The additive energy and the eigenvalues

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Let $G$ be an abelian group, and $A \subseteq G$ be a finite set.

**Sets with small doubling**

$A$ is called a set with *small doubling* if

$$|A + A| \leq K|A|.$$  

**Examples**

- $A = P = \{a, a + s, \ldots, a + d(k - 1)\}$,
- $A = P_1 + \cdots + P_d$ (generalized arithmetic progression),
- large subsets of $P_1 + \cdots + P_d$ or $P$. 
Theorem (Freiman, 1973)

Let $A \subseteq \mathbb{Z}$, and $|A + A| \leq K|A|$. Then there is $Q = P_1 + \cdots + P_d$ such that

$$A \subseteq Q$$

and

$$|Q| \leq C|A|,$$

where $d, C$ depend on $K$ only.

Thus, $A$ is a large subset of a generalized arithmetic progression.
Theorem (Freiman)

Let $A \subseteq \mathbb{F}_2^n$, and $|A + A| \leq K|A|$. Then there is a subspace $Q$ of dimension $d$ such that

$$A \subseteq Q \quad \text{and} \quad |Q| \leq C|A|,$$

where $d, C$ depend on $K$ only ($d(K) \sim 2K$, $C(K) \sim \exp(K)$).

Example

Let $A = \{e_1, \ldots, e_s\}$, $|A + A| \sim |A|^2/2 \sim s^2$. Thus $K \sim s$, and $C(K) \sim \exp(K)$. 
Subsets

Instead of covering $A$ let us find a structural *subset* of $A$.

**Polynomial Freiman–Ruzsa Conjecture**

Let $A \subseteq \mathbb{F}_2^n$, and $|A + A| \leq K|A|$. Then there is a subspace $Q$ such that

$$|A \cap Q| \geq |A|/C_1(K),$$

and

$$|Q| \leq C_2(K)|A|,$$

where $C_1, C_2$ depends on $K$ polynomially.

It is known (Sanders, 2012) for $C_1(K) \sim C_2(K) \sim \exp(\log^4(K))$. 
Additive energy

Let \( A, B \subseteq G \) be sets. The (common) additive energy of \( A \) and \( B \)

\[
E(A, B) = E_2(A, B) := |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.
\]

If \( A = B \) then write \( E(A) \) for \( E(A, A) \).

Example, \( E(A) \) large

\( A \) is an arithmetic progression (\( \mathbb{Z} \)) or subspace (\( \mathbb{F}_2^n \)).

If \( |A + A| \leq K|A| \) then \( E(A) \geq |A|^3 / K \).
Theorem (Balog–Szemerédi–Gowers)

Let $G$ be an abelian group, and $A \subseteq G$ be a finite set. Suppose that $E(A) \geq |A|^3/K$. Then there is $A_* \subseteq A$ such that

$$|A_*| \geq |A|/C_1(K),$$

and

$$|A_* + A_*| \leq C_2(K)|A_*|,$$

where $C_1, C_2$ depend on $K$ polynomially.

So, firstly, we find a structural subset and, secondly, all bounds are polynomial.
So, $|A + A| \leq K|A| \Rightarrow E(A) \geq |A|^3/K$.
But $E(A) \geq |A|^3/K \Rightarrow |A_* + A_*| \leq C(K)|A_*|$ for some polynomially large $A_*$. 

Can we have it for the whole $A$?

**Example**

$A \subseteq \mathbb{F}_2^n,$

\[ A = Q \bigcup \Lambda, \]

where $Q$ is a subspace, $|Q| \sim E^{1/3}(A)$ and $\Lambda$ is a basis ($|\Lambda| \sim |A|$).

$E(Q) \sim E(A)$ but $|A + A| \geq |\Lambda + \Lambda| \gg |A|^2$. 
Example, again

\[ A \subseteq \mathbb{F}_2^n, \]

\[ A = Q \bigcup \Lambda, \]

where \( Q \) is a subspace, \(|Q| \sim E^{1/3}(A)\) and \( \Lambda \) is a basis (\(|\Lambda| \sim |A|\)).

\[ E(Q) \sim E(A) \] and, similarly, \( E(A, Q) \sim E(A) \). Hence

\[ \frac{E(A, Q)}{|Q|} > \frac{E(A, A)}{|A|} = \frac{E(A)}{|A|}. \]
Convolutions

\[(g \circ h)(x) := \sum_y g(y)h(x + y),\]

\[(\chi_A \circ \chi_B)(x) := |\{a - b = x : a \in A, b \in B\}|.\]

Consider the hermitian positively defined operator (matrix)

\[T(x, y) = (A \circ A)(x - y)A(x)A(y),\]

where \(A(x)\) is the characteristic function \(\chi_A\) of the set \(A\), i.e. \(A(x) = 1, x \in A\) and \(A(x) = 0\) otherwise.
Recall

\[ T(x, y) = (A \circ A)(x - y)A(x)A(y). \]

We have

\[ \langle TA, A \rangle = \sum_{x, y} (A \circ A)(x - y)A(x)A(y) = \|A \circ A\|_2^2 = E(A), \]

and, similarly,

\[ \left\langle T \frac{A(x)}{|A|^{1/2}}, \frac{A(x)}{|A|^{1/2}} \right\rangle = \frac{E(A)}{|A|} < \left\langle T \frac{Q(x)}{|Q|^{1/2}}, \frac{Q(x)}{|Q|^{1/2}} \right\rangle = \frac{E(A, Q)}{|Q|}. \]

Thus, the action of \( T \) on (normalized) \( Q \) is larger than the action of \( T \) on (normalized) \( A \).
Let
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|A|} > 0 \]
be the spectrum of \( T \) and
\[ f_1, f_2, \ldots, f_{|A|} \]
the correspondent eigenfunctions.

By Courant–Fisher Theorem
\[ \mu_1 = \max_{\|f\|_2 = 1} \langle Tf, f \rangle. \]

Thus, \( f_1 \) 'sits' on \( Q \) not \( A \)!

Here \( A = Q \bigcup \Lambda \).
Conjecture

The structured pieces of $A \subseteq G$ are (essential) supports of the eigenfunctions of $T$.

Holds

- not for any $A$, $A$ should be a 'popular difference set':

$$A = \{ x \ : \ (B \circ B)(x) \geq c|B| \}$$

for some $B$, $c = c(K) > 0$

- may be we need in some another 'weights'.
Let $\mathbf{G}$ be an abelian group, and $A \subseteq \mathbf{G}$ be a finite set. Take any real function $g$ such that $g(-x) = g(x)$. Put

$$T^g_A(x, y) = g(x - y)A(x)A(y).$$

Let

$$\mu_1(T^g_A) \geq \mu_1(T^g_A) \geq \cdots \geq \mu_{|A|}(T^g_A)$$

be the spectrum of $T^g_A$ and

$$f_1, f_2, \ldots, f_{|A|}$$

the correspondent eigenfunctions.
\[ T^g_A(x, y) = g(x - y)A(x)A(y). \]

**Examples**

- If \( A = G, \ g(x) = B(x), \ B \subseteq G \) then \( T^B_G \) the adjacency matrix of Cayley graph defined by \( B \).
- If \( g(x) = B(x) \) and \( A \) is any set then \( T^B_A \) is a submatrix of Cayley graph.
- Put \( g(x) = (A \circ A)(x). \) Then \( T = T^{A \circ A}_A \). Always

\[ \mu_1(T) \geq \frac{E(A)}{|A|}. \]
Trace of $T_A^g$ and $T_A^g(T_A^g)^*$

\[ |A| g(0) = \sum_{j=1}^{\left| A \right|} \mu_j(T_A^g), \]

\[ \sum_{z} \left| g(z) \right|^2 (A \circ A)(z) = \sum_{j=1}^{\left| A \right|} \left| \mu_j(T_A^g) \right|^2. \]

Example

Let $T = T_A^{A \circ A}$. Then

\[ \sum_{j=1}^{\left| A \right|} \left| \mu_j(T_A^g) \right|^2 = \sum_{z} (A \circ A)^3(z) := E_3(A). \]
Structural $E_2, E_3$ result

Theorem (Shkredov, 2013)

Let $A \subseteq G$ be a set, $E(A) = |A|^3/K$, and $E_3(A) = M|A|^4/K^2$. Then there is $A_* \subseteq A$ s.t.

$|A_*| \geq M^{-C}|A|$, 

and for any $n, m$

$|nA_* - mA_*| \leq K \cdot M^{C(n+m)}|A_*|$. 

Let $Q \subseteq \mathbb{F}_2^n$ be a subspace, $A \subseteq Q$ be a random subset s.t.

$|A| = |Q|/K \Rightarrow E(A) \sim |A|^3/K$, $E_3(A) \sim |A|^4/K^2 \Rightarrow M \sim 1$. 

$|A - A| \sim K|A| \sim |Q|$ as well as $|nA - mA| \sim |Q|$. 

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Let $a \leq b$ and $M \geq 1$.

$a$ and $b$ are $M$–critical if $a \leq b \leq Ma$.

**Structural $E_2, E_3$ result, again**

Let $A \subseteq G$ be a set, and $E_2(A)$, $E_3(A)$ be $M$–critical

$$\frac{E_2^2}{|A|^2} \leq E_3 \leq \frac{ME_2^2}{|A|^2}. $$

Then there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t.

- $A_* - A_*$ has small doubling,
- $|A_* - A_*| \sim_M \frac{|A|^4}{E(A)}$. 

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The additive energy and the eigenvalues
Let $\Gamma \subseteq \mathbb{F}_q^*$ be a subgroup, $q = p^s$, $|\Gamma|$ divides $q - 1$, $n = \frac{q-1}{|\Gamma|}$, $g$ be a primitive root. Then

$$\Gamma = \{1, g^n, g^{2n}, \ldots, g^{(t-1)n}\},$$

Consider the orthonormal family of multiplicative characters on $\Gamma$

$$\chi_\alpha(x) = |\Gamma|^{-1/2} \cdot \Gamma(x) e^{\frac{2\pi i \alpha l}{|\Gamma|}}, \quad x = g^l, \quad 0 \leq l < |\Gamma|. $$
Lemma

Let $\Gamma \subseteq \mathbb{F}_q^*$ be a subgroup, $g$ be any real even $\Gamma$–invariant function 

$$g(\gamma x) = g(x), \quad \gamma \in \Gamma.$$ 

Then $\chi_\alpha$, $\alpha = 0, 1, \ldots, |\Gamma| - 1$ are eigenfunctions of $T^g_\Gamma$.

In particular

$$E(\Gamma) = |\Gamma| \mu_1(T^g_\Gamma),$$

$$E(\Gamma) = \max_{f : \|f\|_2 = |\Gamma|} E(\Gamma, f),$$

and

$$E(\Gamma, A) \geq E(\Gamma) \frac{|A|^2}{|\Gamma|^2}, \quad A \subseteq \Gamma.$$
The additive energy of subgroups

Theorem (Konyagin, 2002)

Let \( \Gamma \subseteq \mathbb{F}_p \) be a multiplicative subgroup, \( |\Gamma| \ll p^{2/3} \). Then

\[
E(\Gamma) := |\{g_1 + g_2 = g_3 + g_4 : g_1, g_2, g_3, g_4 \in \Gamma\}| \ll |\Gamma|^{5/2}.
\]

Theorem (Shkredov, 2012)

Let \( \Gamma \subseteq \mathbb{F}_p \) be a multiplicative subgroup, \( |\Gamma| \ll p^{3/5} \). Then

\[
E(\Gamma) := |\{g_1 + g_2 = g_3 + g_4 : g_1, g_2, g_3, g_4 \in \Gamma\}| \ll |\Gamma|^{5/2-\varepsilon_0}.
\]
Why?

Suppose that $E(\Gamma) \sim |\Gamma|^{5/2} = |\Gamma|^3/K$, $K \sim |\Gamma|^{1/2}$.

**Lemma**

We have

$$E_3(\Gamma) \ll |\Gamma|^3 \log |\Gamma| = \frac{M|\Gamma|^4}{K^2},$$

where $M \sim \log |\Gamma|$.

Thus by our structural result $\Gamma$ stabilized under addition but $k\Gamma = \mathbb{F}_p$ (more delicate arguments give the better bounds).

Thus, $E(\Gamma) = |\Gamma|^{5/2-\varepsilon_0}$, $\varepsilon_0 > 0$. 
Convex sets

\[ A = \{a_1 < a_2 < \cdots < a_n\} \subseteq \mathbb{R} \text{ is called convex if} \]

\[ a_{i+1} - a_i > a_i - a_{i-1} \quad \text{for all} \ i. \]

Example.

\[ A = \{1^2, 2^2, \ldots, n^2\}. \]
Theorem (Iosevich, Konyagin, Rudnev, Ten, 2006)
Let \( A \subseteq \mathbb{R} \) be a convex set. Then
\[
E(A) \ll |A|^{5/2}.
\]

Theorem (Shkredov, 2012–2013)
Let \( A \subseteq \mathbb{R} \) be a convex set. Then
\[
E(A) \ll |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A|.
\]

Proof: a formula for higher moments of eigenvalues and estimation of eigenvalues.
Further applications

Doubling constants for:

- multiplicative subgroups,
- convex sets,
- sets with small product sets $|AA|$.

Combinatorial methods = counting spectrum of $T_A^{A\pm A}$. 
Theorem (Elekes–Nathanson–Rusza, 1999)

Let $A \subseteq \mathbb{R}$ be a convex set. Then

$$|A + A| \gg |A|^{3/2},$$

Theorem (Schoen–Shkredov, 2011)

Let $A \subseteq \mathbb{R}$ be a convex set. Then

$$|A - A| \gg |A|^{8/5 - \varepsilon},$$

and

$$|A + A| \gg |A|^{14/9 - \varepsilon}.$$
Heilbronn’s exponential sums

Let $p$ be a prime number.
Heilbronn’s exponential sum is defined by

$$S(a) = \sum_{n=1}^{p} e^{2\pi i \cdot \frac{an^p}{p^2}}.$$  

Fermat quotients defined as

$$q(n) = \frac{n^{p-1} - 1}{p}, \quad n \neq 0 \pmod{p}.$$
Theorem (Heath–Brown, Konyagin, 2000)

Let \( p \) be a prime, and \( a \neq 0 \pmod{p} \). Then

\[
|S(a)| \ll p^{7/8}.
\]

In the proof an upper bound of the additive energy of Heilbronn’s subgroup

\[
\Gamma = \{m^p : 1 \leq m \leq p - 1\} = \{m^p : m \in \mathbb{Z}/p^2\mathbb{Z}, m \neq 0\}
\]

was used.
Via the additive energy estimate and the operator $T_{A^o A}^A$.

**Theorem (Shkredov, 2012–2013)**

Let $p$ be a prime, and $a \neq 0 \pmod{p}$. Then

$$|S(a)| \ll p^{\frac{7}{8} - \varepsilon_0} p.$$ 

Via direct calculations and the operator with "dual" weights $\hat{T}_A^A$.

**Theorem (Shkredov, 2013)**

Let $p$ be a prime, and $a \neq 0 \pmod{p}$. Then

$$|S(a)| \ll p^{\frac{5}{6} \log \frac{1}{6}} p.$$
By $l_p$ denote the smallest $n$ such that $q(n) \neq 0 \, (\text{mod} \ p)$.

**Theorem (Bourgain, Ford, Konyagin, Shparlinski, 2010)**

One has

$$l_p \leq (\log p)^{\frac{463}{252} + o(1)}$$

as $p \to \infty$.

Previously Lenstra (1979): $l_p \ll (\log p)^{2+o(1)}$.

**Theorem (Shkredov, 2012–2013)**

One has

$$l_p \leq (\log p)^{\frac{463}{252} - \varepsilon_0 + o(1)}, \quad \varepsilon_0 > 0,$$

$\varepsilon_0$ is an absolute (small) constant.
Other applications are:

- discrepancy of Fermat quotients,
- new bound for the size of the image of $q(n)$,
- estimates for Ihara sum,
- better bounds for the sums

$$\sum_{n=1}^{k} \chi(q(n)), \quad \sum_{n=1}^{k} \chi(nq(n)).$$

- Surprising inequalities between $E(A)$ and $E_s(A), \ s \in (1, 2]$. 
$f : \mathbf{G} \rightarrow \mathbb{C}$ be a function, $\hat{\mathbf{G}} = \{\xi\}$, $\xi : \mathbf{G} \rightarrow \mathbb{D}$ be the group of homomorphisms.

### Fourier transform

$$\hat{f}(\xi) := \sum_{x} f(x)\overline{\xi(x)}, \quad \xi \in \hat{\mathbf{G}}.$$

### Properties of $T_{A}^{g}$

We have

- $\text{Spec} \left( T_{A}^{B} \right) = \text{Spec} \left( T_{B}^{A_{c}} \right) = \text{Spec} \left( T_{B}^{A_{c}} \right)$.
- $\text{Spec} \left( T_{A}^{B}(T_{A}^{B})^{*} \right) = |\mathbf{G}| \cdot \text{Spec} \left( T_{A}^{|B|^{2}} \right)$

Here $f_{c}(x) := f(-x)$ for any function $f : \mathbf{G} \rightarrow \mathbb{C}$. 

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Chang Theorem

Let $G$ be an abelian group, and $A \subseteq G$ be a finite set.

**Dissociated sets**

A set $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subseteq G$ is called *dissociated* if any equation of the form

$$\sum_{j=1}^{d} \varepsilon_j \lambda_j = 0,$$

where $\varepsilon_j \in \{0, \pm 1\}$ implies $\varepsilon_j = 0$ for all $j$.

**Exm.** $G = \mathbb{F}_2^n$. 

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Proof of Chang Theorem via operators

**Chang theorem**

For any dissociated set $\Lambda$, any set $A \subseteq G$, $|A| = \delta|G|$ and an arbitrary function $f$, $\text{supp } f \subseteq A$

\[
\sum_{\xi \in \Lambda} |\hat{f}(\xi)|^2 \leq |A| \log(1/\delta) \cdot \|f\|_2^2.
\]

\[
\sum_{\xi \in \Lambda} |\hat{f}(\xi)|^2 = \langle T_A^\Lambda f, f \rangle \leq \mu_1(T_A^\Lambda) \|f\|_2^2 = \mu_1(T_A^\Lambda) \|f\|_2^2
\]
Estimating $\mu_1(T^A_\Lambda)$

$\text{supp } w \subseteq \Lambda$, $k \sim \log(1/\delta)$.

$$\mu_1(T^A_\Lambda) := \max_{\|w\|_2=1} \langle T^A_\Lambda w, w \rangle = \sum_x |\hat{w}(x)|^2 A(x).$$

$$\mu_1^k(T^A_\Lambda) \leq \sum_x |\hat{w}(x)|^{2k} \cdot |A|^{k-1}$$

$$\sum_x |\hat{w}(x)|^{2k} = |G| \sum_{x_1+\cdots+x_k=x'_1+\cdots+x'_k} w(x_1) \cdots w(x_k) \overline{w(x'_1)} \cdots \overline{w(x'_k)}$$

$$\leq NC^k k! \|w\|_2^{2k} = NC^k k!.$$
Advantages of the approach

- Relaxation of dissociativity
  \[ \sum_{j} \varepsilon_j \lambda_j = 0 \quad \text{instead of} \quad \sum_{j=1}^{\Lambda} \varepsilon_j \lambda_j = 0. \]

- Very weak dissociativity (\( \sum_{j} |\varepsilon_j| \leq C \)).

- Other operators \( T_{g}^A \). Higher moments
  \[ \sum_{\xi \in \Lambda} |\hat{A}(\xi)|^l, \quad l > 2, \]

  dual Chang theorems

  \[ \sum_{x \in \Lambda} (A_1 \ast A_2)^2(x) \ll |A_1||A_2| \log (\min\{|A_1|, |A_2|\}) . \]
Structural $E_2, E_3$ result, again

Let $A \subseteq G$ be a set, and $E_2, E_3$ be $M$–critical
Then there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t.
• $A_* - A_*$ has small doubling,
• $|A_* - A_*| \sim_M \frac{|A|^4}{E(A)}$.

Balog–Szemerédi–Gowers, again

$E(A)$ and $|A|$ are $M$–critical ($|A|^3 \ll_M E(A) \leq |A|^3$) iff there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t. $A_*$ has small doubling.

It is a (rough) criterium.
\[ E_k(A) = \left| \{ a_1 - a'_1 = a_2 - a'_2 = \cdots = a_k - a'_k : a_1, a'_1, \ldots, a_k, a'_k \in A \} \right| \]

\[ E_1(A) = |A|^2, \ E_2(A) = T_2(A). \]

\[ T_k(A) := \left| \{ a_1 + \cdots + a_k = a'_1 + \cdots + a'_k : a_1, \ldots, a_k, a'_1, \ldots, a'_k \in A \} \right| \]

\[ \sum_x (\widehat{A \circ \hat{A}})^{2k}(x) = |G|^{2k+1} T_k(A) \text{ and } T_k(|\hat{A}|^2) = |G|^{2k-1} E_{2k}(A). \]

\[ \left( \frac{E_{3/2}(A)}{|A|} \right)^{2k} \leq E_k(A) T_k(A). \]
Criterium for $A, 2A, 3A, \ldots$ stops at the second step (with right doubling constant).

**Criterium**

$T_4(A)$ and $E(A)$ are $M$–critical

$$T_4(A) \sim_M |A|^2 E(A)$$

iff there is $A_* \subseteq A$, $|A_*| \gg_M |A|$ s.t.

- $A_* - A_*$ has small doubling,
- $|A_* - A_*| \sim_M \frac{|A|^4}{E(A)}$. 
Zoo

1) random sets
2) sets with small doubling

Further families

3) $2A$ has small (right) doubling
4) $A = H + \mathcal{L}$, $H$ has small doubling, $\mathcal{L}$ is ”dissociated”
5) $A = H_1 \bigcup H_2 \bigcup \cdots \bigcup H_k$, all $H_j$ with small doubling

and further

6) Intermediate between 4) and 5) (Bateman–Katz)
7) Schoen potatoes and so on.
Characterization of $H + \mathcal{L}$

**Criterion**

$E_2(A)$ and $E_3(A)$ are $M$–critical

\[ E_3(A) \sim_M |A|E(A) \]

iff there are two sets $H, \mathcal{L}$ such that

\[ |H \cap A| \gg_M \frac{E(A)}{|A|^2}, \quad |\mathcal{L}| \ll_M \frac{|A|}{|H|}, \]

\[ |H - H| \ll_M |H|, \]

and

\[ |A \cap (H + \mathcal{L})| \gg_M |A|. \]
Energy of sumsets

\[ A = H + \mathcal{L} \subseteq \mathbb{F}_2^n, \ H \text{ is a subspace, } \mathcal{L} \text{ is a basis, } |\mathcal{L}| = K. \]
Then \( \mathcal{E}(A) \sim |A|^3/K, \)

\[ D = A + A = H + (\mathcal{L} + \mathcal{L}). \]

Katz–Koester:

\[ \mathcal{E}(D) \geq |A|^2|D| = K|A|^3, \quad \mathcal{E}_3(D) \geq K|A|^4. \]

Lower bound for \( \mathcal{E}_3(A \pm A) \)

Let \( A \) be a set, \( D = A \pm A, \ |D| = K|A|. \) Suppose that \( \mathcal{E}(A) \ll |A|^3/K + \) some technical conditions. Then

\[ \mathcal{E}_3(D) \gg K^{7/4}|A|^4. \]

Upper bound \( \mathcal{E}_3(D) \ll K^2|A|^4. \)
Put \( A_s = A \cap (A + s) \).

\[
\|A\|_{U^3} := \sum_{s} E(A_s).
\]

**Criterium**

\( E_3(A) \) and \( \|A\|_{U^3} \) are \( M \)-critical

\[
E_3(A) \sim_M \|A\|_{U^3}
\]

+ some technical conditions. iff

\[
A = H_1 \Delta H_2 \Delta \cdots \Delta H_k,
\]

all \( H_j \subseteq A_s \) with small doubling, \( |H_j| \gg_M |A_s| \).
Weak counterexample to Gowers construction

Recall $A_s = A \cap (A + s)$.

Existence of $A_s$ with small energy

Let $A \subseteq G$ be a set, $E(A) = |A|^3/K$,

$$|A_s| \leq \frac{M|A|}{K},$$

where $M \geq 1$ is a real number. Then $\exists s \neq 0, |A_s| \geq \frac{|A|}{2K}$ s.t.

$$E(A_s) \ll \frac{M^{93/79}}{K^{1/198}} \cdot |A_s|^3.$$
Dichotomy for sumsets

\[ D = A - A. \]

For \( a_1, \ldots, a_k \in A \), we have

\[ A \subseteq (D + a_1) \cap (D + a_2) \cap \cdots \cap (D + a_k). \]

Dichotomy

Suppose that for any \( x_1, \ldots, x_k \)

\[ |A| \leq |(D + x_1) \cap (D + x_2) \cap \cdots \cap (D + x_k)| \leq |A|^{1+\varepsilon}. \]

Then either \( E_k(A) \ll |A|^\varepsilon, k |A|^k \) or \( E(A) \gg |A|^\varepsilon, k |A|^3 \).
Concluding remarks

- Studying the eigenvalues and the eigenfunctions of $T$, we obtain the information about the initial object $E(A)$.

- Our approach tries to emulate Fourier analysis onto $A$ not on the whole group $G$.

Conjecture, again

The structured pieces of $A \subseteq G$ are (essential) supports of the eigenfunctions of $T$. 
Considered examples

In all examples above (multiplicative subgroups, convex sets and so on), we have

$$\mu_1 \gg \mu_2 \geq \mu_3 \geq \ldots, \quad \mu_1 \text{ dominates.}$$

PFRC case

If our set $A$ is a sumset $A = B - B$, $|A| = K|B|$ (or popular difference set) then

$$\mu_1 \sim \mu_2 \sim \ldots \sim \mu_k \gg \mu_{k+1} \geq \ldots, \quad k \sim K.$$ 

So, there many roughly equal eigenvalues. The correspondent eigenfunctions lives on ”disjoint” (sub)sets of $B - b$, $b \in B$. 

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Thank you for your attention!