Social Balance on Networks: The Dynamics of Friendship and Hatred

Basic question:
How do social networks evolve when both friendly and unfriendly relationships exist?

Partial answers: (Heider 1944, Cartwright & Harary 1956, Wasserman & Faust 1994)
Social balanced defined; balanced states on a complete graph must be either utopia or bipolar.

This work:
Endow a network with the simplest dynamics and investigate evolution of relationships.

Main result:
Dynamical phase transition between bipolarity and utopia.

related work: Kulakowski et al.
Socially Balanced States

Social Balance

\[
\begin{align*}
\text{a friend of my friend} & \quad \text{is my friend;} \\
\text{an enemy of my enemy} & \quad \text{is my friend;} \\
\text{a friend of my enemy} & \quad \text{is my enemy.} \\
\text{an enemy of my friend} & \quad \text{is my enemy.}
\end{align*}
\]
how does violence correlate with relations?

violence frequency
Local Triad Dynamics on Arbitrary Networks

(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad
2. Reverse a single link so that the triad becomes balanced
   
   \[
   \text{probability } p: \text{unfriendly } \rightarrow \text{friendly}; \quad \text{probability } 1-p: \text{friendly } \rightarrow \text{unfriendly}
   \]

Fundamental parameter \( p \):
   - \( p=1/3 \): flip a random link in the triad equiprobably
   - \( p>1/3 \): predisposition toward tranquility
   - \( p<1/3 \): predisposition toward hostility
Triad Evolution on the Complete Graph

Basic graph characteristics:

- \( N \) nodes
- \( \frac{N(N-1)}{2} \) links
- \( \frac{N(N-1)(N-2)}{6} \) triads

\( \rho = \) friendly link density

\( n_k = \) density of triads of type \( k \)

\( n_k^\pm = \) density of triads of type \( k \) attached to a \( \pm \) link

\[ \begin{align*}
N_0^+ &\quad n_0^+ \\
n_1^+ &\quad n_2^+ \\
n_1^- &\quad n_2^- \\
n_3^- &\quad \text{N-2 triads}
\end{align*} \]

positive link

negative link
Triad Evolution on the Complete Graph

\( n_k \) = density of triads of type \( k \)

\( n_k^\pm \) = density of triads of type \( k \) attached to a \( \pm \) link

\( \pi^+ = (1 - p) n_1 \) \hspace{1cm} \text{flip rate} + \rightarrow - \hspace{1cm} \triangle \xrightarrow{1-p} \triangle

\( \pi^- = p n_1 + n_3 \) \hspace{1cm} \text{flip rate} - \rightarrow + \hspace{1cm} \triangle \xrightarrow{p} \triangle \hspace{1cm} \triangle \xrightarrow{1} \triangle

Master equations:

\[
\begin{align*}
\frac{dn_0}{dt} &= \pi^- n_1^- - \pi^+ n_0^+, \\
\frac{dn_1}{dt} &= \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+, \\
\frac{dn_2}{dt} &= \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+, \\
\frac{dn_3}{dt} &= \pi^+ n_2^+ - \pi^- n_3^- .
\end{align*}
\]
Steady State Solution

\[
\begin{align*}
\frac{dn_0}{dt} &= \pi^- n_1^- - \pi^+ n_0^+ , \\
\frac{dn_1}{dt} &= \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+ , \\
\frac{dn_2}{dt} &= \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+ , \\
\frac{dn_3}{dt} &= \pi^+ n_2^+ - \pi^- n_3^- .
\end{align*}
\]

impose \( \dot{n}_i \) and \( \pi^+ = \pi^- \\
gives \( n_k^+ = n_{k+1}^- \)

finally, use \( n_k^\pm = \left\{ \begin{array}{l}
\frac{(3-k)n_k}{3n_0+2n_1+n_2} \\
\frac{kn_k}{n_1+2n_2+3n_3}
\end{array} \right. \\
n_j = \binom{3}{j} \rho_\infty^{3-j} (1 - \rho_\infty)^j,

\rho_\infty = \left\{ \begin{array}{l}
1/\left[\sqrt{3(1-2p)} + 1\right] & \quad p \leq 1/2; \\
1 & \quad p \geq 1/2
\end{array} \right. 

Steady State Triad Densities

steady state only for $p \leq \frac{1}{2}$
The Evolving State

rate equation for the friendly link density:

\[
\frac{d\rho}{dt} = 3\rho^2 (1 - \rho) [p - (1 - p)] + (1 - \rho)^3
\]

\[
= 3(2p - 1)\rho^2 (1 - \rho) + (1 - \rho)^3
\]

\[
\rho(t) \sim \begin{cases} 
\rho_\infty + Ae^{-Ct} & p < 1/2; \\
1 - \frac{1 - \rho_0}{\sqrt{1 + 2(1 - \rho_0)^2t}} & p = 1/2; \\
1 - e^{-3(2p-1)t} & p > 1/2.
\end{cases}
\]

rapid onset of frustration
slow relaxation to utopia
rapid attainment of utopia
Fate of a Finite Society

\( p < 1/2: \) effective random walk picture

\[ D \propto N^2 \]

\[ v \propto N \]

\[ \pm N \quad \rightarrow \quad T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2} \]

\[ p > 1/2: \] inversion of the rate equation

\[ u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim \frac{\ln N}{2p-1} \]

\( u=1-\rho, \) the unfriendly link density
p = 1/2

naive rate equation estimate:

\[ u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \quad \rightarrow \quad T_N \sim N^4 \]

incorporating fluctuations as balance is approached:

\[ U = Lu + \sqrt{L} \eta \sim \frac{L}{\sqrt{t}} + \sqrt{L} t^{1/4} \]

equating the 2 terms in U:

\[ T_N \sim L^{2/3} \sim N^{4/3} \]
Simulations for a Finite Society

\[ p < \frac{1}{2}, \quad T_N \sim e^{N^2} \]

\[ p = \frac{1}{2}, \quad T_N \sim N^{4/3} \]

\[ p > \frac{1}{2}, \quad T_N \sim \frac{\ln N}{2p - 1} \]
Constrained (Socially Aware) Triad Dynamics

1. Pick a random imbalanced (frustrated) triad

2. Reverse a random link \((p=\frac{1}{3})\) to eliminate a frustrated triad
   \textit{only if the total number of frustrated triads does not increase}

\textbf{Outcome:} Quick approach to a final static state
\textbf{Typically:} \(T_N \sim \ln N\)
Final Clique Sizes

\[ \langle \delta^2 \rangle \]

\[ \delta = \frac{C_1 - C_2}{N} \]

balance: two equal size cliques

utopia \approx \frac{2}{3}

\[ N = 256, 512, 1024, 2048 \]
Origin of the Balance/Utopia Transition

First consider evolution of an uncorrelated network:

\[
\begin{align*}
\rho^2 \\
2\rho(1-\rho) \\
(1-\rho)^2
\end{align*}
\]

we need:

\[
\begin{align*}
\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} & > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}, \text{ with } \vec{n}_+ = [\rho^2, 2\rho(1-\rho), (1-\rho)^2, 0] \\
\rightarrow 1 - 4\rho(1 - \rho) < 0, \text{ impossible, so + links never flip}
\end{align*}
\]
for $- \rightarrow +$

we need:

\[ n_1^- + n_3^- > n_0^- + n_2^-, \quad \text{with } \vec{n}_- = [0, \rho^2, 2\rho(1-\rho), (1-\rho)^2] \]

\[ \rightarrow 1 - 4\rho(1-\rho) > 0, \quad \text{valid when } \rho \neq 1/2 \]

**Conclusion:** only negative links flip, except when $\rho \rightarrow \frac{1}{2}$

flow diagram for $\rho$: 

```
0 -> 1/2 -> 1
```
Instability near $\rho = \frac{1}{2}$

intraclique relationship evolution

$\rho_1 = \frac{1}{2}^+$
$\rho_2 = \frac{1}{2}^+$

$\beta = \frac{1}{2}^-$

for $\rightarrow +$, we need:

$n^-_1 + n^-_3 > n^-_0 + n^-_2$, with $\vec{n}^- = \begin{cases} [0, \rho^2_i, 2\rho_i(1-\rho_i), (1-\rho_i)^2] & \text{intraclique} \\ [0, \beta^2, 2\beta(1-\beta), (1-\beta)^2] & \text{interclique} \end{cases}$

$\rightarrow C_1[1 - 4\rho_i(1 - \rho_i)] + C_2[1 - 4\beta(1 - \beta)] > 0$, always true

negative intraclique links disappear

increased cohesiveness within cliques
interclique relationship evolution

\[ \rho_1 = \frac{1^+}{2} \quad \rho_2 = \frac{1^+}{2} \]
\[ \beta = \frac{1^-}{2} \]

for \( + \rightarrow - \), we need:

\[ n_1^+ + n_3^+ > n_0^+ + n_2^+ \]

frustrated \quad unfrustrated

\[ \vec{n}_+ = [\beta \rho_i, \beta(1 - \rho_i) + \rho_i(1 - \beta), (1 - \beta)(1 - \rho_i), 0] \]

\[ \rightarrow [C_1(2\rho_1 - 1) + C_2(2\rho_2 - 1)](1 - 2\beta) > 0, \quad \text{true if } \rho_1, \rho_2 > \frac{1}{2}, \beta < \frac{1}{2} \]

positive interclique links disappear

increased emnity between cliques
A Historical Lesson

3 Emperor’s League 1872-81
Triple Alliance 1882
German-Russian Lapse 1890

French-Russian Alliance 1891-94
Entente Cordiale 1904
British-Russian Alliance 1907
Long Beach Gang Lesson

Gang relations

violence frequency

Less likely to attack enemy’s friend
More likely to attack friend’s enemy

love
hate

Low incidence
High incidence
Summary & Outlook

If we can’t all love each other → social balance

Local triad dynamics:
finite network: social balance, with the time until balance strongly dependent on $p$

infinite network: phase transition between utopia and social balance at $p=\frac{1}{2}$

Global triad dynamics ($p=\frac{1}{3}$):
jammed states possible but never occur

infinite network: two cliques always emerge, with utopia when $\rho_0 \approx \frac{2}{3}$ (rough argument gives $\rho_0 = \frac{1}{2}$)

Open questions:
incomplete graphs, indifference, continuous interactions

allow → Machiavellian society
asymmetric relations
gang control?