Social Balance on Networks: The Dynamics of Friendship and Hatred

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Crime Hotspots: Behavioral, Computational & Mathematical Models, IPAM, 2007

Basic question:

How do social networks evolve when both friendly and unfriendly relationships exist?

Partial answers: *(Heider 1944, Cartwright & Harary 1956, Wasserman & Faust 1994)*

Social balanced defined; balanced states on a complete graph must be either utopia or bipolar.

This work:

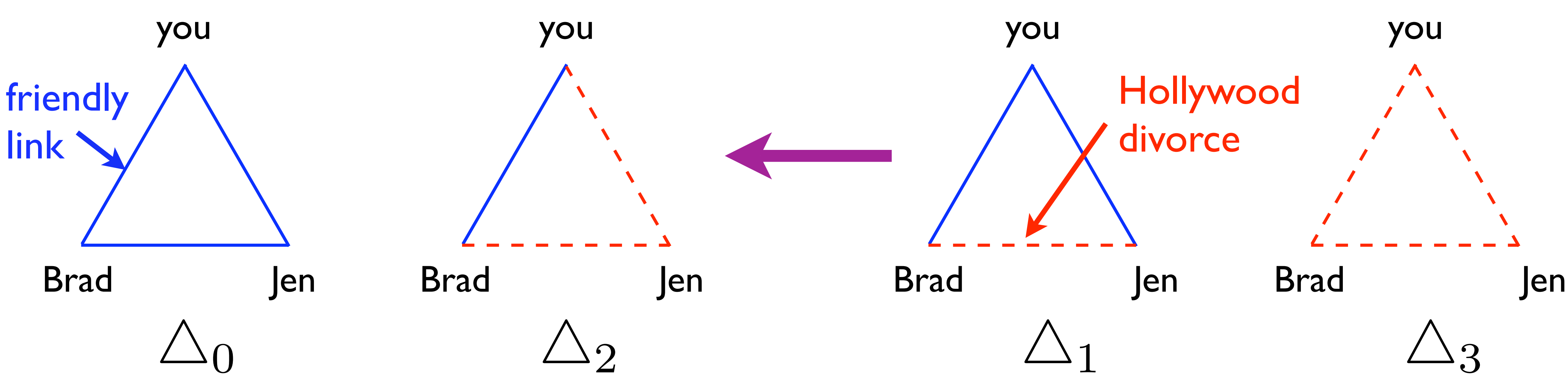
Endow a network with the simplest dynamics and investigate evolution of relationships.

related work: Kulakowsi et al.

Main result:

Dynamical phase transition between bipolarity and utopia.

Socially Balanced States



unfrustrated/balanced

frustrated/imbalanced

Social Balance

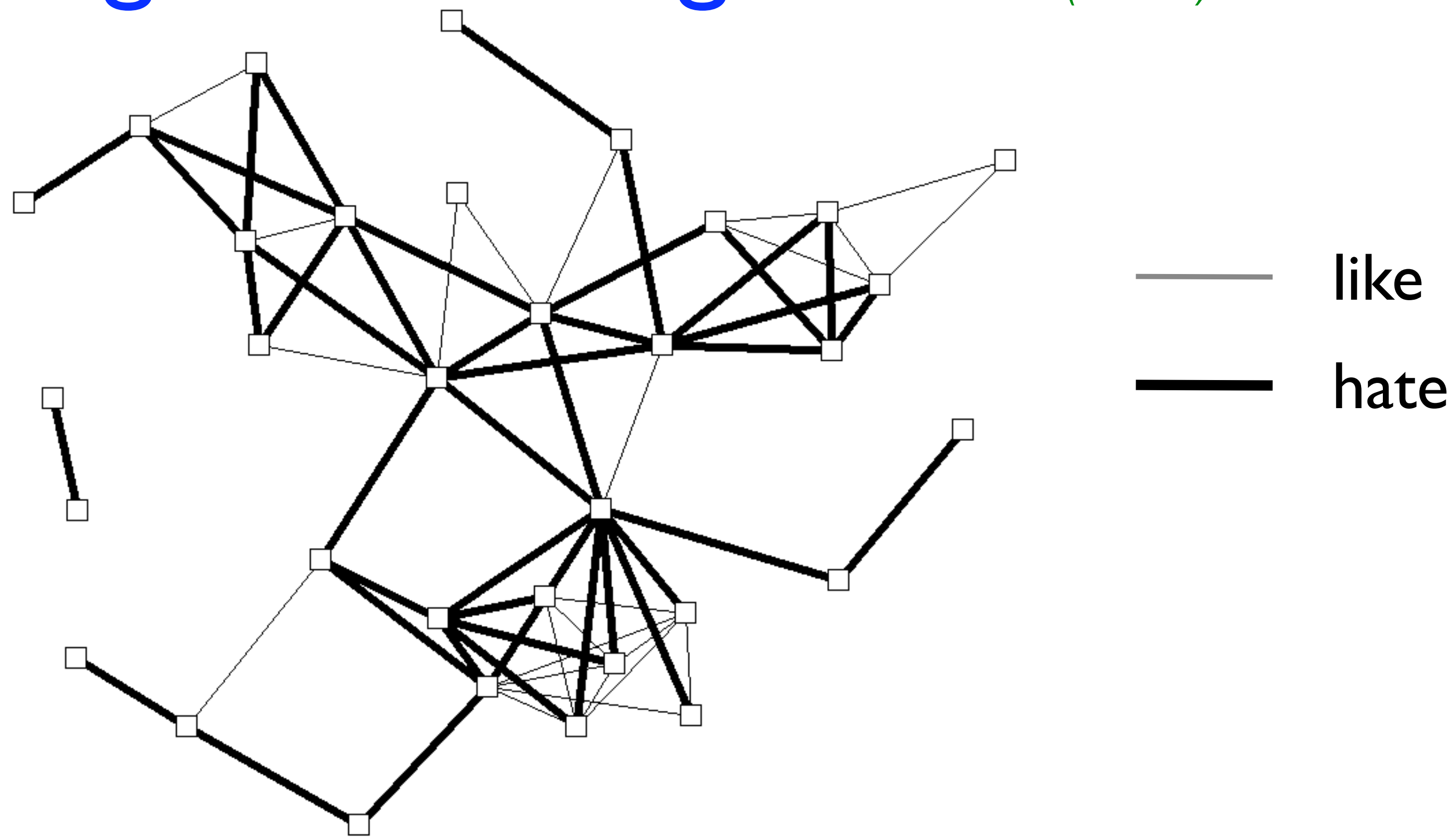
a friend of my friend
an enemy of my enemy } *is my friend;*

a friend of my enemy
an enemy of my friend } *is my enemy.*

Long Beach Gangs

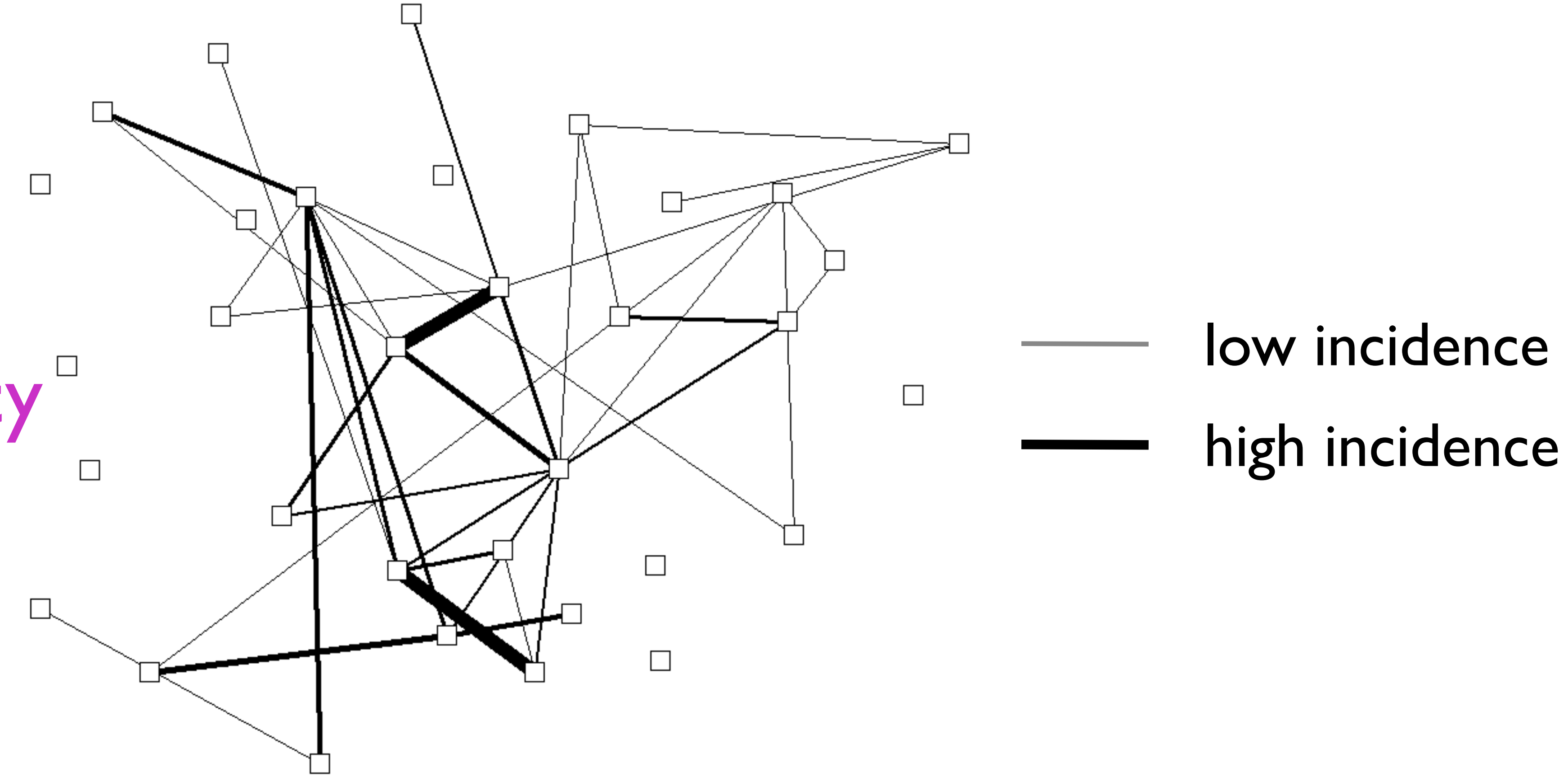
Nakamura, Tita, & Krackhardt (2007)

gang relations



how does violence correlate with relations?

violence frequency

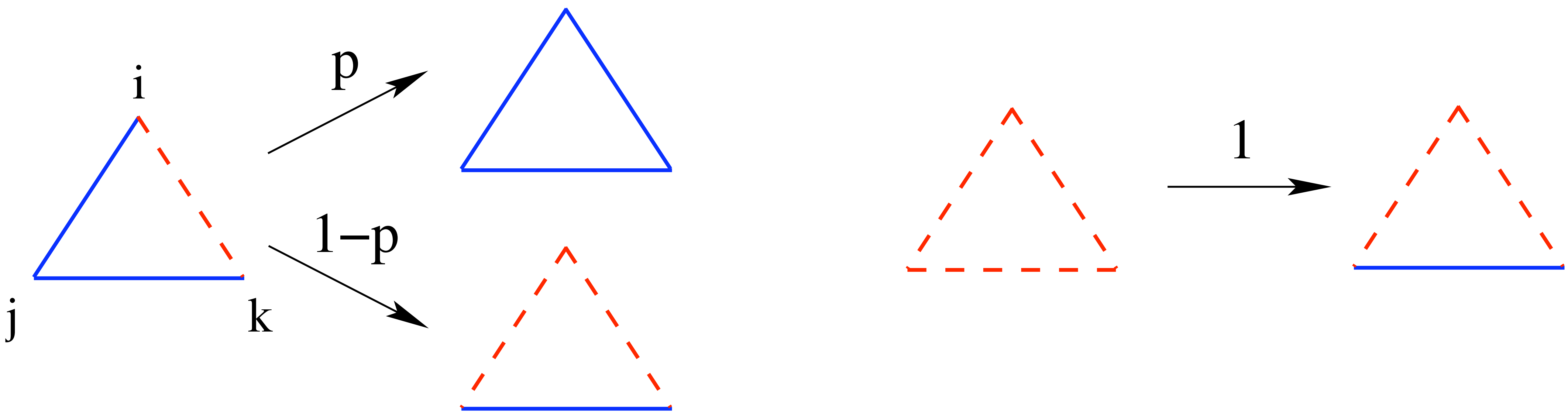


Local Triad Dynamics on Arbitrary Networks

(social graces of the clueless)

1. Pick a random imbalanced (frustrated) triad
2. Reverse a single link so that the triad becomes balanced

   probability p: unfriendly → friendly; probability 1-p: friendly → unfriendly



Fundamental parameter p:

- $p=1/3$: flip a random link in the triad equiprobably
- $p>1/3$: predisposition toward tranquility
- $p<1/3$: predisposition toward hostility

Triad Evolution on the Complete Graph

Basic graph characteristics:

N nodes

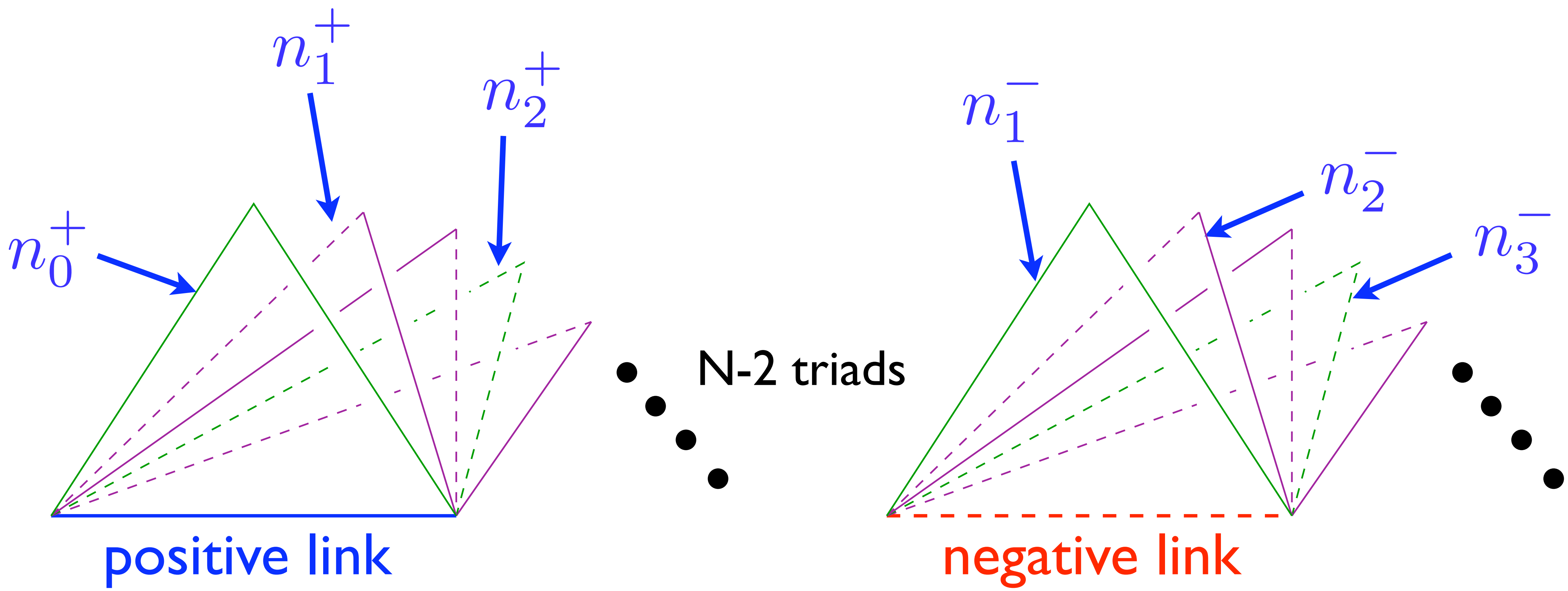
$\frac{N(N-1)}{2}$ links

$\frac{N(N-1)(N-2)}{6}$ triads

$\rho =$ friendly link density

$n_k =$ density of triads of type k

$n_k^{\pm} =$ density of triads of type k attached to a $\pm$ link



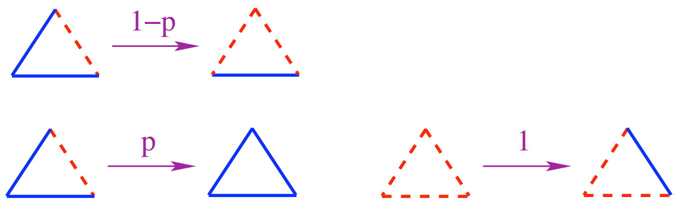
Triad Evolution on the Complete Graph

n_k = density of triads of type k

$n_k^{\pm}$ = density of triads of type k attached to a $\pm$ link

$\pi^+ = (1-p)\, n_1$ flip rate $+ \rightarrow -$

$\pi^- = p\, n_1 + n_3$ flip rate $- \rightarrow +$



Master equations:

$\triangle_1 \rightarrow \triangle_0$ $\triangle_0 \rightarrow \triangle_1$

$$
\begin{aligned}
\frac{dn_0}{dt} &= \pi^- n_1^- - \pi^+ n_0^+ \,, \\
\frac{dn_1}{dt} &= \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+ \,, \\
\frac{dn_2}{dt} &= \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+ \,, \\
\frac{dn_3}{dt} &= \pi^+ n_2^+ - \pi^- n_3^- \,.
\end{aligned}
$$

Steady State Solution

$$\frac{dn_0}{dt} = \pi^- n_1^- - \pi^+ n_0^+ ,$$
$$\frac{dn_1}{dt} = \pi^+ n_0^+ + \pi^- n_2^- - \pi^- n_1^- - \pi^+ n_1^+ ,$$
$$\frac{dn_2}{dt} = \pi^+ n_1^+ + \pi^- n_3^- - \pi^- n_2^- - \pi^+ n_2^+ ,$$
$$\frac{dn_3}{dt} = \pi^+ n_2^+ - \pi^- n_3^- .$$

impose $\dot{n}_i$ and $\pi^+ = \pi^-$

gives $n_k^+ = n_{k+1}^-$

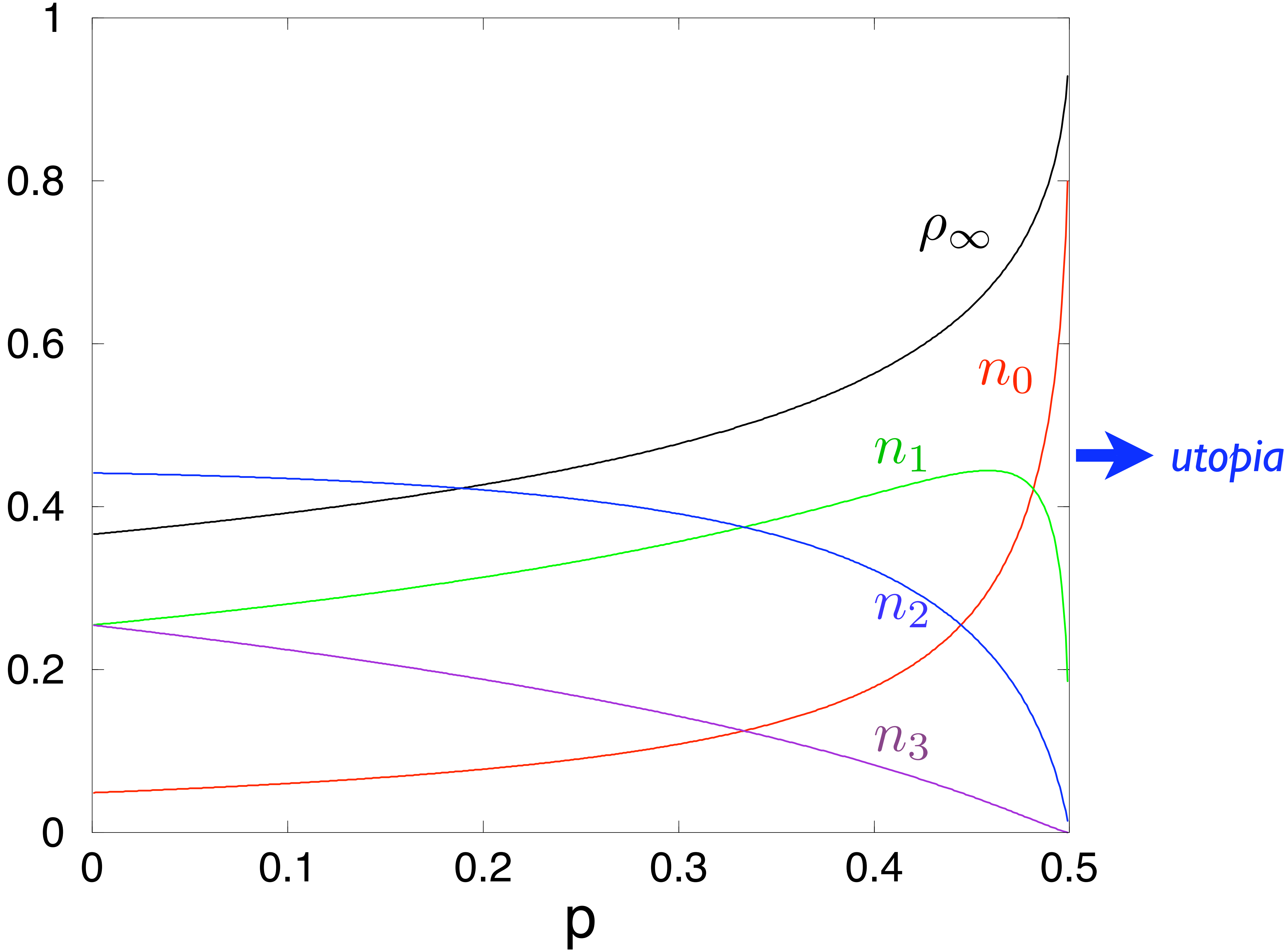
finally, use $n_k^\pm = \begin{cases} \frac{(3-k)n_k}{3n_0+2n_1+n_2} \\ \frac{kn_k}{n_1+2n_2+3n_3} \end{cases}$

$$n_j = \binom{3}{j} \rho_\infty^{3-j} (1 - \rho_\infty)^j ,$$

$$\rho_\infty = \begin{cases} 1/[\sqrt{3(1-2p)} + 1] & p \le 1/2; \\ 1 & p \ge 1/2 \end{cases}$$

Steady State Triad Densities

steady state only for p≤½



The Evolving State

rate equation for the friendly link density:

$- \to +$ in $\triangle_1$ $+ \to -$ in $\triangle_1$ $- \to +$ in $\triangle_3$

$$\begin{aligned}\frac{d\rho}{dt} &= 3\rho^2(1-\rho)[p-(1-p)] + (1-\rho)^3 \\ &= 3(2p-1)\rho^2(1-\rho) + (1-\rho)^3\end{aligned}$$

$$\rho(t) \sim \begin{cases} \rho_\infty + Ae^{-Ct} & p < 1/2; \\ 1 - \dfrac{1-\rho_0}{\sqrt{1+2(1-\rho_0)^2 t}} & p = 1/2; \\ 1 - e^{-3(2p-1)t} & p > 1/2. \end{cases}$$

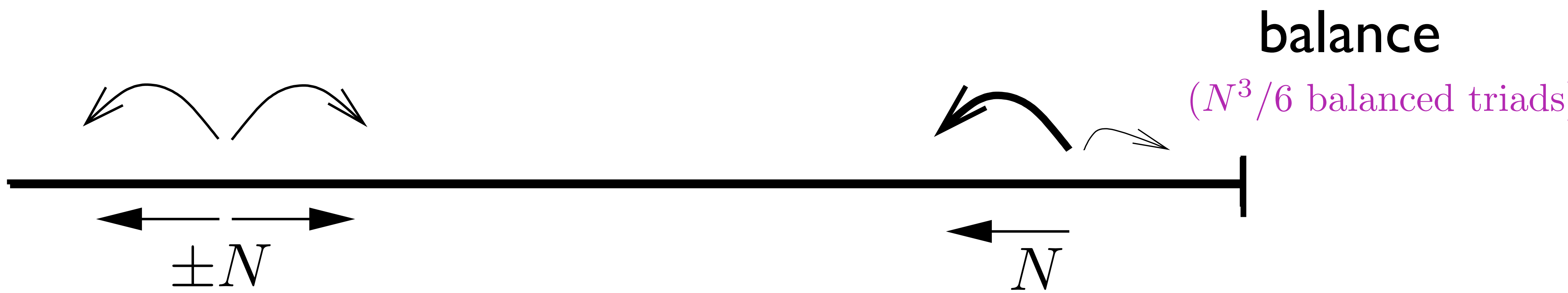
rapid onset of frustration ($p < 1/2$)

slow relaxation to utopia ($p = 1/2$)

rapid attainment of utopia ($p > 1/2$)

Fate of a Finite Society

p<1/2: effective random walk picture



$D \propto N^2$ $\qquad$ $v \propto N$

$$\rightarrow T_N \sim e^{v\mathcal{L}_N/D} \sim e^{N^2}$$

p>1/2: inversion of the rate equation

$$u \sim e^{-3(2p-1)t} \approx N^{-2} \quad \rightarrow \quad T_N \sim \frac{\ln N}{2p-1}$$

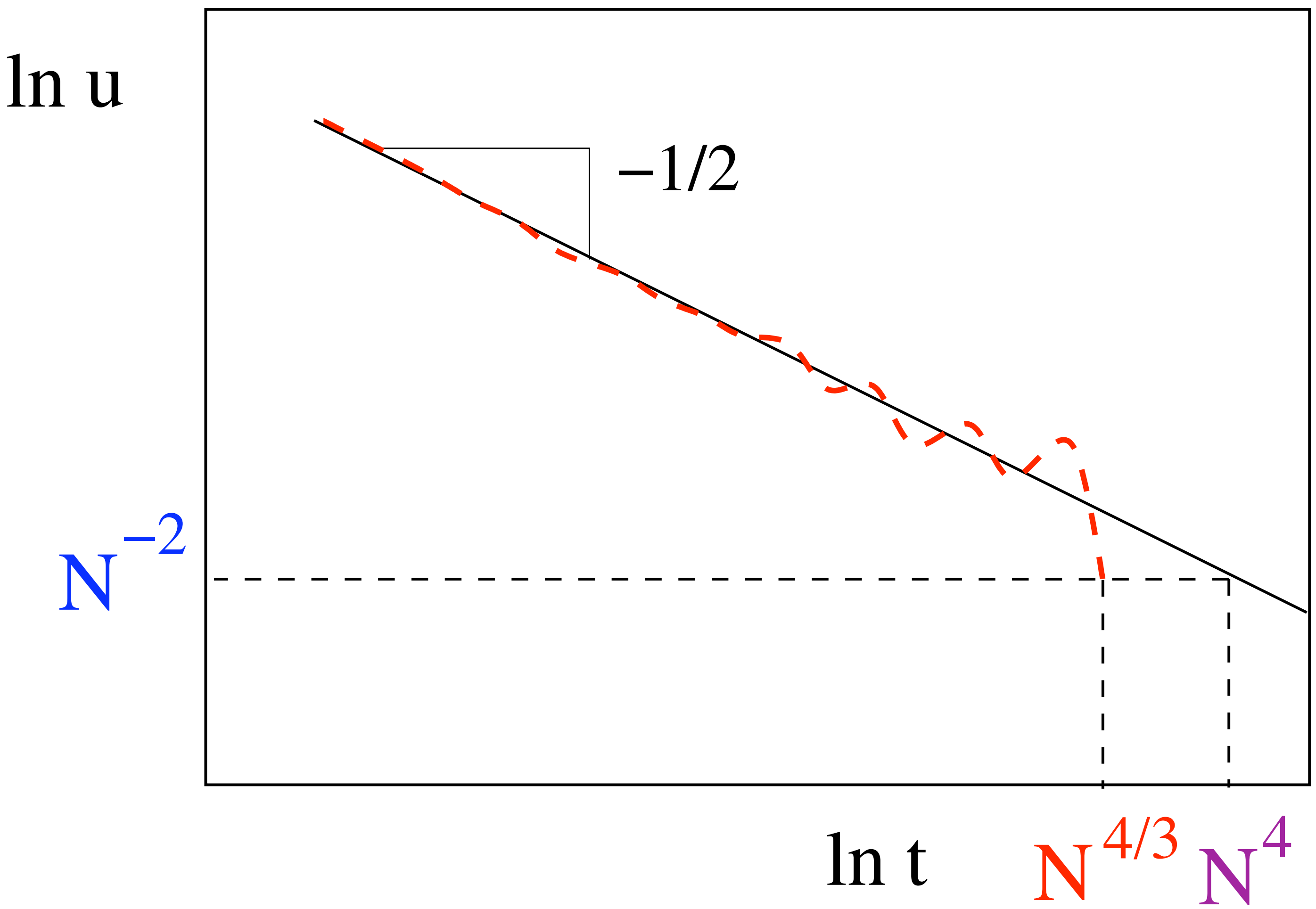
u=1-ρ, the unfriendly link density

p=1/2

naive rate equation estimate:

$$u \equiv 1 - \rho \propto t^{-1/2} \approx N^{-2} \qquad \to \; T_N \sim N^4$$

incorporating fluctuations as balance is approached:

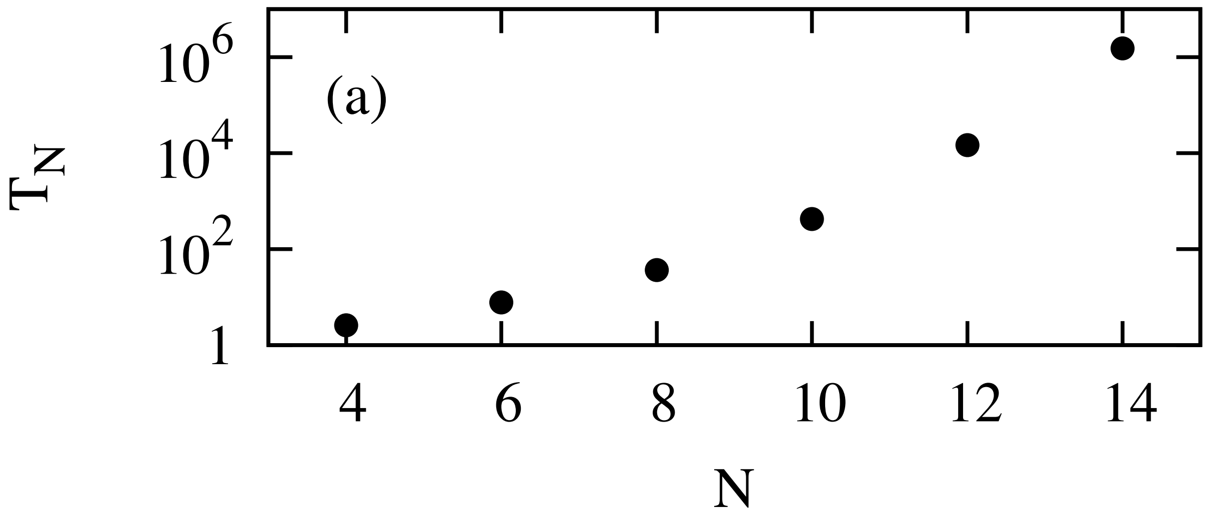


$$\begin{aligned} U &= Lu + \sqrt{L}\,\eta \\ &\sim \frac{L}{\sqrt{t}} + \sqrt{L}\,t^{1/4} \end{aligned}$$

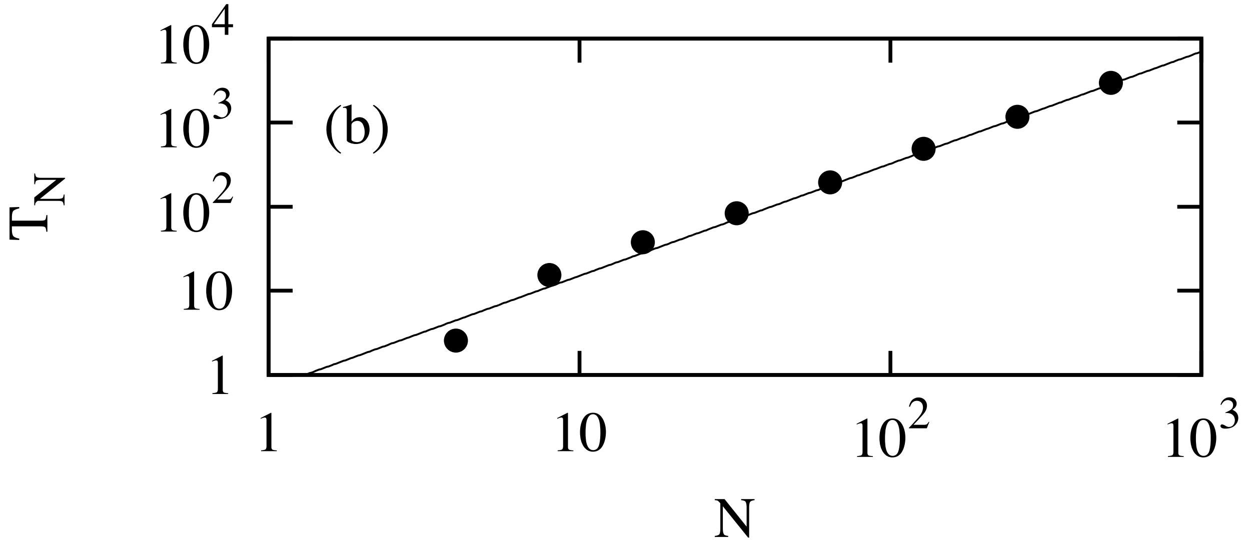
equating the 2 terms in U:

$$T_N \sim L^{2/3} \sim N^{4/3}$$

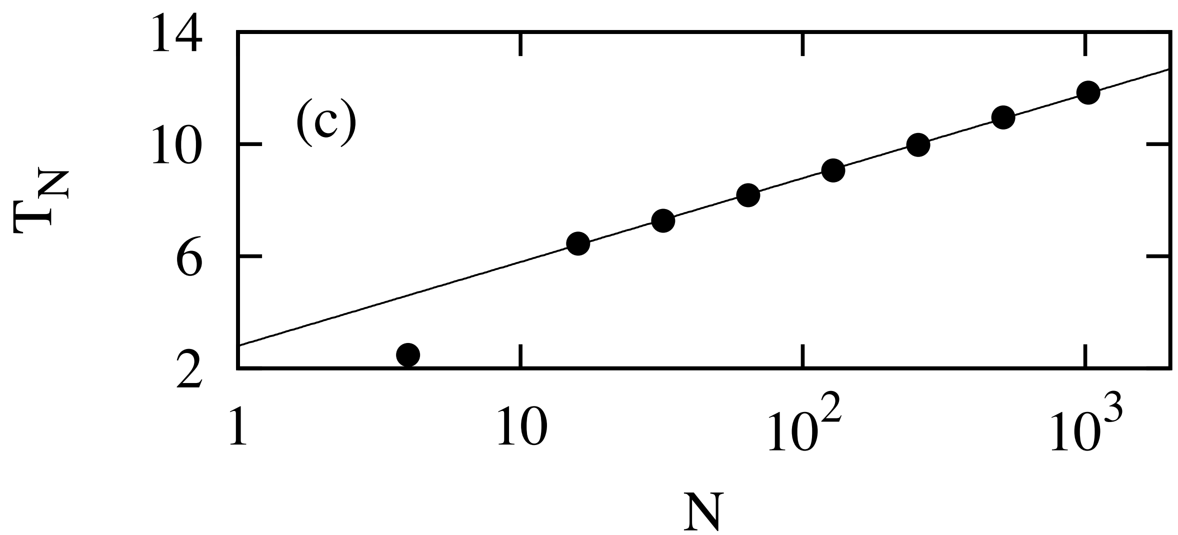
Simulations for a Finite Society



$$p < \frac{1}{2}, \qquad T_N \sim e^{N^2}$$



$$p = \frac{1}{2}, \qquad T_N \sim N^{4/3}$$



$$p > \frac{1}{2}, \qquad T_N \sim \frac{\ln N}{2p - 1}$$

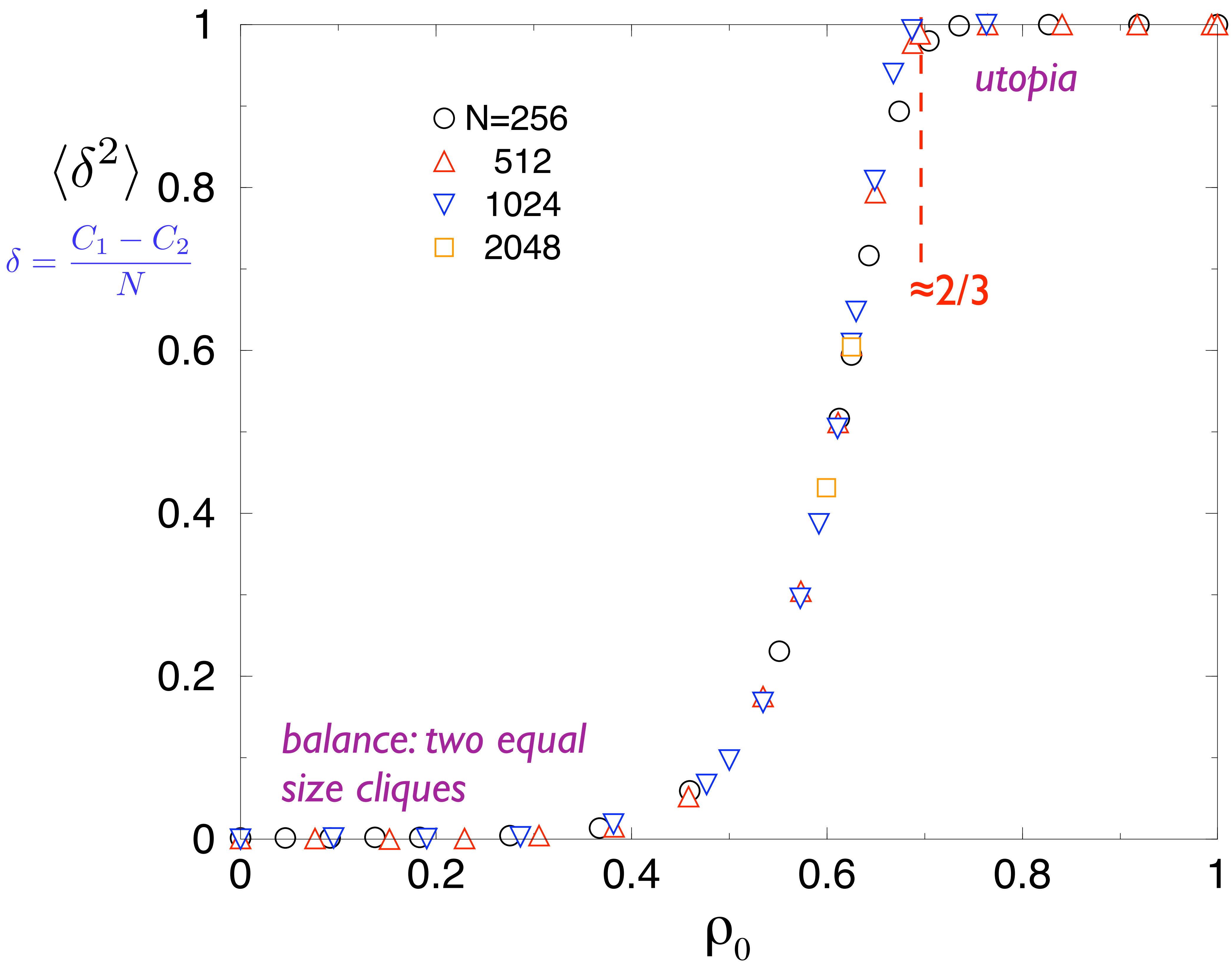
Constrained (Socially Aware) Triad Dynamics

1. Pick a random imbalanced (frustrated) triad

2. Reverse a random link (p=⅓) to eliminate a frustrated triad
   only if the total number of frustrated triads does not increase

Outcome: Quick approach to a final static state
Typically: $T_N \sim \ln N$

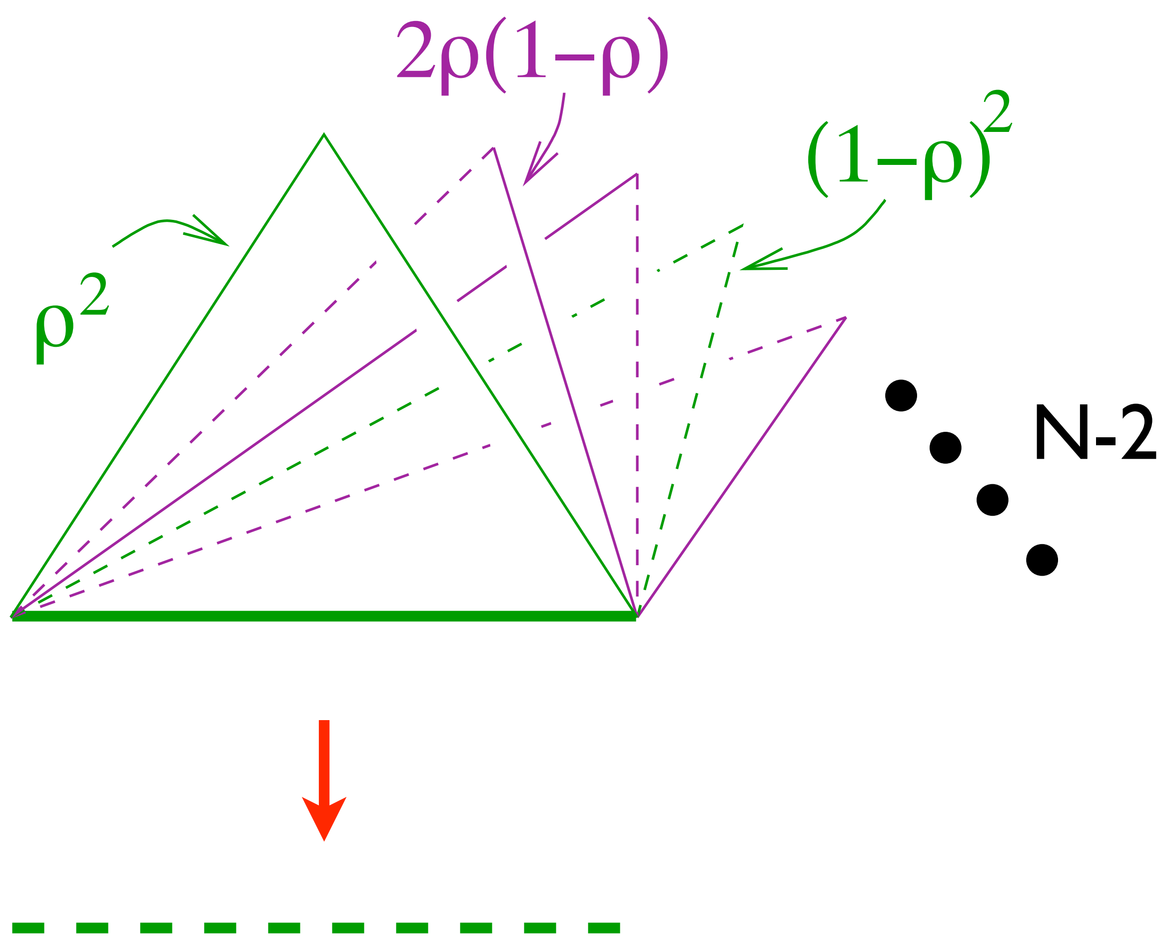
Final Clique Sizes



Origin of the Balance/Utopia Transition

First consider evolution of an uncorrelated network:

for + → −

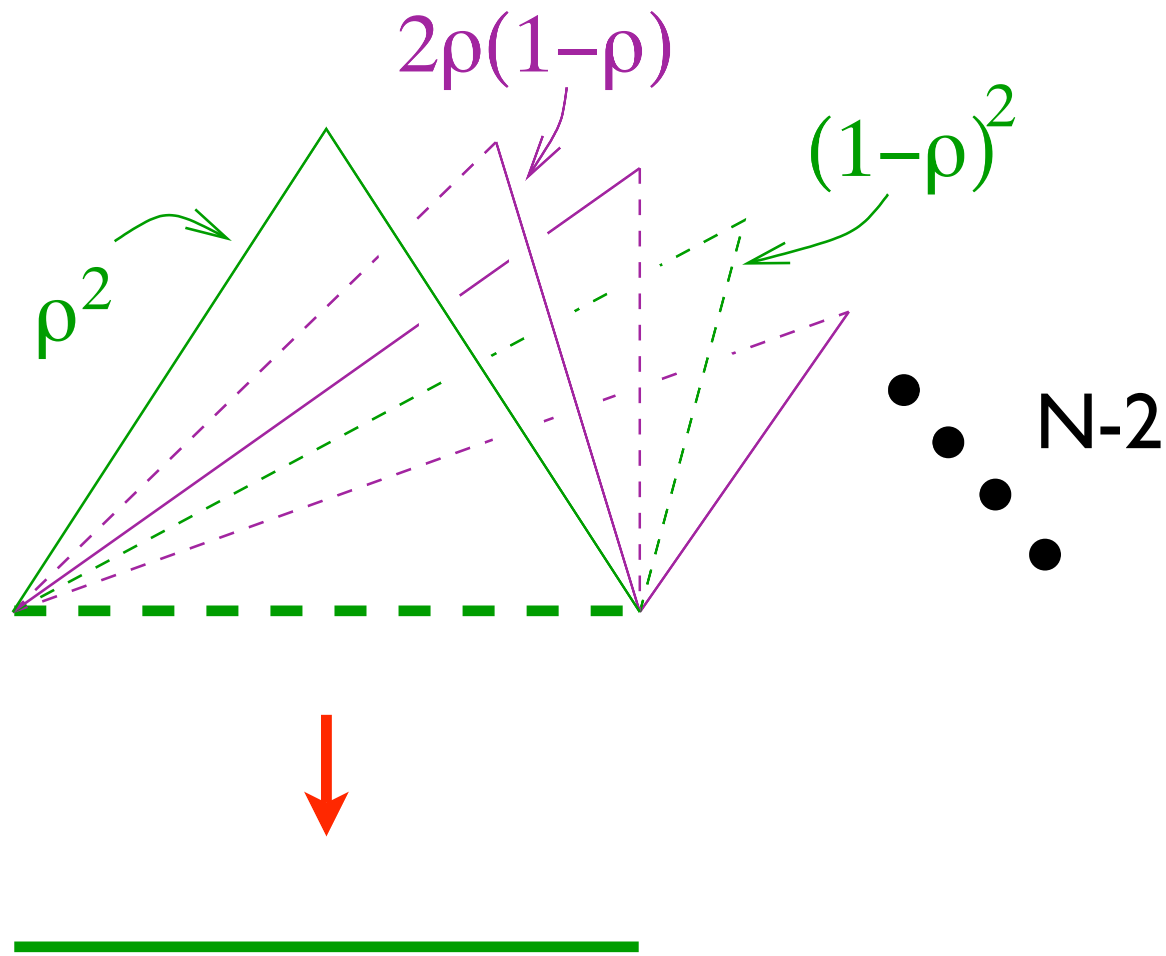


we need:

$$\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}, \text{ with } \vec{n}_+ = [\rho^2, 2\rho(1-\rho), (1-\rho)^2, 0]$$

$$\rightarrow 1 - 4\rho(1-\rho) < 0, \text{ impossible, so + links never flip}$$

for $- \to +$



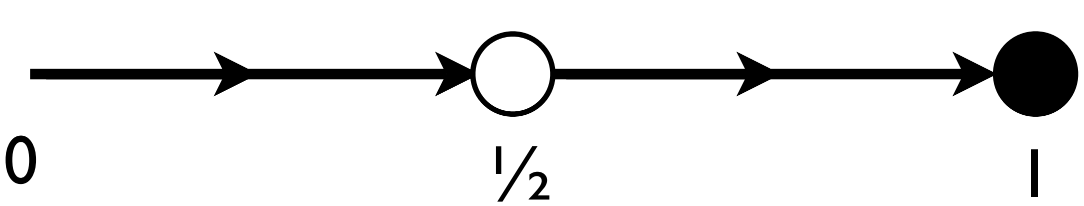
we need:

$$\underbrace{n_1^- + n_3^-}_{\text{frustrated}} > \underbrace{n_0^- + n_2^-}_{\text{unfrustrated}}, \text{ with } \vec{n}_- = [0, \rho^2, 2\rho(1{-}\rho), (1{-}\rho)^2]$$

$$\to 1 - 4\rho(1-\rho) > 0, \text{ valid when } \rho \neq 1/2$$

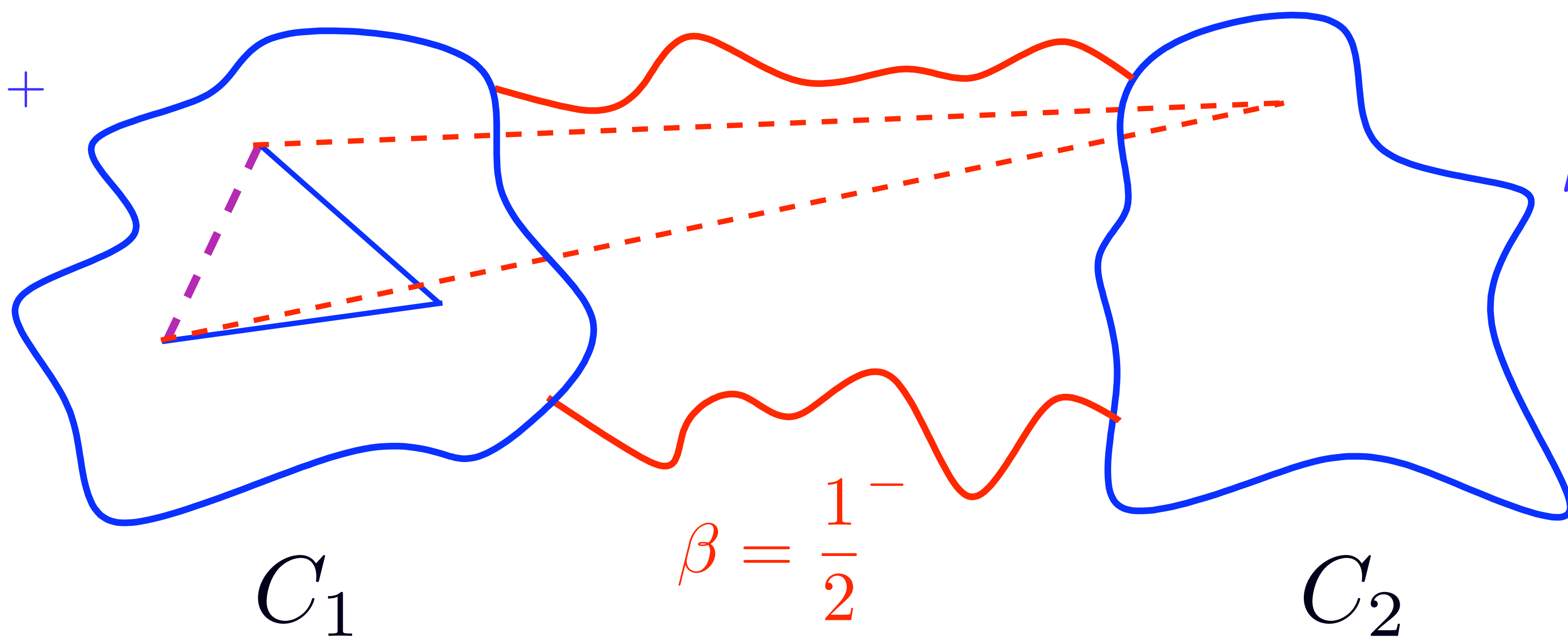
Conclusion: only negative links flip, except when ρ→½

flow diagram for ρ:



Instability near ρ=½

intraclique relationship evolution



for $-\rightarrow+$, we need:

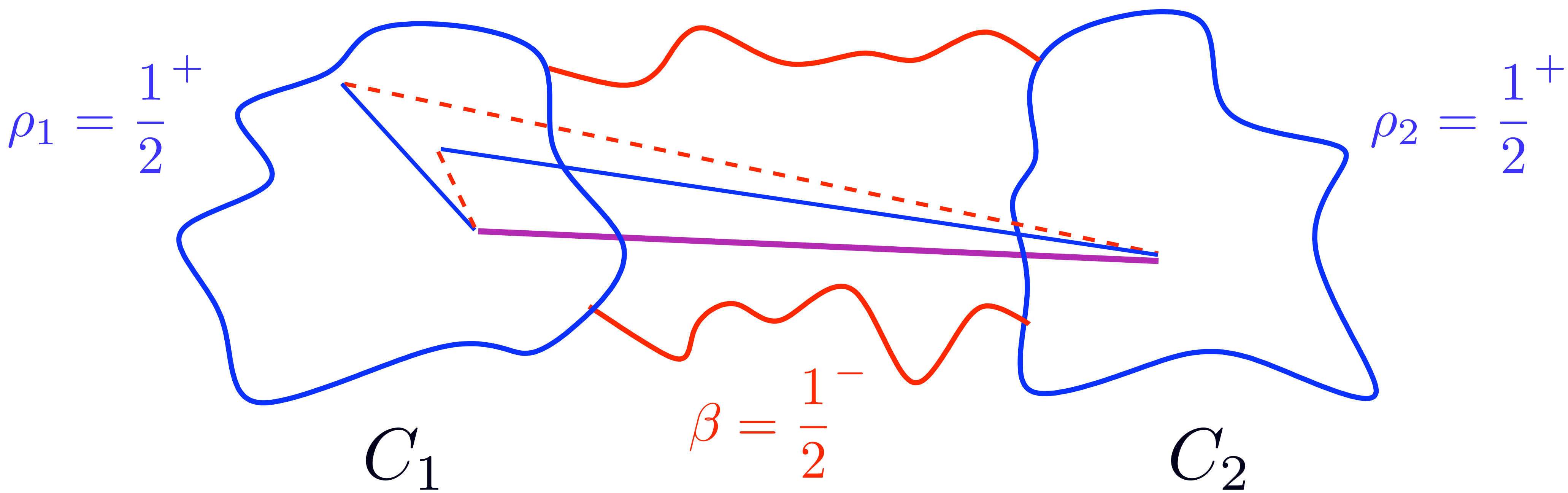
$$\underbrace{n_1^- + n_3^-}_{\text{frustrated}} > \underbrace{n_0^- + n_2^-}_{\text{unfrustrated}}, \text{ with } \vec{n}_- = \begin{cases} [0, \rho_i^2, 2\rho_i(1-\rho_i), (1-\rho_i)^2] & \text{intraclique} \\ [0, \beta^2, 2\beta(1-\beta), (1-\beta)^2] & \text{interclique} \end{cases}$$

$$\rightarrow C_1[1-4\rho_i(1-\rho_i)] + C_2[1-4\beta(1-\beta)] > 0, \text{ always true}$$

negative intraclique links disappear

increased cohesiveness within cliques

interclique relationship evolution



for + → − , we need:

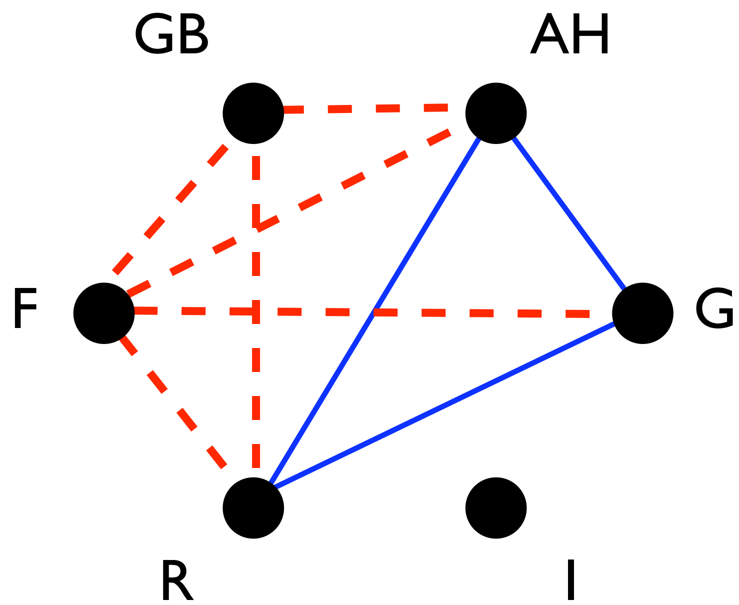
$$\underbrace{n_1^+ + n_3^+}_{\text{frustrated}} > \underbrace{n_0^+ + n_2^+}_{\text{unfrustrated}}, \text{ with } \vec{n}_+ = [\beta\rho_i, \beta(1-\rho_i) + \rho_i(1-\beta), (1-\beta)(1-\rho_i), 0]$$

$$\rightarrow [C_1(2\rho_1 - 1) + C_2(2\rho_2 - 1)](1 - 2\beta) > 0, \quad \text{true if } \rho_1, \rho_2 > 1/2, \beta < 1/2$$

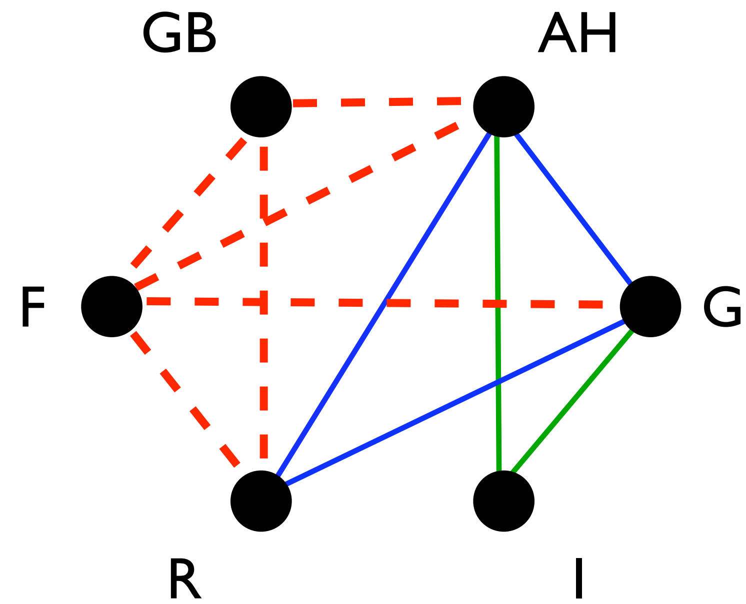
positive interclique links disappear

increased emnity between cliques

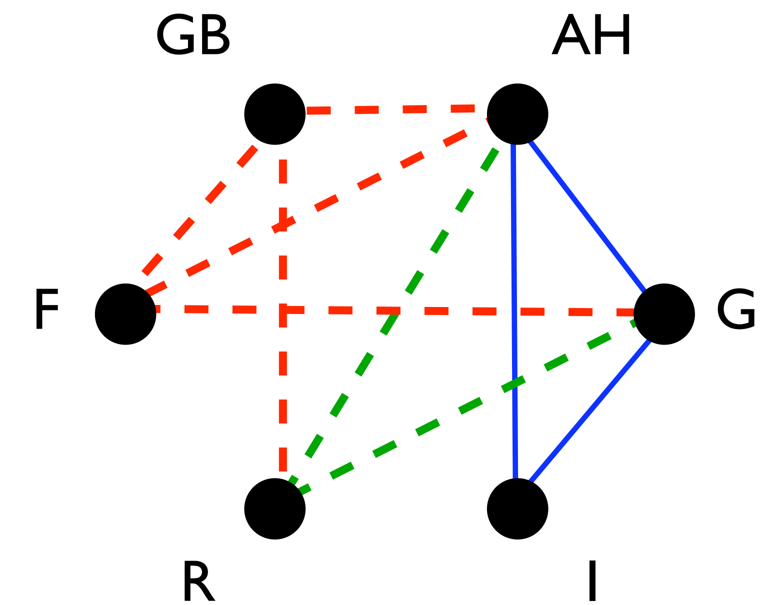
A Historical Lesson



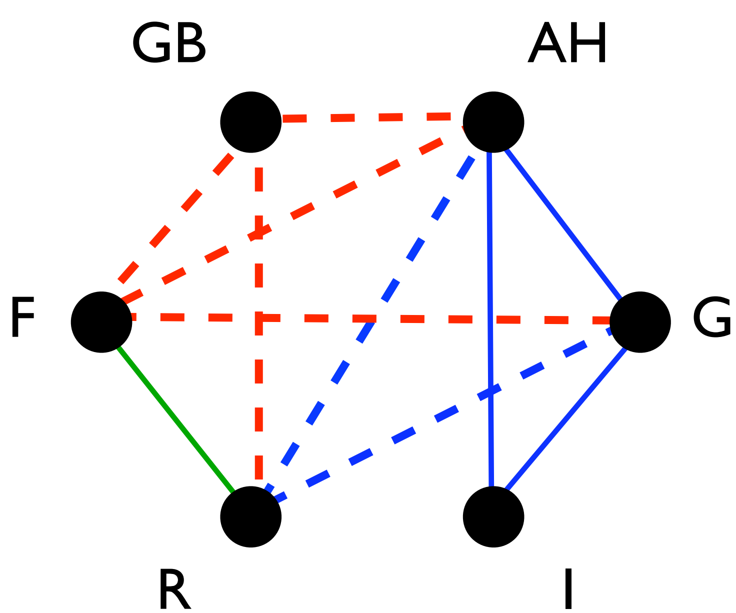
3 Emperor's League 1872-81



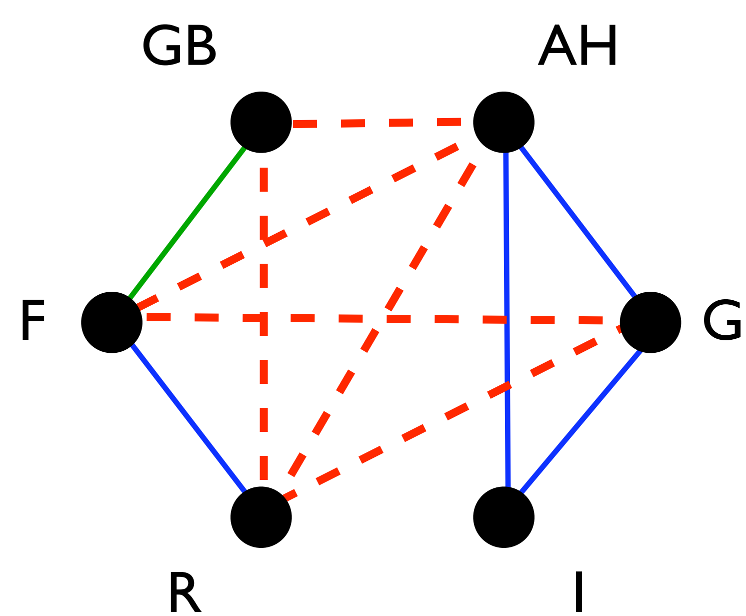
Triple Alliance 1882



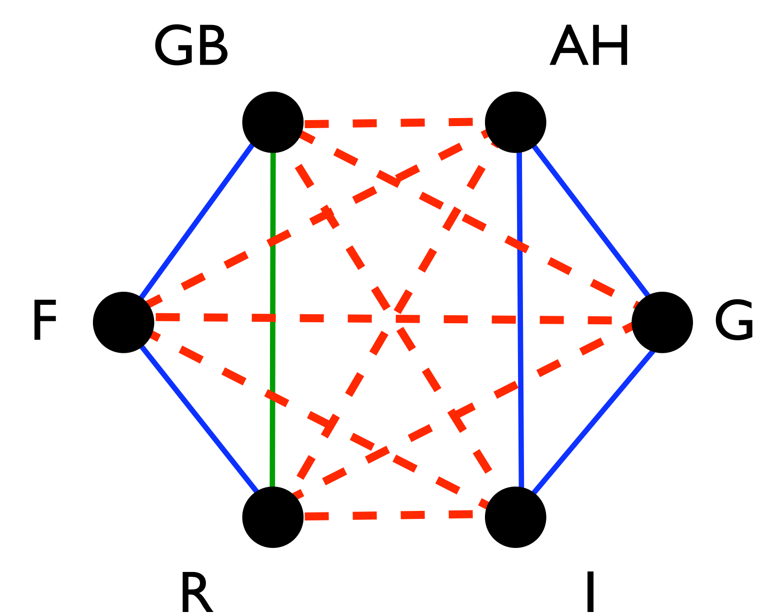
German-Russian Lapse 1890



French-Russian Alliance 1891-94



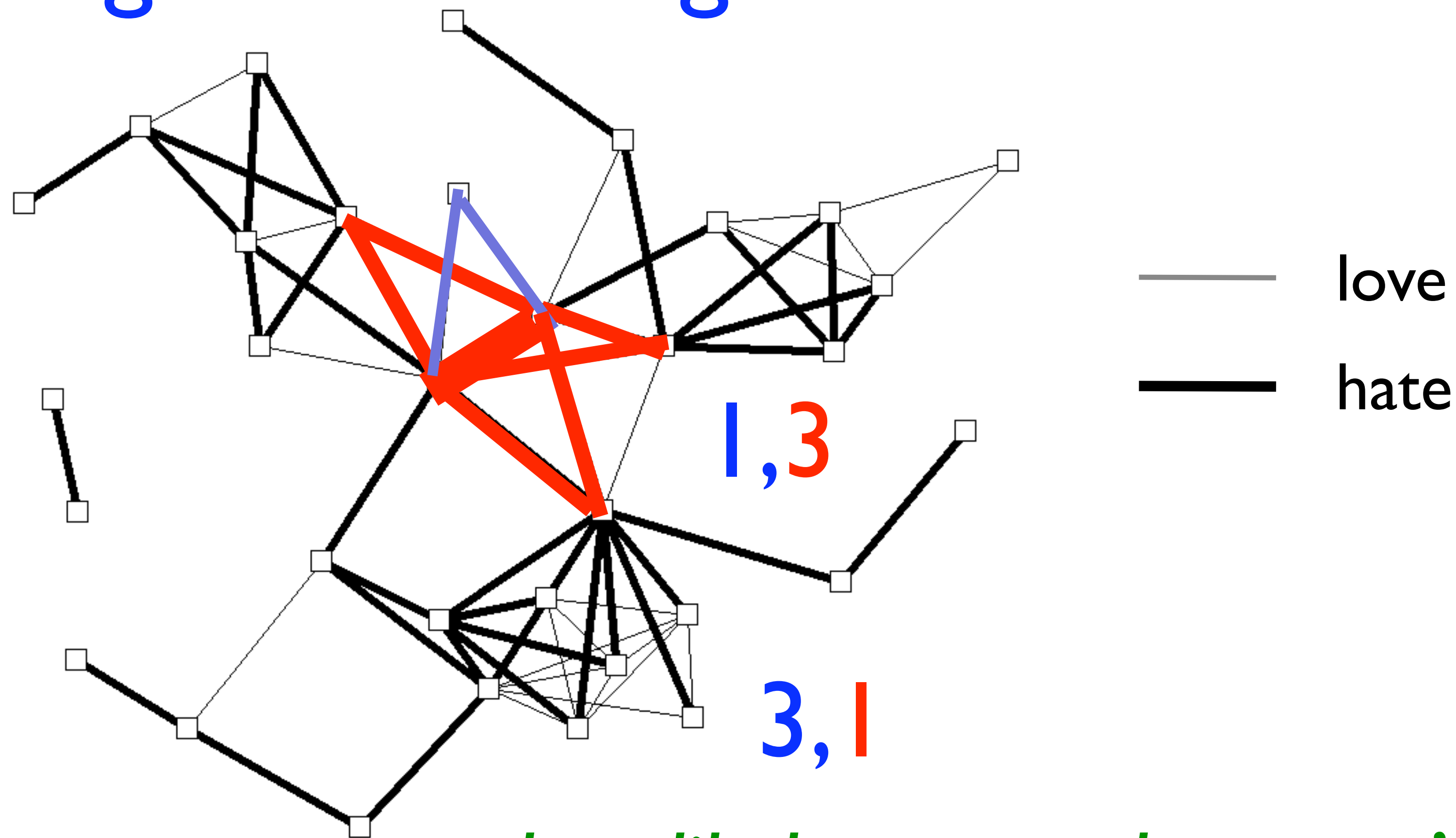
Entente Cordiale 1904



British-Russian Alliance 1907

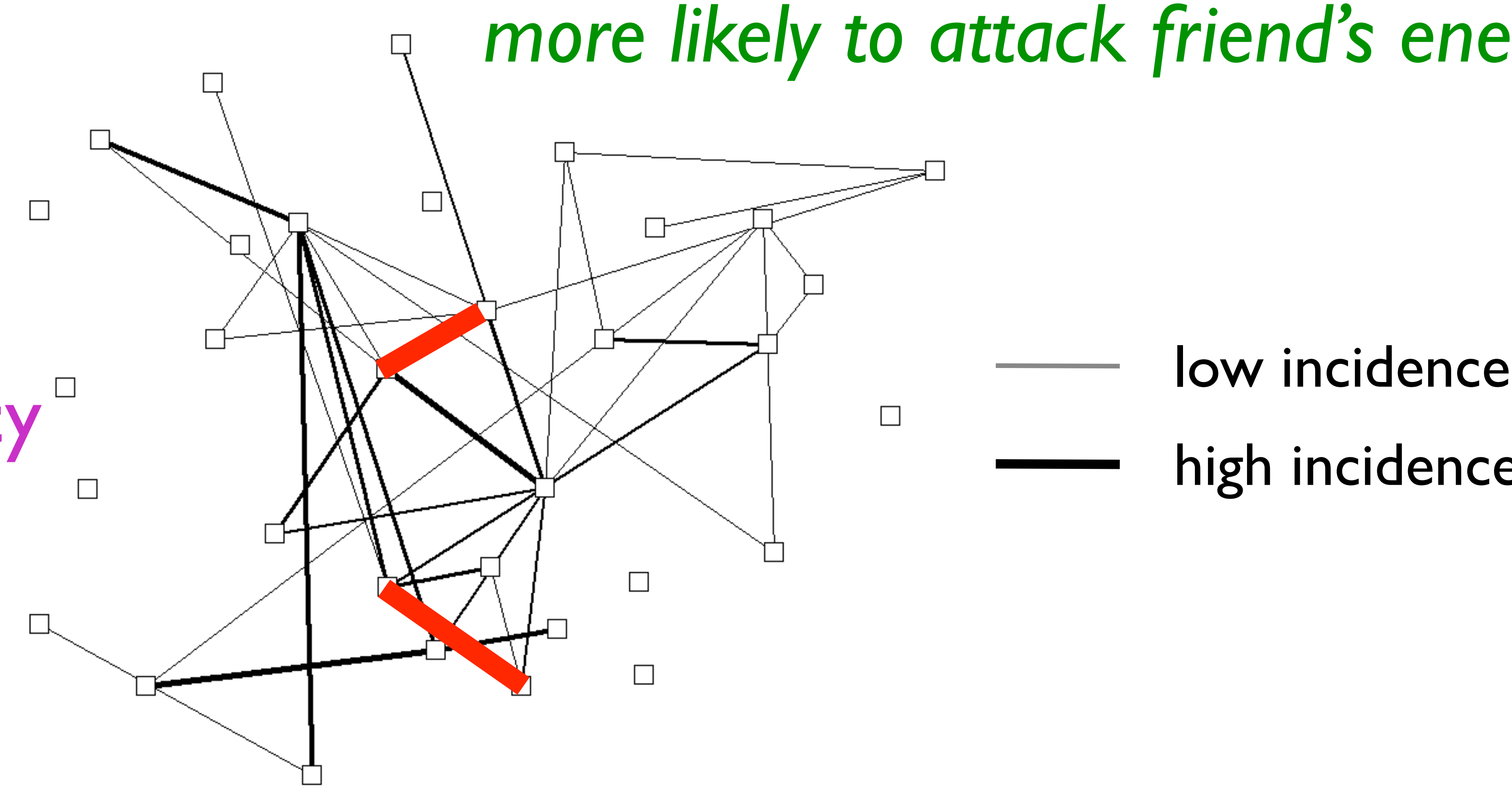
Long Beach Gang Lesson

gang relations



less likely to attack enemy's friend
more likely to attack friend's enemy

violence frequency



Summary & Outlook

If we can't all love each other → *social balance*

Local triad dynamics:

finite network: social balance, with the time until balance strongly dependent on p

infinite network: phase transition between utopia and social balance at $p=\frac{1}{2}$

Global triad dynamics ($p=\frac{1}{3}$):

jammed states possible but never occur

infinite network: two cliques always emerge, with utopia when $\rho_0 \cong 2/3$ (rough argument gives $\rho_0=\frac{1}{2}$)

Open questions:

incomplete graphs, indifference, continuous interactions

allow  → Machiavellian society

asymmetric relations

gang control?