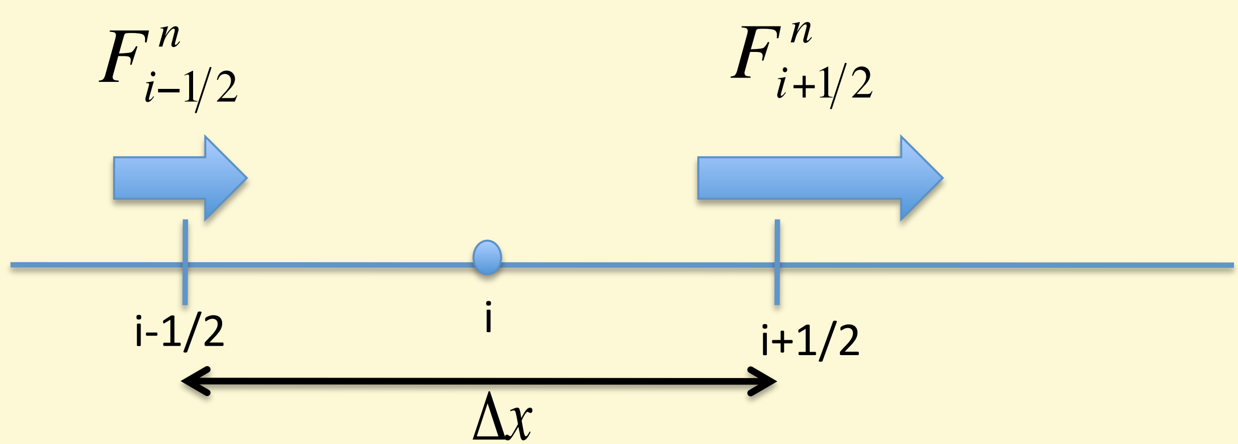
Mass conservation in flux form

- Rewrite the equation with numerical fluxes F in the x direction:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

- The density variations at the next time step n+1 are determined by the **balance of incoming and outgoing fluxes** at the grid interfaces with indices i+1/2, i-1/2
- Picture this:

assume u > 0



Choice of the model variables

- We can choose (within limits) the model variables
- Hydrostatic models lose the ability to forecast the vertical velocity (becomes diagnostic)
- The choices are also determined by the numerical schemes (e.g. vertical coordinate system)
- A common set is **u, v, T, $\mathbf{p_s}$,ρ**
- Another common set is **ζ, δ, T, $\mathbf{p_s}$, ρ** where ζ, δ are the relative vorticity and horizontal divergence
- The thermodynamic variable is sometimes the potential temperature θ instead of T
- Advantage: built-in conservation $\frac{\partial(\rho\theta)}{\partial t} + \nabla \bullet (\rho\theta\vec{v}) = 0$

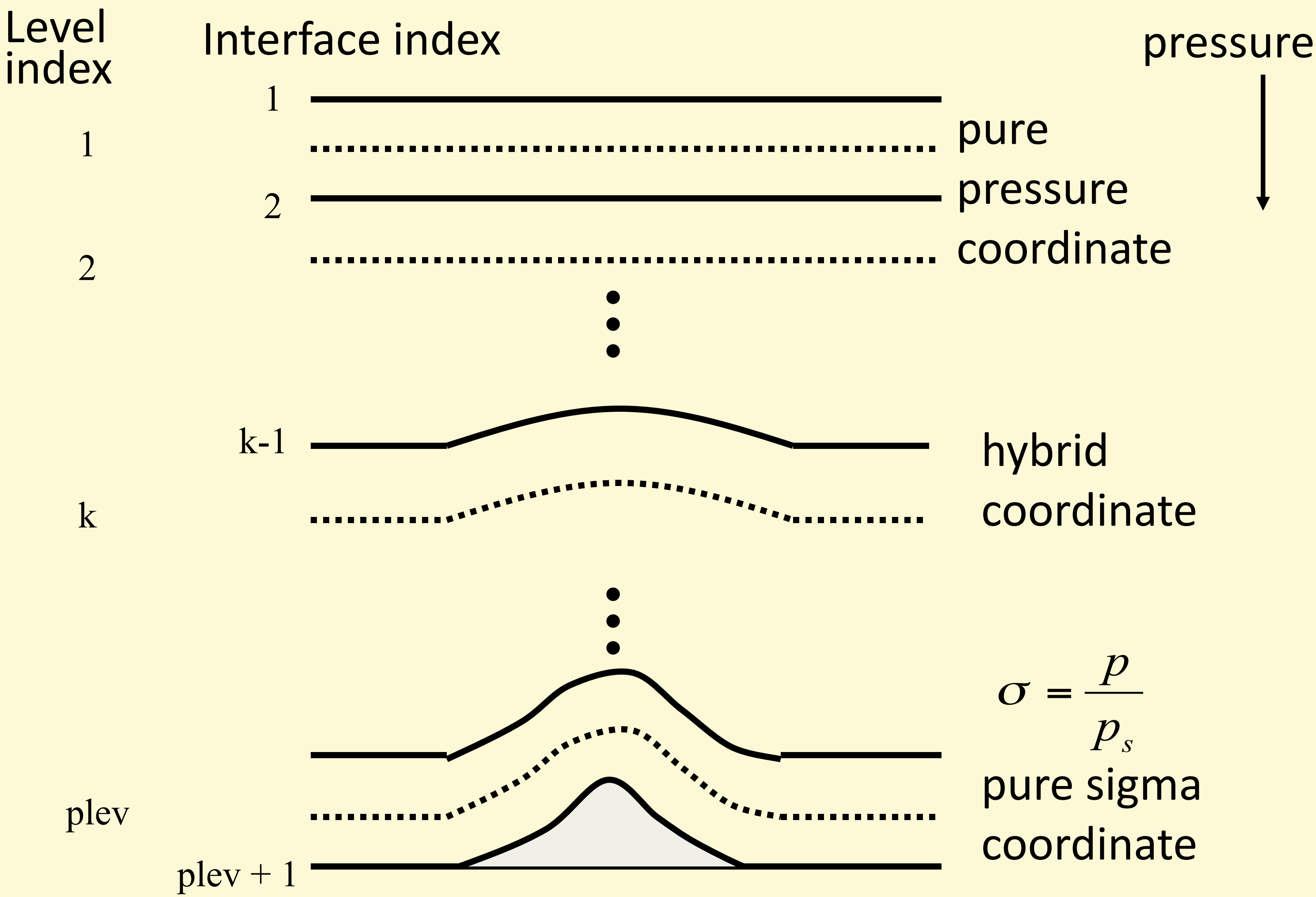
Choice of the vertical coordinate

- First decision to make:
  Orography-intersecting model levels or **orography-following** coordinate?

Most common choice: **orography following,** e.g.

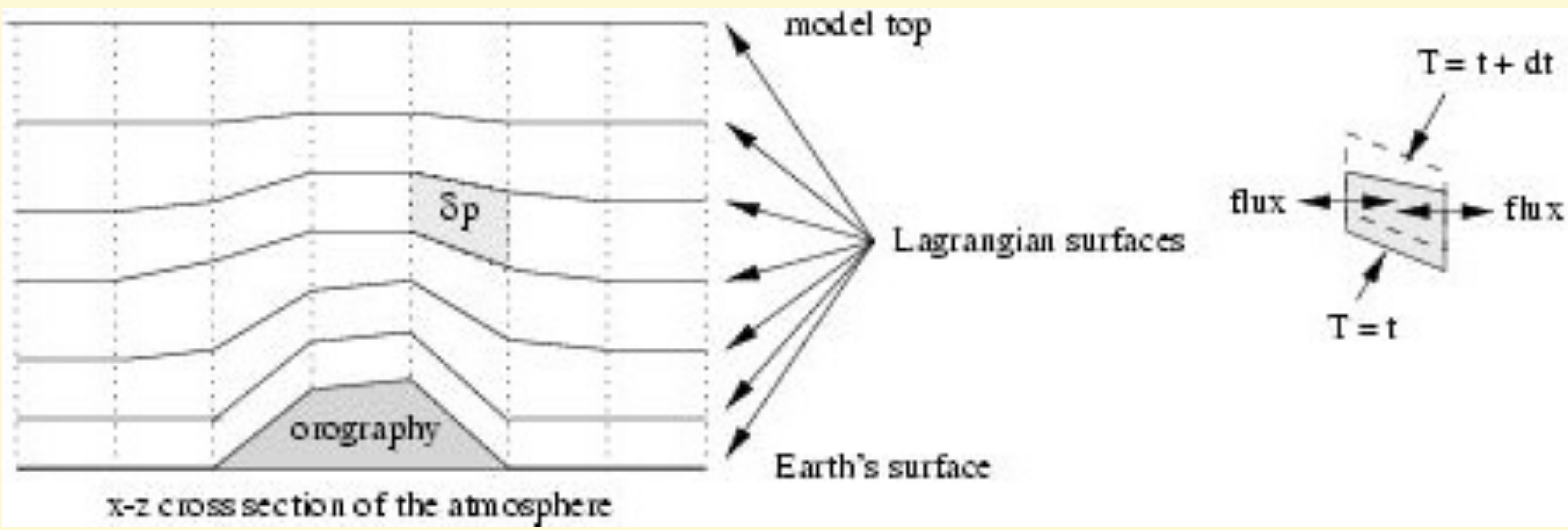
- Pressure-based, so-called σ-coordiante:
  $$\sigma = (p-p_t)/(p_s-p_t)$$
  with p_t (p at the model top), p_s is surface pressure
- Hybrid σ-p coordinate called η-coordinate:
  $$\eta = A\,p_0 + B\,p_s$$
  with prescribed coefficients A and B (dependent on vertical position), constant p_0=1000 hPa, used in many GCMs

Hybrid (η) vertical coordinates



Floating Lagrangian vertical coordinate

- 2D transport calculations, let layers expand
- Layers are material surfaces, no vertical advection
- Periodic re-mapping of the Lagrangian layers onto reference grid



Choice of the vertical coordinate

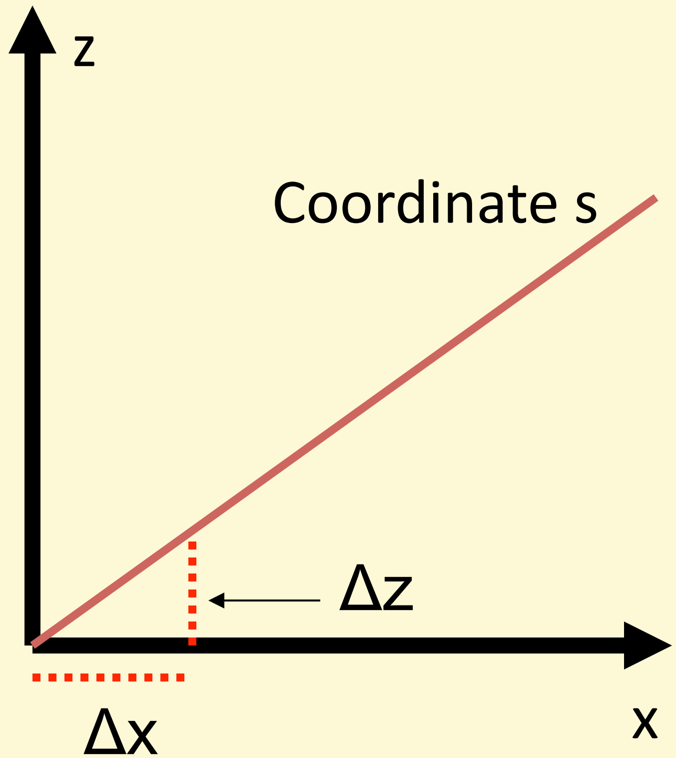
Other choices might be:

- Height based coordinates
- Floating Lagrangian coordinate (Lin (2004))
- Isentropic based (hybrid θ-p)
- Shaved cells, step coordinate

- **Requirement:** the new vertical coordinate needs to be **monotonic**

- Whatever we choose it requires a coordinate transformation and modifies the equations of motion, e.g. see the example on the next slide for a pressure-based coordinate s=p

Vertical coordinate transformations



Pressure gradient along 's':

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z$$

Example:

For s = p:

(pressure levels)

$$\left(\frac{\partial p}{\partial x}\right)_z = \overbrace{\cancel{\left(\frac{\partial p}{\partial x}\right)_p}}^{=\,0} - \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_p$$

$$= -(-\rho g)\left(\frac{\partial z}{\partial x}\right)_p = \rho\left(\frac{\partial gz}{\partial x}\right)_p = \rho\left(\frac{\partial \Phi}{\partial x}\right)_p$$

Vertical coordinate transformations

- New vertical coordinates introduce new vertical velocities
- Example: in a pressure-based system the vertical velocity becomes

$$\omega = \frac{dp}{dt} = \dot{p}$$

- In a hybrid σ-p (η) system the vertical velocity becomes

$$\dot{\eta} = \frac{d\eta}{dt}$$

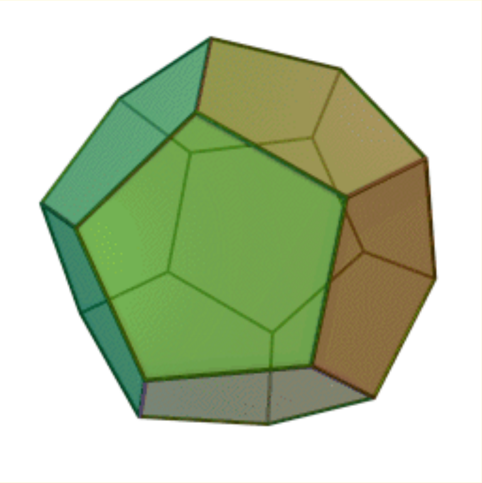
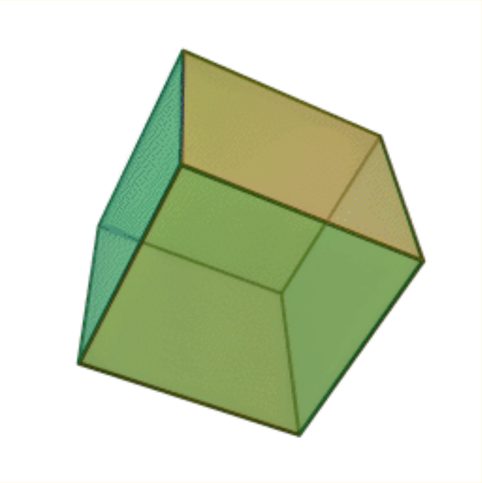
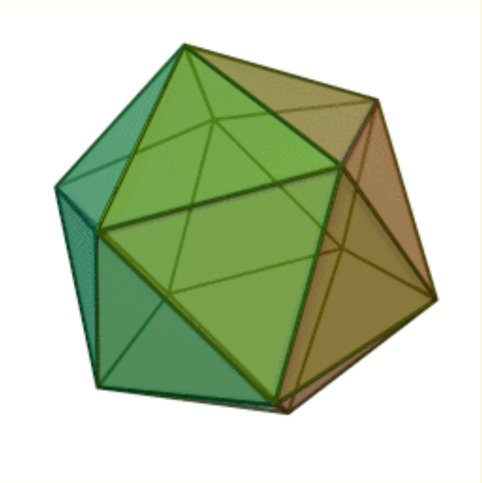
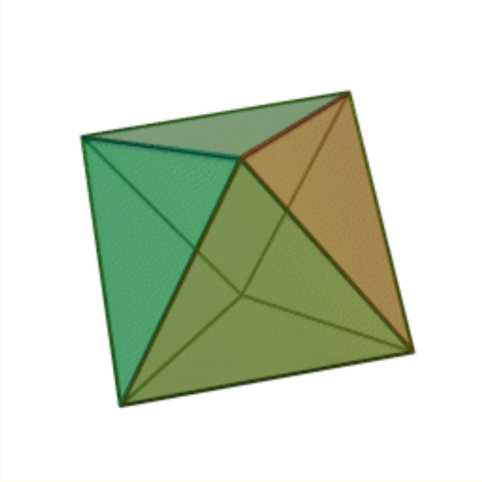
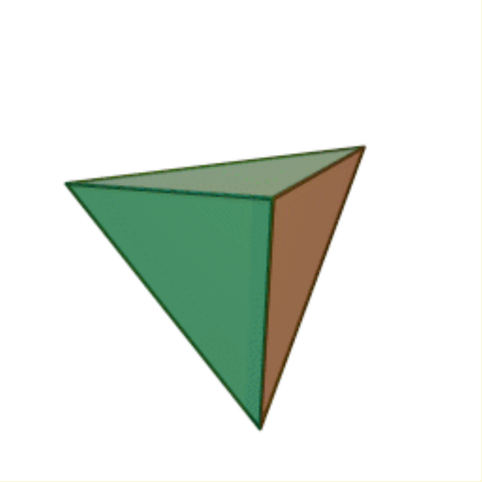
- The new vertical velocity enters the equations of motion, e.g.

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial(\ )}{\partial\lambda} + \frac{v}{a}\frac{\partial(\ )}{\partial\phi} + \boxed{\dot{\eta}\frac{\partial(\ )}{\partial\eta}}$$

The pursuit of the ‘perfect’ model grid

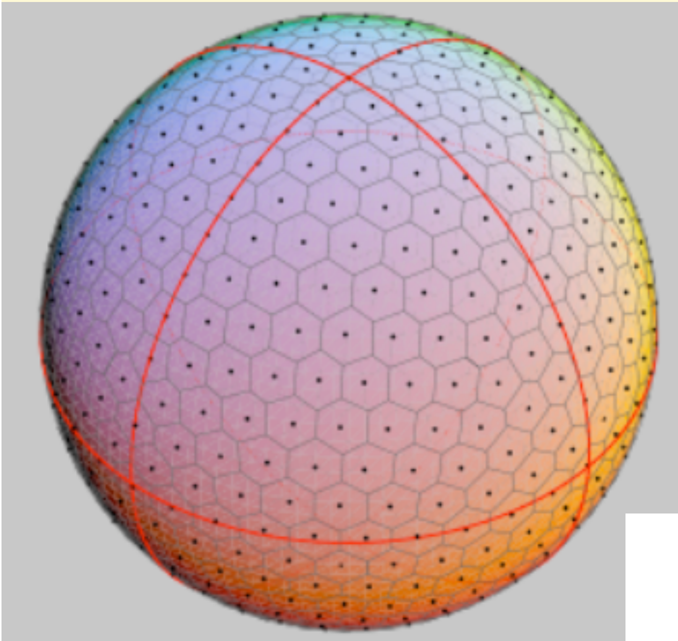
- How to distribute grid points over the sphere: yet to be solved
- Possible design criteria:
  - Highly uniform coverage
  - Orthogonal
  - Structured versus unstructured
  - Adaptive mesh

Platonic solids - Regular grid structures

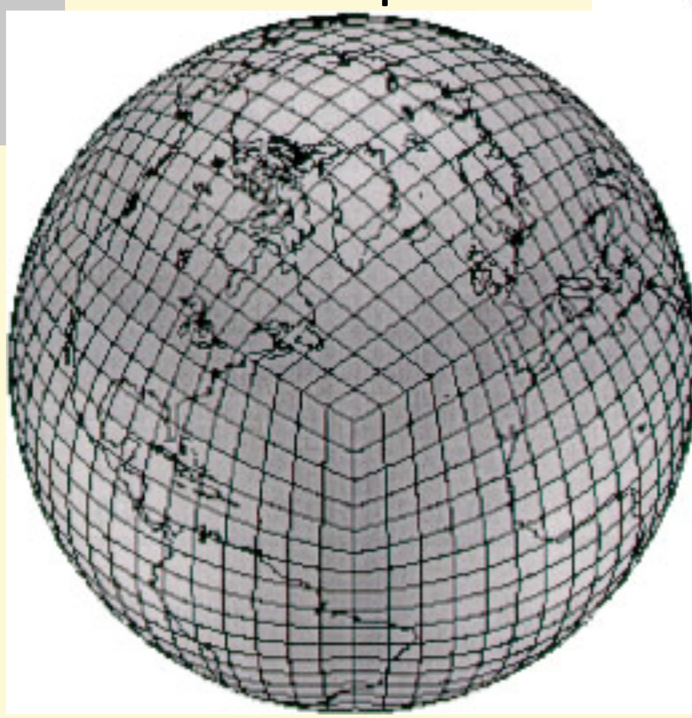
- Platonic solids can be enclosed in a sphere

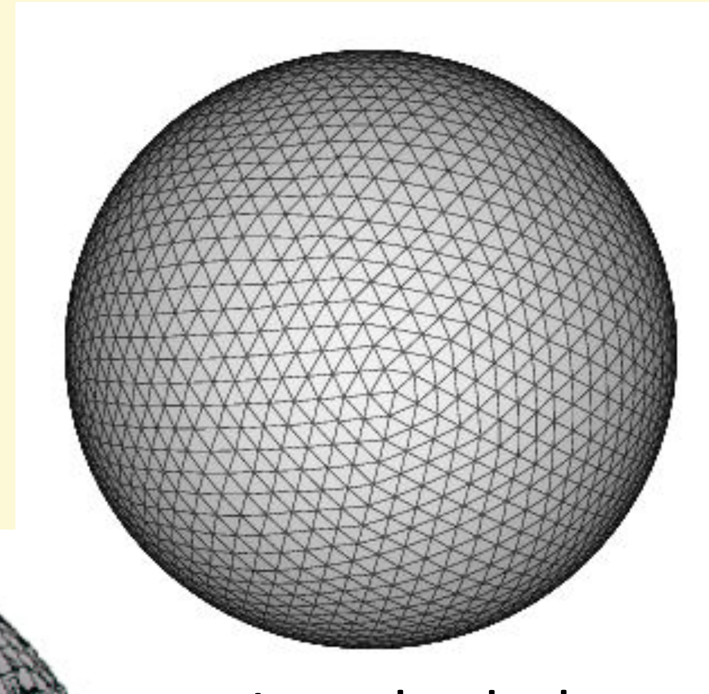
Computational grids (horizontal)



Hexagonal grid

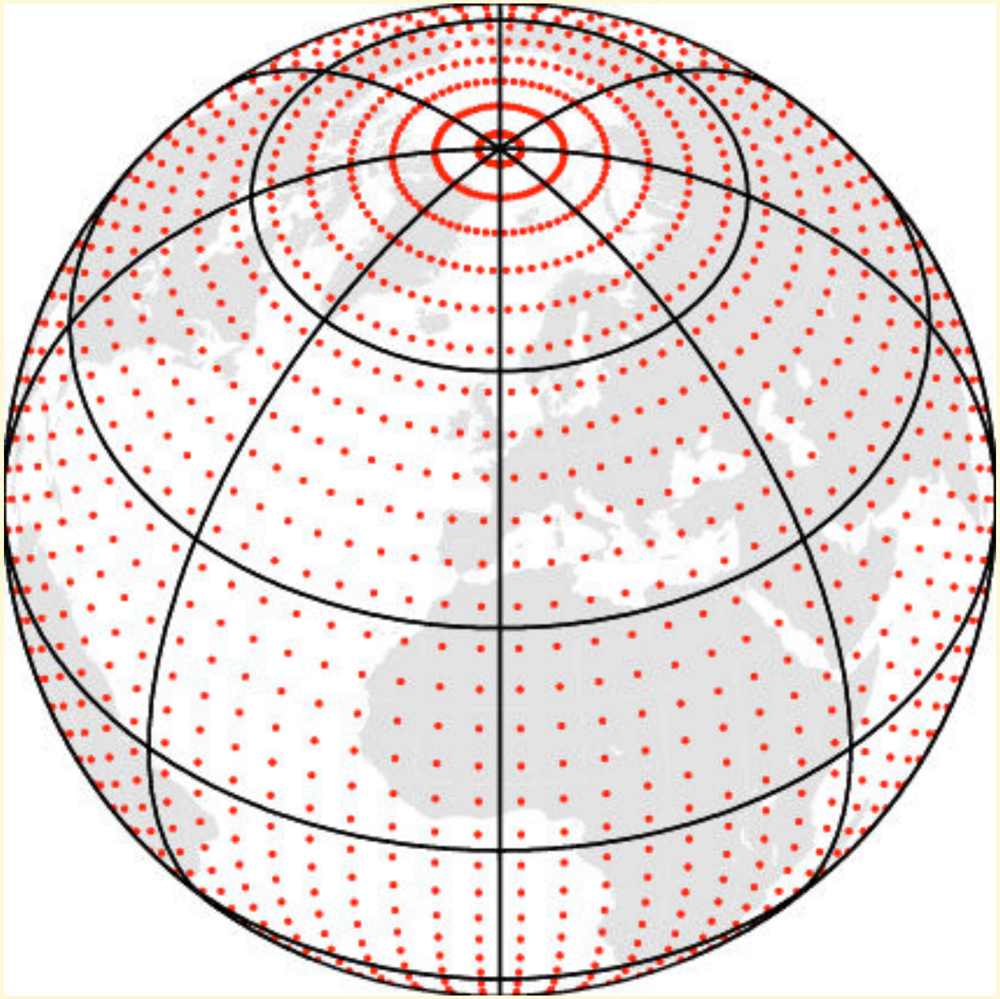
Cubed sphere





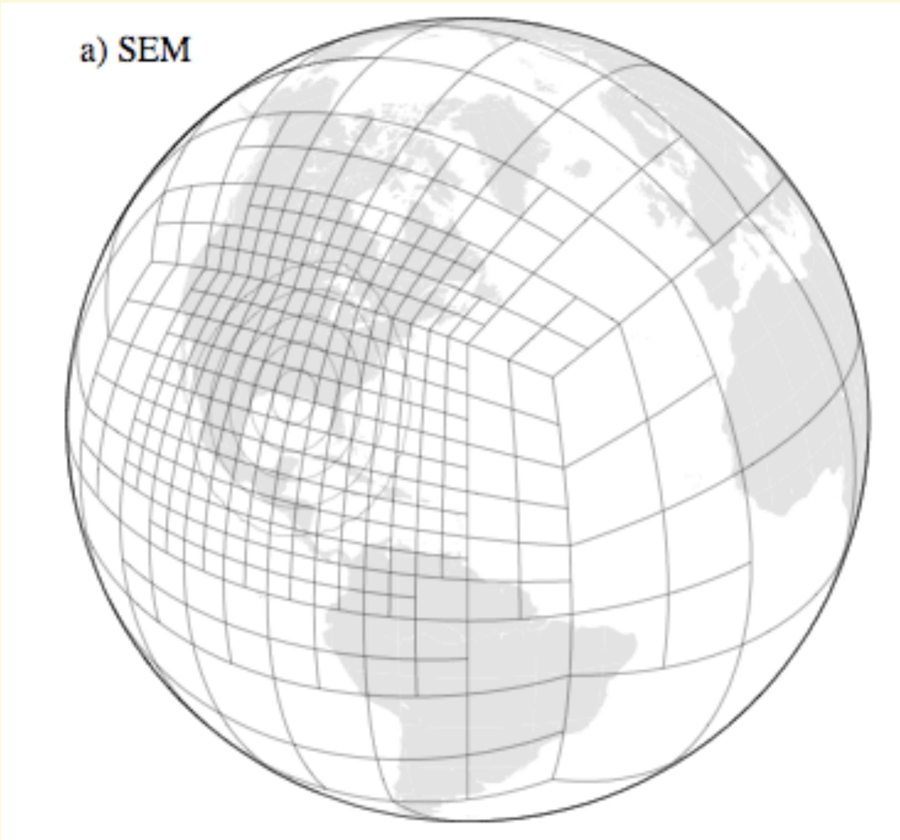
Icosahedral grid

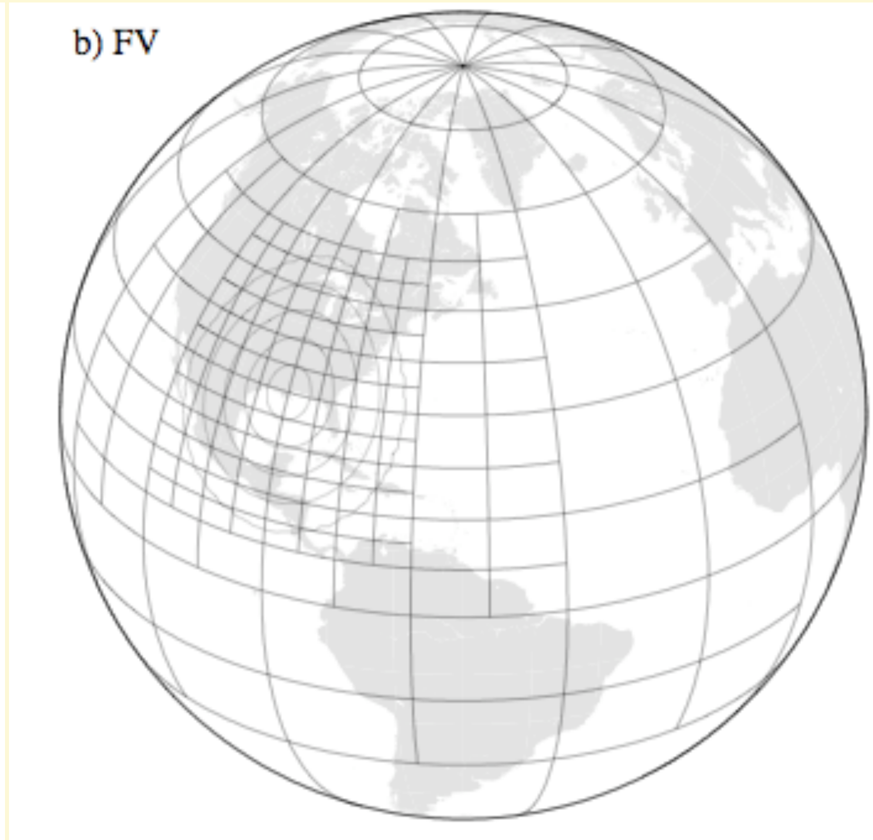
Latitude-Longitude Grid



- Popular choice in the past
- Meridians converge: requires polar filters or/and
  small time steps
- Orthogonal

Adaptive Mesh Refinements (AMR)



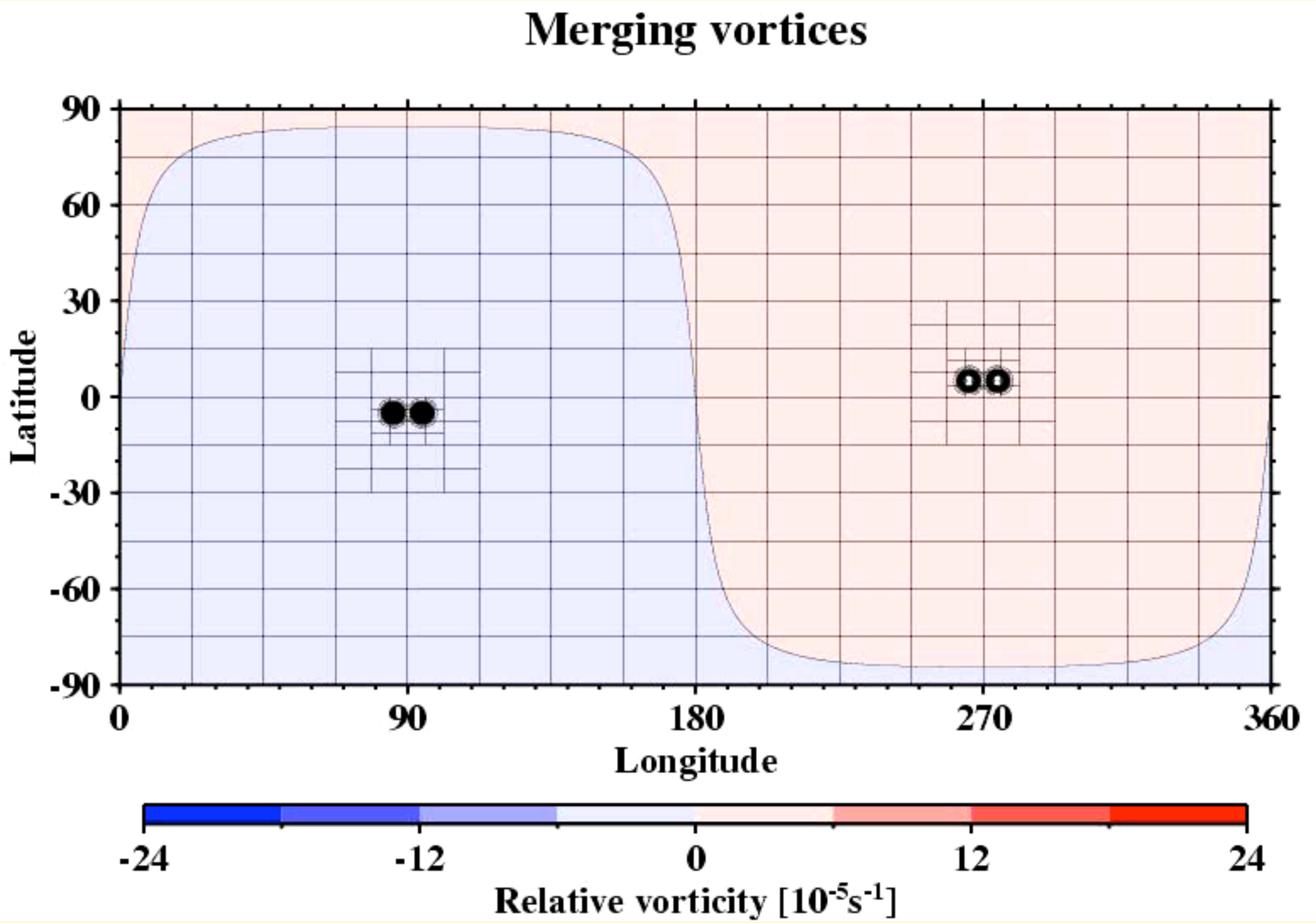


AMR on a cubed-sphere mesh

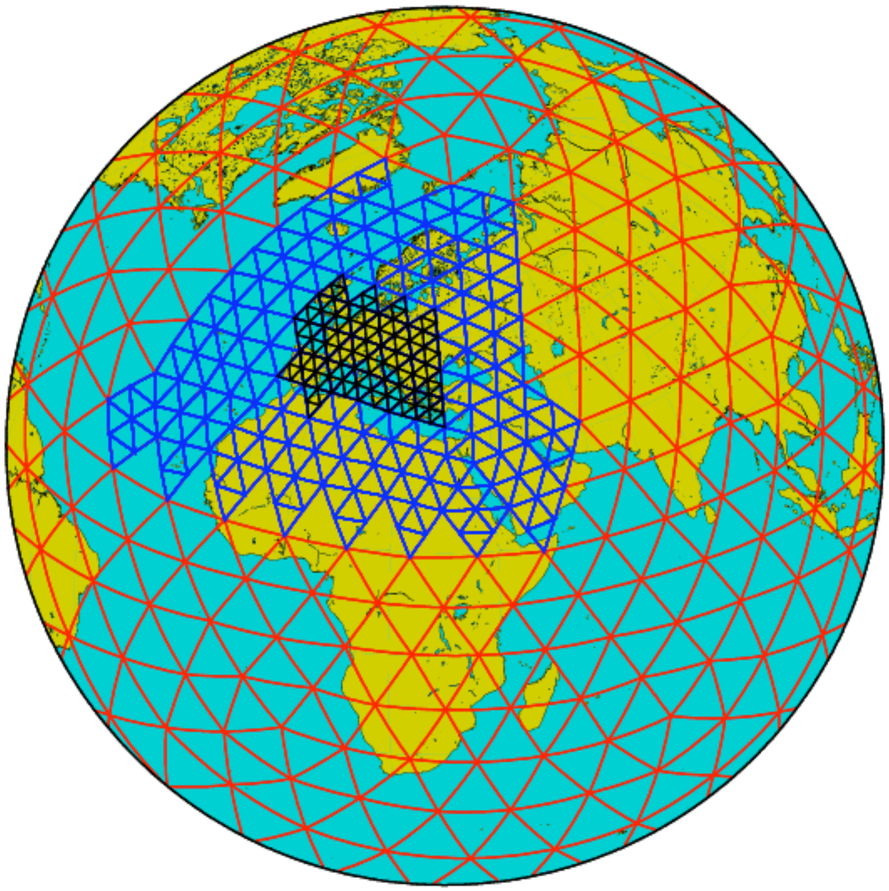
AMR on a latitude-longitude grid

St-Cyr, Jablonowski, et.al (MWR, 2008)

Example of an AMR simulation



Other non-uniform (nested) grids



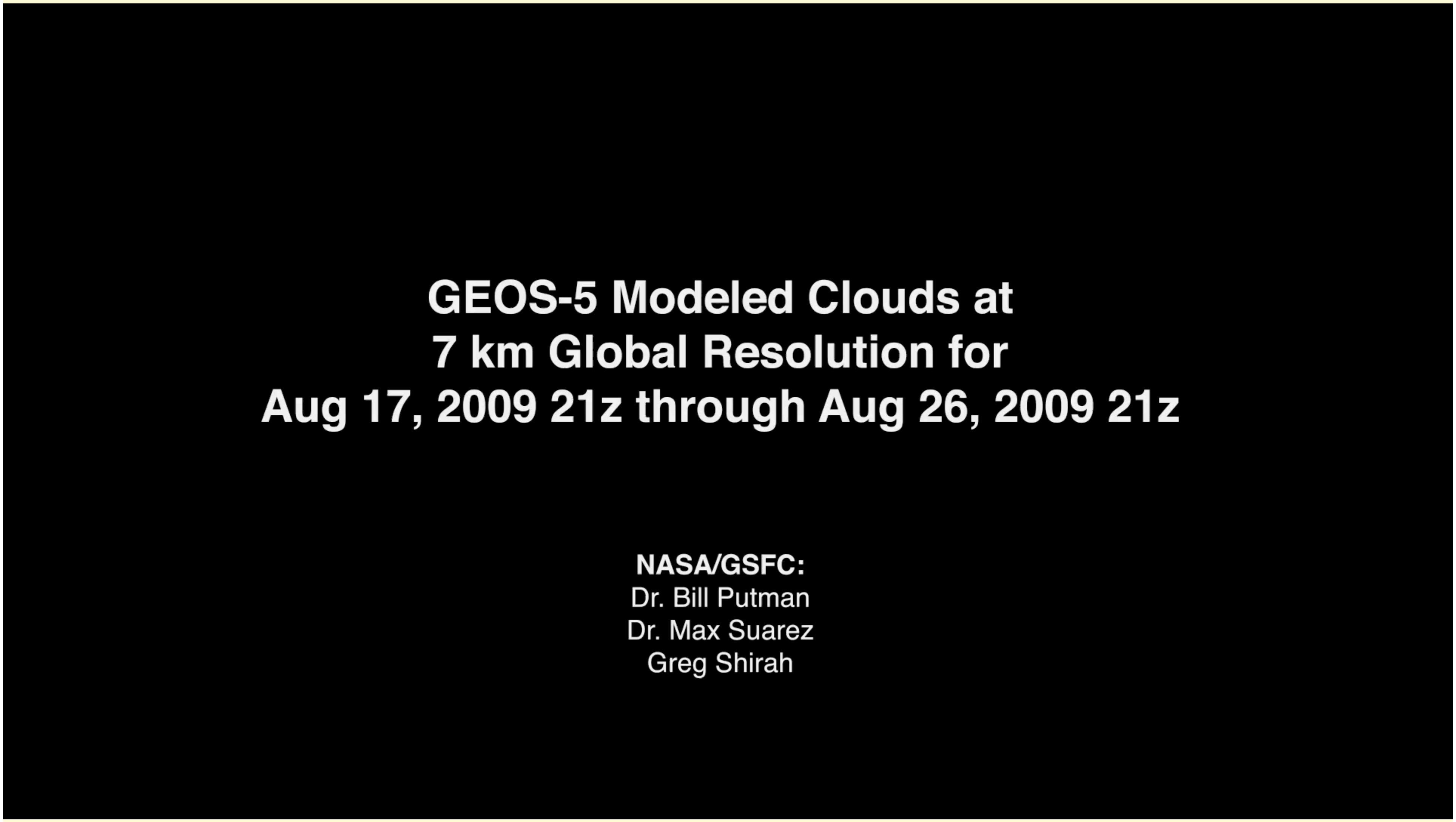
Source: DWD

Model ICON

Icosahedral grid with nested high-resolution regions

under development at the German Weather Service (DWD) and MPI, Hamburg, Germany

Why do we might want AMR grids in GCMs?

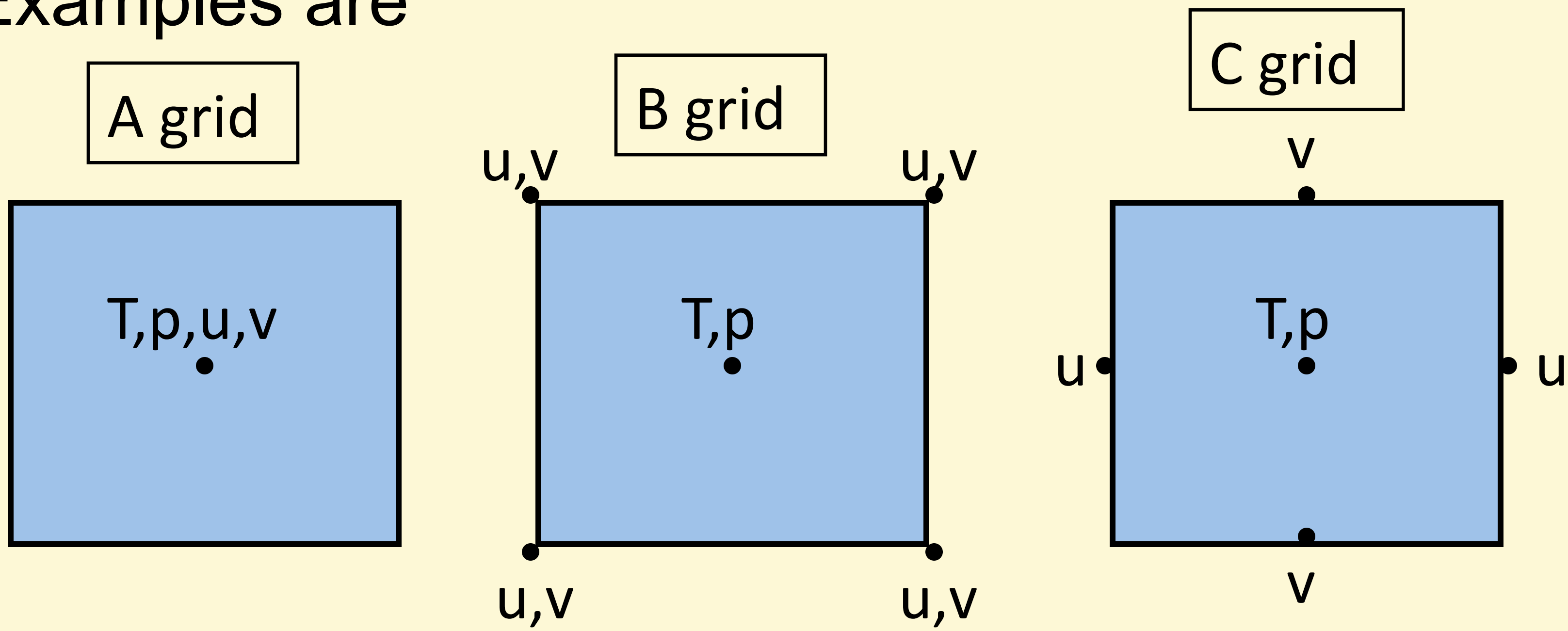


The pursuit of the ‘perfect’ positions of variables in the discrete system

- Having decided on the basic distribution of grid points, a choice has to be made as to how to arrange the different prognostic variables on the grid
- Most obvious choice of representing all variables at the same point has disadvantages
- There are many choices, called:
  A, B, C, D, E, Z or ZM grid (the first five are based on a classification by Akio Arakawa (UCLA))

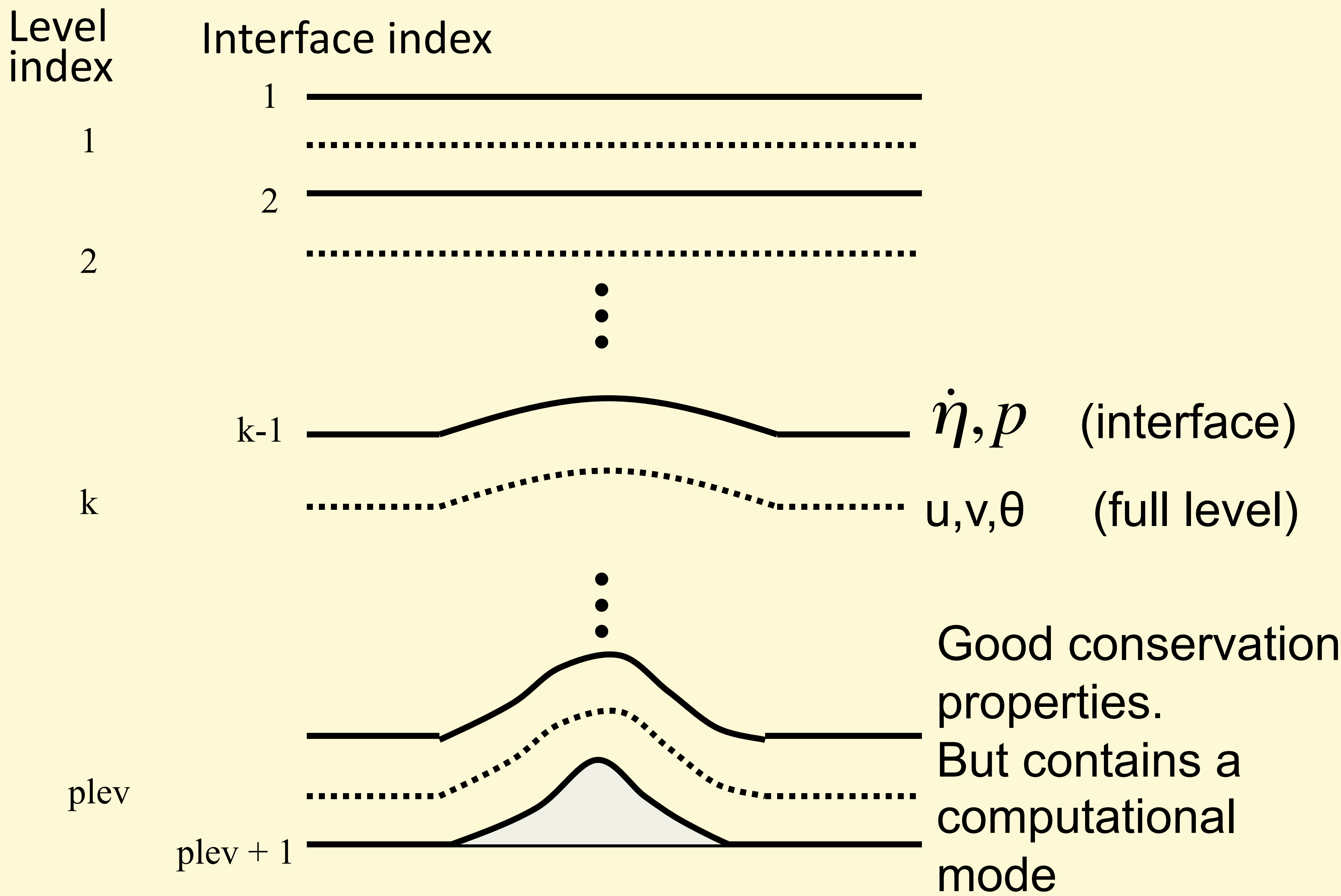
Example: Grid staggerings

- Many choices how to place scalars and vector winds in the computational grid
- Examples are

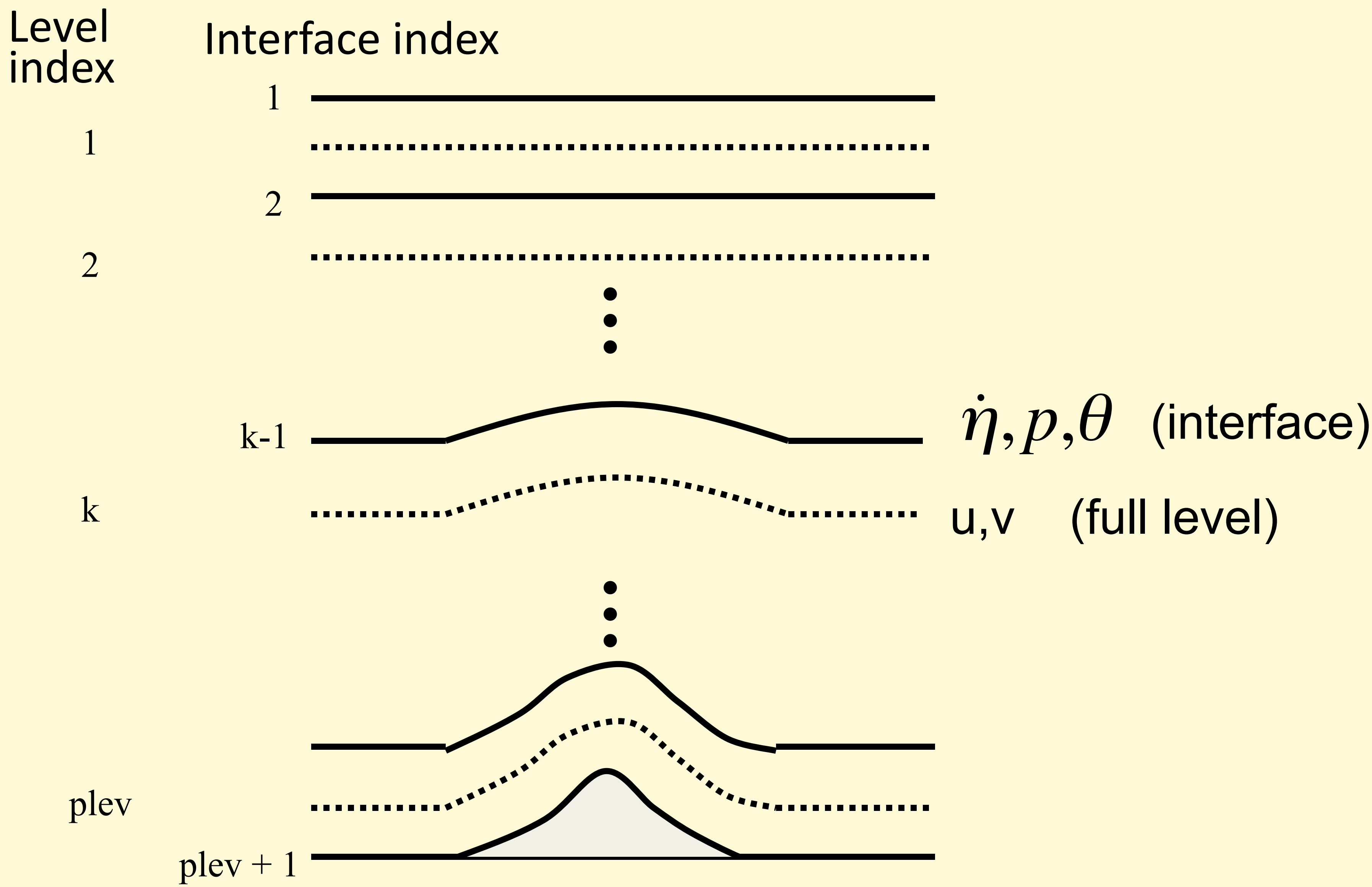


- Staggerings determine properties of the numerical schemes: dispersion and diffusion properties
- Additional staggering options in the vertical

Vertical grid staggerings: Lorenz



Vertical grid staggerings: Charney-Phillips



The pursuit of the ‘perfect’ numerical scheme

- We want: high order of accuracy, but computationally cheap method
- We need: discretizations in space and time
- Many space discretization philosophies:
  - Finite difference methods (FD)
  - Finite volume methods (FV)
  - Finite (spectral) element methods (FE, SE)
  - Spectral methods
- Sometimes different spatial methods are used in the horizontal and vertical directions

The pursuit of the ‘perfect’ numerical scheme

- Phase errors and damping should be small (often a compromise)
- Explicit scheme is ‘easy’ to program, but it will only be conditionally stable and so the choice of time step is limited
- Implicit schemes are absolutely stable; however at every time step a system of simultaneous equations has to be solved
- More than two time levels: additional computational modes and possibly separation of the solution at odd and even time steps. Higher storage (memory) requirements.

Discretizations in time: explicit versus implicit

- The earlier example used a so-called explicit time stepping scheme with two time levels n+1 and n

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

- How about rewriting this equation as an implicit scheme

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1}\right)$$

- It uses the (unknown) fluxes at the future time step, requires sophisticated numerical methods to solve (more expensive, e.g. iterative methods)
- Big advantage: increased numerical stability, allows longer time steps

Multi-level time discretizations

- We can increase the order of accuracy of a time stepping scheme by using **multiple time levels**, e.g. three time levels n+1, n, n-1
- Picture an equation like (describes Rayleigh friction)

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = -\tau u$$

- Possible time discretizations are:

$$\frac{u^{n+1} - u^n}{\Delta t} = -\tau u^n \quad \text{or} \quad \frac{u^{n+1} - u^n}{\Delta t} = -\tau u^{n+1} \quad \text{(explicit / implicit)}$$

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = -\tau u^n \qquad \text{(3 - step method, called Leapfrog method)}$$

Multi-level time discretizations

- Leapfrog is a popular choice in today's GCMs
- Uses 3 time levels and computes the forcing (here friction) at the center time with index n, forcing is applied over a time span of 2Δt

$$u^{n+1} = u^{n-1} - 2\Delta t \tau u^{n}$$

- 2nd-order accurate
- Unfortunately, the Leapfrog method can be numerically unstable (has a computational mode, separates odd and even time steps)
- But it then can be stabilized by applying a time filter that mimics diffusion in time (Asselin filter)

Time-splitting

- Sometimes is equations are separated into two parts (e.g. horizontal and vertical part) and solved independently (in a time-split fashion)
- Example: Consider the tracer conservation equation

$$\frac{\partial q}{\partial t} + \vec{v} \bullet \nabla q = 0$$

- Can be split like (using Leapfrog)

$$q^{*} = q^{n-1} + 2\Delta t \left(\vec{v}_h \bullet \nabla_h q\right)^n$$

$$q^{n+1} = q^{*} + 2\Delta t \left(w \frac{\partial q}{\partial z}\right)^n$$

- Allows to use different techniques for the horizontal (h) and vertical advection, introduces a splitting error

The flavor of FV spatial discretizations

- Finite volume discretization are based on an integrated version of the equations of motion
- Consider 2D example with h (=height of shallow water system), this is a conservation law:

$$\frac{\partial h}{\partial t} + \nabla \bullet \left(h\vec{v} \right) = 0$$

- Conservation equation can be integrated over spatial domain Ω with “volume” (here an area) A_Ω and time t

$$\int_{t_n}^{t_{n+1}} \int_\Omega \left(\frac{\partial}{\partial t} h \right) d\Omega \, dt + \int_\Omega \int_{t_n}^{t_{n+1}} \nabla \cdot (h \, \vec{v}) \, dt \, d\Omega = 0$$

The flavor of FV spatial discretizations

- Integration over spatial domain Ω with area/volume A_Ω and time t, can be rearranged:

$$\int_{t_n}^{t_{n+1}} \left(\frac{\partial}{\partial t}\bar{h}\right) dt + \frac{\Delta t}{A_\Omega} \int_\Omega \nabla \cdot \vec{F}\, d\Omega = 0$$

- Overbar denotes the spatial mean, $\vec{F}$ denotes time averaged fluxed across the interface of a volume
- Apply Gauss' divergence theorem to second term:

$$\int_{t_n}^{t_{n+1}} \left(\frac{\partial}{\partial t}\bar{h}\right) dt + \frac{\Delta t}{A_\Omega} \oint_{\partial\Omega} \vec{F} \cdot d\vec{n} = 0$$

Introduces line integral, $\vec{n}$ is the line segment vector normal to the boundary

The flavor of FV spatial discretizations

- Leads to discretized forecast equation:

$$\bar{h}^{n+1} = \bar{h}^{n} - \frac{\Delta t}{A_\Omega} \sum_{i=1}^{4} l_i \, \vec{F}_i \cdot \vec{n}_i$$

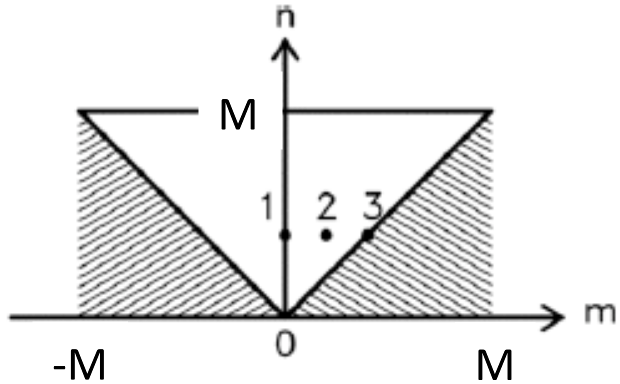
- Where l_i denotes the length of a line segment, $\vec{n}_i$ is the unit vector normal to the boundary 'i'
- The future time step is determined by the sum of all fluxes across the boundaries of a finite volume
- Order of accuracy determined by the fluxes $\vec{F}$, rely on subgrid-scale representations (constant, linear, quadratic, cubic) of transported variable (here h)
- Idea: express subgrid structure of h with polynomials

The flavor of spectral transform models

- Spectral transform methods on latitude-longitude grids have been very popular in the past
- Some GCM (ECHAM, ECMWF's weather model IFS) still use it
- Idea:
  - Use a model formulation with vorticity and divergence as prognostic variables
  - Use Fourier and Legendre transformation to transform/ represent the flow in spectral space
  - Solve the linear parts of the Eqs. in spectral space (exact)
  - Solve the nonlinear parts in grid point space
- Highly accurate, but suffers from Gibb's ringing, non-local discretization

The flavor of spectral transform models

Triangular wave number range m, n



Spectral representations of variable q:

$$q(\lambda,\varphi,t) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} q_n^m(t) Y_n^m(\lambda,\varphi)$$

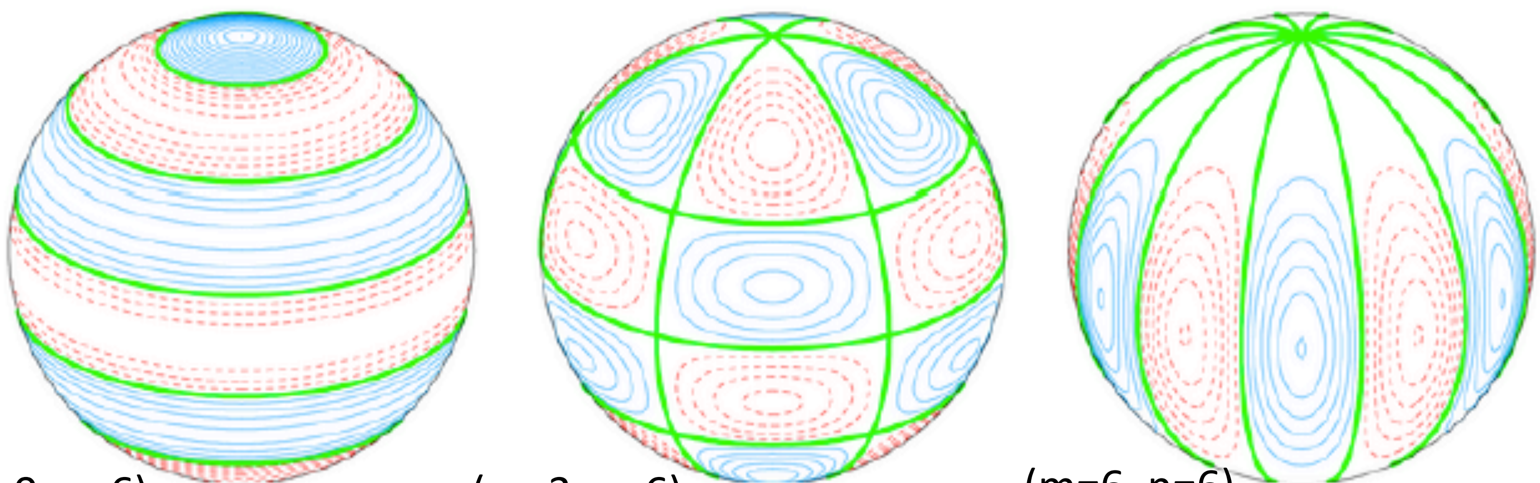
Spherical harmonics

$$Y_n^m(\lambda,\varphi) = P_n^m(\sin\varphi)\exp(im\lambda)$$

associated Legendre functions

Fourier modes

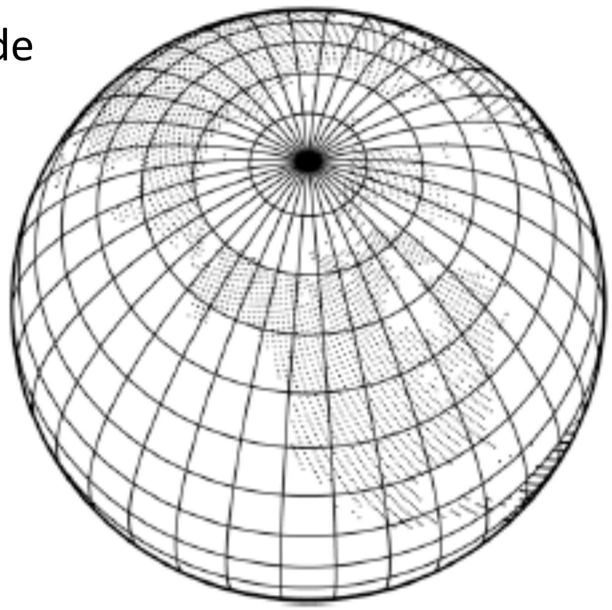
Structure of the global basis functions:



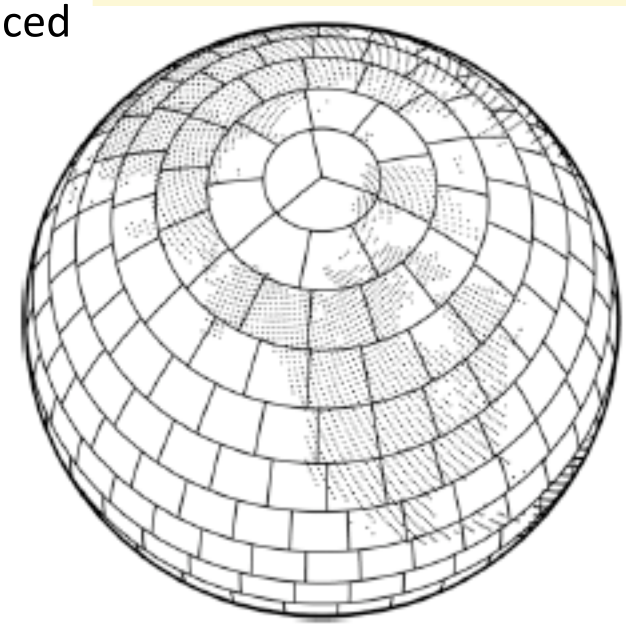
(m=0, n=6) (m=3, n=6) (m=6, n=6)

The flavor of spectral transform models

Latitude-Longitude grid



Reduced grid



- Triangular truncation T… with N(m)=M is unique, provides uniform spatial resolution over the entire surface of the sphere, e.g. M=42, 85, 170
- Eliminates pole problem in latitude-longitude grids
- Allows reduced grids with fewer points towards poles

Summary

Today we reviewed:

- Scientific and numerical aspects in the design process
- Different forms of the equations and variables
- Computational meshes, staggering options
- Characteristics (and accuracy) of numerical discretizations
- Suitable time-stepping schemes and vertical coordinates