The additive energy and the eigenvalues

I. D. Shkredov

Steklov Mathematical Institute

Let $\mathbf{G}$ be an abelian group, and $A \subseteq \mathbf{G}$ be a finite set.

Sets with small doubling

A is called a set with *small doubling* if

$$|A + A| \le K|A|\,.$$

Examples

$$A = P = \{a, a+s, \dots, a+d(k-1)\}\,,$$

$$A = P_1 + \dots + P_d \text{ (generalized arithmetic progression)},$$

$$\text{large subsets of } P_1 + \dots + P_d \ \text{ or } \ P\,.$$

Theorem (Freiman, 1973)

Let $A \subseteq \mathbb{Z}$, and $|A+A| \le K|A|$. Then there is $Q = P_1 + \dots + P_d$ such that

$$A \subseteq Q$$

and

$$|Q| \le C|A| \,,$$

where d, C depend on K only.

Thus, A is a large subset of a generalized arithmetic progression.

Freiman, $\mathbb{F}_2^n$

Theorem (Freiman)

Let $A \subseteq \mathbb{F}_2^n$, and $|A+A| \le K|A|$. Then there is a subspace Q of dimension d such that

$$A \subseteq Q \quad \text{and} \quad |Q| \le C|A|\,,$$

where d, C depend on K only ($d(K) \sim 2K$, $C(K) \sim \exp(K)$).

Example

Let $A = \{e_1, \dots, e_s\}$, $|A+A| \sim |A|^2/2 \sim s^2$.
Thus $K \sim s$, and $C(K) \sim \exp(K)$.

Subsets

Instead of covering A let us find a structural *subset* of A.

Polynomial Freiman–Ruzsa Conjecture

Let $A \subseteq \mathbb{F}_2^n$, and $|A+A| \le K|A|$. Then there is a subspace Q such that

$$|A \cap Q| \ge |A|/C_1(K),$$

and

$$|Q| \le C_2(K)|A|,$$

where C_1, C_2 depends on K polynomially.

It is known (Sanders, 2012) for $C_1(K) \sim C_2(K) \sim \exp(\log^4(K))$.

Balog–Szemerédi–Gowers

Additive energy

Let $A, B \subseteq \mathbf{G}$ be sets. The (common) *additive energy* of A and B

$$\mathrm{E}(A, B) = \mathrm{E}_2(A, B) :=$$

$$|\{a_1 + b_1 = a_2 + b_2 \;:\; a_1, a_2 \in A,\; b_1, b_2 \in B\}|\,.$$

If $A = B$ then write $\mathrm{E}(A)$ for $\mathrm{E}(A, A)$.

Example, $\mathrm{E}(A)$ large

A is an arithmetic progression ($\mathbb{Z}$) or subspace ($\mathbb{F}_2^n$).

If $|A + A| \le K|A|$ then $\mathrm{E}(A) \ge |A|^3/K$.

Balog–Szemerédi–Gowers

Theorem (Balog–Szemerédi–Gowers)

Let $\mathbf{G}$ be an abelian group, and $A \subseteq \mathbf{G}$ be a finite set. Suppose that $\mathrm{E}(A) \geq |A|^3/K$. Then there is $A_* \subseteq A$ such that

$$|A_*| \geq |A|/C_1(K)\,,$$

and

$$|A_* + A_*| \leq C_2(K)|A_*|\,,$$

where C_1, C_2 depend on K polynomially.

So, firstly, we find a structural subset and, secondly, all bounds are polynomial.

So, $|A+A| \le K|A| \Rightarrow \mathrm{E}(A) \ge |A|^3/K$.
But $\mathrm{E}(A) \ge |A|^3/K \Rightarrow |A_* + A_*| \le C(K)|A_*|$ for some polynomially large A_*.

Can we have it for the whole A?

Example

$A \subseteq \mathbb{F}_2^n$,

$$A = Q \bigsqcup \Lambda\,,$$

where Q is a subspace, $|Q| \sim \mathrm{E}^{1/3}(A)$ and Λ is a basis ($|\Lambda| \sim |A|$).

$\mathrm{E}(Q) \sim \mathrm{E}(A)$ but $|A+A| \ge |\Lambda + \Lambda| \gg |A|^2$.

Example, again

$A \subseteq \mathbb{F}_2^n$,

$$A = Q \bigsqcup \Lambda \,,$$

where Q is a subspace, $|Q| \sim \mathrm{E}^{1/3}(A)$ and Λ is a basis $(|\Lambda| \sim |A|)$.

$\mathrm{E}(Q) \sim \mathrm{E}(A)$ and, similarly, $\mathrm{E}(A, Q) \sim \mathrm{E}(A)$. Hence

$$\frac{\mathrm{E}(A, Q)}{|Q|} > \frac{\mathrm{E}(A, A)}{|A|} = \frac{\mathrm{E}(A)}{|A|} \,.$$

Convolutions

$$(g \circ h)(x) := \sum_y g(y)h(x+y)\,,$$

$$(\chi_A \circ \chi_B)(x) := |\{a - b = x \ : \ a \in A,\, b \in B\}\,.$$

Consider the hermitian positively defined operator (matrix)

$$\mathrm{T}(x, y) = (A \circ A)(x - y)A(x)A(y)\,,$$

where $A(x)$ is the characteristic function χ_A of the set A, i.e. $A(x) = 1$, $x \in A$ and $A(x) = 0$ otherwise.

Recall

$$\mathrm{T}(x,y) = (A \circ A)(x-y)A(x)A(y)\,.$$

We have

$$\langle \mathrm{T}A, A\rangle = \sum_{x,y} (A\circ A)(x-y)A(x)A(y) = \|A\circ A\|_2^2 = \mathrm{E}(A)\,,$$

and, similarly,

$$\left\langle \mathrm{T}\frac{A(x)}{|A|^{1/2}}, \frac{A(x)}{|A|^{1/2}} \right\rangle = \frac{\mathrm{E}(A)}{|A|} < \left\langle \mathrm{T}\frac{Q(x)}{|Q|^{1/2}}, \frac{Q(x)}{|Q|^{1/2}} \right\rangle = \frac{\mathrm{E}(A,Q)}{|Q|}\,.$$

Thus, the action of T on (normalized) Q is larger then the action of T on (normalized) A.

Let

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_{|A|} > 0$$

be the spectrum of T and

$$f_1, f_2, \dots, f_{|A|}$$

the correspondent eigenfunctions.
By Courant–Fisher Theorem

$$\mu_1 = \max_{\|f\|_2 = 1} \langle \mathrm{T} f, f \rangle \,.$$

Thus, f_1 'sits' on Q not A !
Here $A = Q \bigsqcup \Lambda$.

Conjecture

The structured pieces of $A \subseteq \mathbf{G}$ are (essential) supports of the eigenfunctions of T.

Holds

- not for any A, A should be a 'popular difference set' :

$$A = \{x \ : \ (B \circ B)(x) \geq c|B|\}$$

for some B, $c = c(K) > 0$

- may be we need in some another 'weights'.

Operators

Let **G** be an abelian group, and $A \subseteq \mathbf{G}$ be a finite set. Take any real function g such that $g(-x) = g(x)$. Put

$$\mathrm{T}^g_A(x,y) = g(x-y)A(x)A(y)\,.$$

Let

$$\mu_1(\mathrm{T}^g_A) \geq \mu_1(\mathrm{T}^g_A) \geq \dots \geq \mu_{|A|}(\mathrm{T}^g_A)$$

be the spectrum of T^g_A and

$$f_1, f_2, \dots, f_{|A|}$$

the correspondent eigenfunctions.

$$\mathrm{T}^{g}_{A}(x,y) = g(x-y)A(x)A(y)\,.$$

Examples

- If $A = \mathbf{G}$, $g(x) = B(x)$, $B \subseteq \mathbf{G}$ then $\mathrm{T}^{B}_{\mathbf{G}}$ the adjacency matrix of Cayley graph defined by B.
- If $g(x) = B(x)$ and A is any set then T^{B}_{A} is a submatrix of Cayley graph.
- Put $g(x) = (A \circ A)(x)$. Then $\mathrm{T} = \mathrm{T}^{A \circ A}_{A}$. Always

$$\mu_1(\mathrm{T}) \ge \frac{\mathrm{E}(A)}{|A|}\,.$$

Trace of T^g_A and $\mathrm{T}^g_A(\mathrm{T}^g_A)^*$

$$|A|g(0) = \sum_{j=1}^{|A|} \mu_j(\mathrm{T}^g_A)\,,$$

$$\sum_z |g(z)|^2 (A \circ A)(z) = \sum_{j=1}^{|A|} |\mu_j(\mathrm{T}^g_A)|^2\,.$$

Example

Let $\mathrm{T} = \mathrm{T}^{A\circ A}_A$. Then

$$\sum_{j=1}^{|A|} |\mu_j(\mathrm{T}^g_A)|^2 = \sum_z (A \circ A)^3(z) := \mathrm{E}_3(A)\,.$$

Structural E_2, E_3 result

Theorem (Shkredov, 2013)

Let $A \subseteq \mathbf{G}$ be a set, $\mathrm{E}(A) = |A|^3/K$, and $\mathrm{E}_3(A) = M|A|^4/K^2$. Then there is $A_* \subseteq A$ s.t.

$$|A_*| \geq M^{-C}|A|\,,$$

and for any n, m

$$|nA_* - mA_*| \leq K \cdot M^{C(n+m)}|A_*|\,.$$

Let $Q \subseteq \mathbb{F}_2^n$ be a subspace, $A \subseteq Q$ be a random subset s.t. $|A| = |Q|/K \Rightarrow \mathrm{E}(A) \sim |A|^3/K$, $\mathrm{E}_3(A) \sim |A|^4/K^2 \Rightarrow M \sim 1$.

$$|A - A| \sim K|A| \sim |Q| \quad \text{as well as} \quad |nA - mA| \sim |Q|\,.$$

Let $a \leq b$ and $M \geq 1$.

a and b are M–critical if $a \leq b \leq Ma$.

Structural E_2, E_3 result, again

Let $A \subseteq \mathbf{G}$ be a set, and $\mathrm{E}_2(A)$, $\mathrm{E}_3(A)$ be M–critical

$$\frac{\mathrm{E}_2^2}{|A|^2} \leq \mathrm{E}_3 \leq \frac{M\mathrm{E}_2^2}{|A|^2}.$$

Then there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t.

- $A_* - A_*$ has small doubling,
- $|A_* - A_*| \sim_M \frac{|A|^4}{\mathrm{E}(A)}$.

Multiplicative subgroups

Let $\Gamma \subseteq \mathbb{F}_q^*$ be a subgroup, $q = p^s$, $|\Gamma|$ divides $q-1$, $n = \frac{q-1}{|\Gamma|}$, $\mathbf{g}$ be a primitive root. Then

$$\Gamma = \{1, \mathbf{g}^n, \mathbf{g}^{2n}, \dots, \mathbf{g}^{(t-1)n}\},$$

Consider the orthonormal family of multiplicative characters on Γ

$$\chi_\alpha(x) = |\Gamma|^{-1/2} \cdot \Gamma(x) e^{\frac{2\pi i \alpha l}{|\Gamma|}}, \quad x = \mathbf{g}^l, \quad 0 \le l < |\Gamma|.$$

Lemma

Let $\Gamma \subseteq \mathbb{F}_q^*$ be a subgroup, g be any real even Γ–invariant function

$$g(\gamma x) = g(x)\,, \qquad \gamma \in \Gamma\,.$$

Then χ_α, $\alpha = 0, 1, \dots, |\Gamma| - 1$ are eigenfunctions of T^g_Γ.

In particular

$$\mathrm{E}(\Gamma) = |\Gamma| \mu_1(\mathrm{T}^g_\Gamma)\,,$$

$$\mathrm{E}(\Gamma) = \max_{f \,:\, \|f\|_2 = |\Gamma|} \mathrm{E}(\Gamma, f)\,,$$

and

$$\mathrm{E}(\Gamma, A) \ge \mathrm{E}(\Gamma) \frac{|A|^2}{|\Gamma|^2}\,, \quad A \subseteq \Gamma\,.$$

The additive energy of subgroups

Theorem (Konyagin, 2002)

Let $\Gamma \subseteq \mathbb{F}_p$ be a multiplicative subgroup, $|\Gamma| \ll p^{2/3}$. Then

$$\mathrm{E}(\Gamma) := |\{g_1 + g_2 = g_3 + g_4 \ : \ g_1, g_2, g_3, g_4 \in \Gamma\}| \ll |\Gamma|^{5/2} .$$

Theorem (Shkredov, 2012)

Let $\Gamma \subseteq \mathbb{F}_p$ be a multiplicative subgroup, $|\Gamma| \ll p^{3/5}$. Then

$$\mathrm{E}(\Gamma) := |\{g_1 + g_2 = g_3 + g_4 \ : \ g_1, g_2, g_3, g_4 \in \Gamma\}|$$

$$\ll |\Gamma|^{5/2-\varepsilon_0} .$$

Why ?

Suppose that $\mathrm{E}(\Gamma) \sim |\Gamma|^{5/2} = |\Gamma|^3/K$, $K \sim |\Gamma|^{1/2}$.

Lemma

We have
$$\mathrm{E}_3(\Gamma) \ll |\Gamma|^3 \log |\Gamma| = \frac{M|\Gamma|^4}{K^2} \,,$$
where $M \sim \log |\Gamma|$.

Thus by our structural result Γ stabilized under addition but $k\Gamma = \mathbb{F}_p$ (more delicate arguments give the better bounds).

Thus, $\mathrm{E}(\Gamma) = |\Gamma|^{5/2-\varepsilon_0}$, $\varepsilon_0 > 0$.

Convex sets

$A = \{a_1 < a_2 < \cdots < a_n\} \subseteq \mathbf{R}$ is called *convex* if

$$a_{i+1} - a_i > a_i - a_{i-1} \qquad \text{for all } i\,.$$

Example.

$$A = \{1^2, 2^2, \dots, n^2\}\,.$$

Theorem (Iosevich, Konyagin, Rudnev, Ten, 2006)

Let $A \subseteq \mathbb{R}$ be a convex set. Then

$$\mathrm{E}(A) \ll |A|^{5/2}\,.$$

Theorem (Shkredov, 2012–2013)

Let $A \subseteq \mathbb{R}$ be a convex set. Then

$$\mathrm{E}(A) \ll |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A|\,.$$

Proof : a formula for higher moments of eigenvalues and estimation of eigenvalues.

Further applications

Doubling constants for :

- multiplicative subgroups,
- convex sets,
- sets with small product sets $|AA|$.

Combinatorial methods $=$ counting spectrum of $\mathrm{T}_A^{A\pm A}$.

Theorem (Elekes–Nathanson–Rusza, 1999)

Let $A \subseteq \mathbb{R}$ be a convex set. Then

$$|A+A| \gg |A|^{3/2},$$

Theorem (Schoen–Shkredov, 2011)

Let $A \subseteq \mathbb{R}$ be a convex set. Then

$$|A-A| \gg |A|^{8/5-\varepsilon},$$

and

$$|A+A| \gg |A|^{14/9-\varepsilon}.$$

Heilbronn's exponential sums

Let p be a prime number.
Heilbronn's exponential sum is defined by

$$S(a) = \sum_{n=1}^{p} e^{2\pi i \cdot \frac{an^p}{p^2}} \,.$$

Fermat quotients defined as

$$q(n) = \frac{n^{p-1}-1}{p}\,, \quad n \neq 0 \pmod p\,.$$

Theorem (Heath–Brown, Konyagin, 2000)

Let p be a prime, and $a \neq 0 \pmod p$. Then

$$|S(a)| \ll p^{\frac{7}{8}}\,.$$

In the proof an upper bound of the additive energy of Heilbronn's subgroup

$$\Gamma = \{m^p \;:\; 1 \le m \le p-1\} = \{m^p \;:\; m \in \mathbb{Z}/p^2\mathbb{Z}\,, m \neq 0\}$$

was used.

Via the additive energy estimate and the operator $\mathrm{T}_A^{A\circ A}$.

Theorem (Shkredov, 2012–2013)

Let p be a prime, and $a \neq 0 \pmod p$. Then

$$|S(a)| \ll p^{\frac{7}{8}-\varepsilon_0} p\,.$$

Via direct calculations and the operator with "dual" weights $\mathrm{T}_A^{\widehat{A}}$.

Theorem (Shkredov, 2013)

Let p be a prime, and $a \neq 0 \pmod p$. Then

$$|S(a)| \ll p^{\frac{5}{6}} \log^{\frac{1}{6}} p\,.$$

By l_p denote the smallest n such that $q(n) \neq 0 \pmod p$.

Theorem (Bourgain, Ford, Konyagin, Shparlinski, 2010)

One has

$$l_p \leq (\log p)^{\frac{463}{252}+o(1)}$$

as $p \to \infty$.

Previously Lenstra (1979) : $l_p \ll (\log p)^{2+o(1)}$.

Theorem (Shkredov, 2012–2013)

One has

$$l_p \leq (\log p)^{\frac{463}{252}-\varepsilon_0+o(1)}\,, \quad \varepsilon_0 > 0\,,$$

ε_0 is an absolute (small) constant.

Other applications are :

- discrepancy of Fermat quotients,
- new bound for the size of the image of $q(n)$,
- estimates for Ihara sum,
- better bounds for the sums

$$\sum_{n=1}^{k} \chi(q(n))\,, \quad \sum_{n=1}^{k} \chi(nq(n))\,.$$

- Surprising inequalities between $\mathrm{E}(A)$ and $\mathrm{E}_s(A)$, $s \in (1,2]$.

$f : \mathbf{G} \to \mathbb{C}$ be a function, $\widehat{\mathbf{G}} = \{\xi\}$, $\xi : \mathbf{G} \to \mathbb{D}$ be the group of homomorphisms.

Fourier transform

$$\widehat{f}(\xi) := \sum_{x} f(x)\overline{\xi(x)}\,, \quad \xi \in \widehat{\mathbf{G}}\,.$$

Properties of T^{g}_{A}

We have

- $\mathrm{Spec}\,(\mathrm{T}^{\widehat{B}}_{A}) = \mathrm{Spec}\,(\mathrm{T}^{\widehat{A}}_{B^c}) = \mathrm{Spec}\,(\mathrm{T}^{\widehat{A}^c}_{B})$.
- $\mathrm{Spec}\,(\mathrm{T}^{\widehat{B}}_{A}(\mathrm{T}^{\widehat{B}}_{A})^{*}) = |\mathbf{G}| \cdot \mathrm{Spec}\,(\mathrm{T}^{|\widehat{B}|^2}_{A})$

Here $f^c(x) := f(-x)$ for any function $f : \mathbf{G} \to \mathbb{C}$.

Chang Theorem

Let $\mathbf{G}$ be an abelian group, and $A \subseteq \mathbf{G}$ be a finite set.

Dissociated sets

A set $\Lambda = \{\lambda_1, \dots, \lambda_d\} \subseteq \mathbf{G}$ is called *dissociated* if any equation of the form

$$\sum_{j=1}^{d} \varepsilon_j \lambda_j = 0\,, \qquad \text{where} \qquad \varepsilon_j \in \{0, \pm 1\}$$

implies $\varepsilon_j = 0$ for all j.

Exm. $\mathbf{G} = \mathbb{F}_2^n$.

Proof of Chang Theorem via operators

Chang theorem

For any dissociated set Λ, any set $A \subseteq \mathbf{G}$, $|A| = \delta|\mathbf{G}|$ and an arbitrary function f, $\operatorname{supp} f \subseteq A$

$$\sum_{\xi \in \Lambda} |\widehat{f}(\xi)|^2 \le |A| \log(1/\delta) \cdot \|f\|_2^2 \,.$$

$$\sum_{\xi \in \Lambda} |\widehat{f}(\xi)|^2 = \langle \mathrm{T}_A^{\widehat{\Lambda}} f, f \rangle \le \mu_1(\mathrm{T}_A^{\widehat{\Lambda}}) \|f\|_2^2 = \mu_1(\mathrm{T}_{\Lambda}^{\widehat{A}}) \|f\|_2^2$$

Estimating $\mu_1(\mathrm{T}^{\widehat{A}}_{\Lambda})$

$\mathrm{supp}\, w \subseteq \Lambda$, $k \sim \log(1/\delta)$.

$$\mu_1(\mathrm{T}^{\widehat{A}}_{\Lambda}) := \max_{\|w\|_2=1} \langle \mathrm{T}^{\widehat{A}}_{\Lambda} w, w\rangle = \sum_x |\widehat{w}(x)|^2 A(x)\,.$$

$$\mu_1^k(\mathrm{T}^{\widehat{A}}_{\Lambda}) \le \sum_x |\widehat{w}(x)|^{2k} \cdot |A|^{k-1}$$

$$\sum_x |\widehat{w}(x)|^{2k} = |\mathbf{G}| \sum_{x_1+\dots+x_k = x'_1+\dots+x'_k} w(x_1)\dots w(x_k)\overline{w(x'_1)}\dots\overline{w(x'_k)}$$

$$\le N C^k k! \|w\|_2^{2k} = N C^k k!\,.$$

Advantages of the approach

- Relaxation of dissociativity

$$\sum_{\sum_j |\varepsilon_j| \ll \log(1/\delta)} \varepsilon_j \lambda_j = 0 \quad \text{instead of} \quad \sum_{j=1}^{|\Lambda|} \varepsilon_j \lambda_j = 0\,.$$

- Very weak dissociativity ($\sum_j |\varepsilon_j| \le C$).
- Other operators T^g_A. Higher moments

$$\sum_{\xi \in \Lambda} |\widehat{A}(\xi)|^l\,, \quad l > 2\,,$$

dual Chang theorems

$$\sum_{x \in \Lambda} (A_1 * A_2)^2(x) \ll |A_1||A_2| \log\left(\min\{|A_1|, |A_2|\}\right)\,.$$

Structural E_2, E_3 result, again

Let $A \subseteq \mathbf{G}$ be a set, and E_2, E_3 be M–critical
Then there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t.

- $A_* - A_*$ has small doubling,
- $|A_* - A_*| \sim_M \frac{|A|^4}{\mathrm{E}(A)}$.

Balog–Szemerédi–Gowers, again

$\mathrm{E}(A)$ and $|A|$ are M–critical ($|A|^3 \ll_M \mathrm{E}(A) \le |A|^3$) iff there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t. A_* has small doubling.

It is a (rough) criterium.

$$\mathrm{E}_k(A) = |\{a_1 - a'_1 = a_2 - a'_2 = \dots = a_k - a'_k \;:\; a_1, a'_1, \dots, a_k, a'_k \in A\}|$$

$$\mathrm{E}_1(A) = |A|^2,\ \mathrm{E}_2(A) = \mathsf{T}_2(A).$$

$$\mathsf{T}_k(A) := |\{a_1 + \dots + a_k = a'_1 + \dots + a'_k \;:\; a_1, \dots, a_k, a'_1, \dots, a'_k \in A\}|$$

$$\sum_x (\widehat{A} \circ \widehat{A})^{2k}(x) = |\mathbf{G}|^{2k+1} \mathsf{T}_k(A) \quad \text{and} \quad \mathsf{T}_k(|\widehat{A}|^2) = |\mathbf{G}|^{2k-1} \mathrm{E}_{2k}(A)\,.$$

$$\left(\frac{\mathrm{E}_{3/2}(A)}{|A|} \right)^{2k} \le \mathrm{E}_k(A) \mathsf{T}_k(A)\,.$$

Criterium for $A, 2A, 3A, \dots$ stops at the second step (with right doubling constant).

Criterium

$\mathsf{T}_4(A)$ and $\mathrm{E}(A)$ are M–critical

$$\mathsf{T}_4(A) \sim_M |A|^2 \mathrm{E}(A)$$

iff there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t.

- $A_* - A_*$ has small doubling,
- $|A_* - A_*| \sim_M \frac{|A|^4}{\mathrm{E}(A)}$.

Zoo

1) random sets
2) sets with small doubling

Further families

3) $2A$ has small (right) doubling
4) $A = H \dotplus \mathcal{L}$, H has small doubling, $\mathcal{L}$ is "dissociated"
5) $A = H_1 \bigsqcup H_2 \bigsqcup \dots \bigsqcup H_k$, all H_j with small doubling

and further

6) Intermediate between 4) and 5) (Bateman–Katz)
7) Schoen potatoes and so on.

Characterization of $H \dot{+} \mathcal{L}$

Criterium

$\mathrm{E}_2(A)$ and $\mathrm{E}_3(A)$ are M–critical

$$\mathrm{E}_3(A) \sim_M |A|\mathrm{E}(A)$$

iff there are two sets $H, \mathcal{L}$ such that

$$|H \cap A| \gg_M \frac{\mathrm{E}(A)}{|A|^2}\,, \qquad |\mathcal{L}| \ll_M \frac{|A|}{|H|}\,,$$

$$|H - H| \ll_M |H|\,,$$

and

$$|A \bigcap (H + \mathcal{L})| \gg_M |A|\,.$$

Energy of sumsets

$A = H \dotplus \mathcal{L} \subseteq \mathbb{F}_2^n$, H is a subspace, $\mathcal{L}$ is a basis, $|\mathcal{L}| = K$. Then $\mathrm{E}(A) \sim |A|^3/K$,

$$D = A + A = H \dotplus (\mathcal{L} + \mathcal{L}) .$$

Katz–Koester:

$$\mathrm{E}(D) \ge |A|^2|D| = K|A|^3 , \quad \mathrm{E}_3(D) \ge K|A|^4 .$$

Lower bound for $\mathrm{E}_3(A \pm A)$

Let A be a set, $D = A \pm A$, $|D| = K|A|$. Suppose that $\mathrm{E}(A) \ll |A|^3/K$ + some technical conditions. Then

$$\mathrm{E}_3(D) \gg K^{7/4}|A|^4 .$$

Upper bound $\mathrm{E}_3(D) \ll K^2|A|^4$.

Put $A_s = A \cap (A+s)$.

$$\|A\|_{\mathcal{U}^3} := \sum_s \mathrm{E}(A_s)\,.$$

Criterium

$\mathrm{E}_3(A)$ and $\|A\|_{\mathcal{U}^3}$ are M–critical

$$\mathrm{E}_3(A) \sim_M \|A\|_{\mathcal{U}^3}$$

$+$ some technical conditions. iff

$$A = H_1 \bigsqcup H_2 \bigsqcup \dots \bigsqcup H_k\,,$$

all $H_j \subseteq A_{s_j}$ with small doubling, $|H_j| \gg_M |A_{s_j}|$.

Weak counterexample to Gowers construction

Recall $A_s = A \cap (A+s)$.

Existence of A_s with small energy

Let $A \subseteq \mathbf{G}$ be a set, $\mathrm{E}(A) = |A|^3/K$,

$$|A_s| \le \frac{M|A|}{K},$$

where $M \ge 1$ is a real number. Then $\exists s \neq 0$, $|A_s| \ge \frac{|A|}{2K}$ s.t.

$$\mathrm{E}(A_s) \ll \frac{M^{93/79}}{K^{1/198}} \cdot |A_s|^3 .$$

Dichotomy for sumsets

$D = A - A$.
For $a_1, \dots, a_k \in A$, we have

$$A \subseteq (D + a_1) \bigcap (D + a_2) \cdots \bigcap (D + a_k) \,.$$

Dichotomy

Suppose that for any $x_1, \dots, x_k$

$$|A| \le |(D + x_1) \bigcap (D + x_2) \cdots \bigcap (D + x_k)| \le |A|^{1+\varepsilon} \,.$$

Then either $\mathrm{E}_k(A) \ll_{|A|^\varepsilon, k} |A|^k$ or $\mathrm{E}(A) \gg_{|A|^\varepsilon, k} |A|^3$.

Concluding remarks

- Studying the eigenvalues and the eigenfunctions of T, we obtain the information about the initial object $\mathrm{E}(A)$.

- Our approach tries to emulate Fourier analysis *onto* A not on the whole group $\mathbf{G}$.

Conjecture, again

The structured pieces of $A \subseteq \mathbf{G}$ are (essential) supports of the eigenfunctions of T.

Considered examples

In all examples above (multiplicative subgroups, convex sets and so on), we have

$$\mu_1 \gg \mu_2 \geq \mu_3 \geq \dots\,, \quad \mu_1 \text{ dominates.}$$

PFRC case

If our set A is a sumset $A = B - B$, $|A| = K|B|$ (or popular difference set) then

$$\mu_1 \sim \mu_2 \sim \dots \sim \mu_k \gg \mu_{k+1} \geq \dots\,, \quad k \sim K\,.$$

So, there many roughly equal eigenvalues.
The correspondent eigenfunctions lives on "disjoint" (sub)sets of $B - b$, $b \in B$.

Thank you for your attention!