Bounded Degree High Dimensional Expanders

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Expander graphs and their Cheeger constant

Graph: $X=(V,E)$.

$X$ is an $\varepsilon$-expander iff its Cheeger constant, $h(x) \geq \varepsilon$.

$$h(x) = \min_{\phi \neq A \subseteq V} \frac{|E(A, \overline{A})|}{\min(|A|,|\overline{A}|)} = \frac{|\delta(A)|}{|[A]_{B0}|}$$

$\delta(A)$: Edges whose vertices in $A$ sum to 1 (mod 2);
Cocycles - $\{V, \phi\}$: $|\delta(V)|=0; |\delta(\phi)|=0$,
$|[A]_{B0}|$ - the distance of $A$ from cocycles.
What is the generalization of Cheeger to higher dimensional simplicial complexes?
Generalizing Cheeger to 2-dimensions

[Linial Meshulam, Gromov]

2 dimensional simplicial complex: \( X = (V,E,T) \)

\[
\varepsilon_1^{LM}(X) = \min \phi \neq A \subseteq E, A \notin B^1(X) = \frac{|\delta(A)|/|T|}{|[A]_{B^1}|/|E|}
\]

\( \delta(A) \): \( \Delta \)'s whose sum of edges in \( A \) is 1 (mod 2);

Cocycles : \(|\delta(A)|=0\): cuts

\( |[A]_{B^1}| \) - the distance of \( A \) from the cuts - \( B^1(X) \).
Brief introduction to $F_2$-cohomology

Let $X$ be a $d$-dimensional simplicial complex on $V$ vertices.

$F \subseteq V$, $F \in X$, $F' \subseteq F$ implies $F' \in X$.

$X(i) = \{F \in X \mid |F| = i + 1\}$.

$X(-1) = \{\Phi\}$, $X(0)$ the vertices, $X(1)$ the edges, $X(2)$ the triangles, etc.

$C^i = C^i(X, F_2)$ is the $F_2$ vector space of functions from $X(i)$ to $F_2$. 
The coboundary map

\[ \delta_i: C^i (X,F_2) \rightarrow C^{i+1} (X,F_2) \]

\[ \delta_i(f)(G) = \sum_{F \subseteq G; |F|=|G|-1} f(F) \] where \( f \in C^i \) and \( G \in X(i+1) \).

\[ B^i (X,F_2) = \text{Image}( \delta_{i-1} ) \] the space of i-coboundaries.

E.g., \( B^0 (X,F_2) = \{0,1\} \), \( B^1 (X,F_2) \) is the "cut-space" of \( X \).

\[ Z^i (X,F_2) = \text{Ker}(\delta_i) \] the space of i-cocycles.

\[ \delta_i \circ \delta_{i-1} = 0, \text{Thus, } B^i \subseteq Z^i, H^i = Z^i / B^i \] - i-cohomology.
Example of non trivial cocycle \((H^1(X) \neq 0)\)

2 dimensional simplicial complex: \(X=(V,E,T)\)

![Diagram of a 2-dimensional simplicial complex.](image)
Definition of high dimensional expansion

$X$ - $d$ dimensional complex. For $i=0,\ldots, d-1$, $f \in C^i(X,F_2)$

Normalized support: $\|f\| = |\{F \in X(i) | f(F) \neq 0\}| / |X(i)|$.

Normalized distance from coboundaries/cocycles

$\|[f]_B\| = \text{dist}(f, B^i(X,F_2))$; $\|[f]_Z\| = \text{dist}(f, Z^i(X,F_2))$

[Gromov] $\varepsilon_i^G(X) = \min_{f \in C^i(X,F_2) \setminus Z^i(X,F_2)} \frac{\| \delta_i f \|}{\|[f]_Z\|}$

$X$ is a $\varepsilon$-cocycle expander if for all $i$, $\varepsilon_i^G(X) > \varepsilon > 0$

[Linial Meshulam]: $\varepsilon_i^{LM}(X) = \min_{f \in C^i(X,F_2) \setminus B^i(X,F_2)} \frac{\| \delta_i f \|}{\|[f]_B\|}$

$X$ is a $\varepsilon$-coboundary expander if for all $i$, $\varepsilon_i^{LM}(X) > \varepsilon > 0$
On the definitions of high dimensional expanders

$X$ - $d$ dimensional complex. For $i=0,\ldots,d-1$, $f \in \mathbb{C}^i(X,F_2)$

$$\varepsilon_i^G(X) = \min_{f \in \mathbb{C}^i(X,F_2) \setminus \mathbb{Z}^i(X,F_2)} \frac{||\delta_i f||}{||[f]_Z||}$$

$$\varepsilon_i^{LM}(X) = \min_{f \in \mathbb{C}^i(X,F_2) \setminus \mathbb{B}^i(X,F_2)} \frac{||\delta_i f||}{||[f]_B||}$$

- Coboundary expander is a cocycle expander: $\varepsilon_i^G(X) \geq \varepsilon_i^{LM}(X)$
- If no non-trivial cocycles, i.e., $H^i(X) = 0$, then: $\varepsilon_i^{LM}(X) = \varepsilon_i^G(X)$
- Generalize Cheeger: $\varepsilon_0^{LM}(X) = h(X)|X(0)|/|X(1)|$
- If $X$ is connected $\varepsilon_0^{LM}(X) = \varepsilon_0^G(X)$

Linial Meshulam: study theory of random graphs in higher dims

Gromov: study topological overlapping.
Background - Topological overlapping

[Boros, Furedi]: For any set $P$ of $n$ points in $\mathbb{R}^2$ there exists a point $z \in \mathbb{R}^2$ contained in at least $2/9$ of the triangles determined by $P$.

[Barany]: For any set $P$ of $n$ points in $\mathbb{R}^d$ there exists a point $z \in \mathbb{R}^d$ contained in at least $\varepsilon(>0)$-fraction of the $d$-simplicies determined by $P$.

[Gromov]: The above results hold, with any drawing of the triangles (simplicies) (i.e., any continues triangle through 3 points).
Topological overlapping

[Gromov]: A $d$-dimensional complex $X$ has $\varepsilon$-topological (geometric) overlapping if for every $f:X(0) \to \mathbb{R}^d$ and every continuous (affine) extension of it to $f':X \to \mathbb{R}^d$ there exists $z \in \mathbb{R}^d$ covered by at least $\varepsilon$-fraction of the images of the facets in $X(d)$ under $f'$.

[Boros, Furedi] and [Barany]: The complete $d$-dimensional complex has a geometric overlapping property.

[Gromov]: The complete $d$-dimensional complex has a topological overlapping property.

The overlapping is a property of complex, not of $\mathbb{R}^d$. 
Gromov’s Question

**Question [Gromov]**: Is there a bounded degree $d$-dimensional complex with a $\varepsilon$-geometric/topological overlapping property for $\varepsilon > 0$?

**[Fox,Gromov,Lafforgue, Naor, Pach]**: There are bounded degree $d$-dimensional complexes with geometric overlapping.

The question about bounded degree $d$-dimensional complexes with the topological overlapping property remained open.
High dim expanders and topological overlapping

**Thm [Gromov]:** A $d$-dimensional complex $X$ that is an $\varepsilon$-coboundary expander has $\varepsilon'$-topological overlapping property.

Trivially holds for $d=1$.

Gromov result showing that the complete $d$-dimensional complex has the topological overlapping property is obtained by showing that this complex is a coboundary expander.
Major questions

Is there a bounded degree d-dimension complex that is an ε-coboundary expander? (known only for d=1)

Is there a bounded degree d-dimension complex that has an ε-topological overlapping property?

Is there a bounded degree d-dimension complex that is an ε-cocycle expander?
Our results

Explicit 2-dimensional bounded degree complex with the topological overlapping property.

Explicit 2-dimensional bounded degree complex which is a cocycle expander.

Assuming Serre’s conjecture on the congruence subgroup property our complex is also a coboundary expander.
Ramanujan complexes

[Lubotzky, Samuels, Vishne] constructed d-dimensional bounded degree complexes that generalize the Ramanujan graph construction of [Lubotzky, Philips, Sarnak].

[LSV] complexes obtained as a quotient of the Bruhat-Tits building associated with $\text{PGL}_{d+1}(F)$ where $F$ is a local field,

Our 2-dimensional complex is the 2-skeleton of the 3-dimensional Ramanujan complex.
Systoles and topological overlapping

Some of our complexes have $H^i(X) \neq 0$, i.e., not coboundary expanders!
How to prove topological overlapping for them?

**Systole:** minimal size of a non-trivial cocycle

$$\text{syst}_i(X) = \min_{f \in Z^i(X,F_2) \setminus B^i(X,F_2)} \frac{|f|}{|X(i)|}$$

A $d$-dim complex $X$ has an $\eta$-systole if $\text{syst}_i(X) > \eta$ for $i=0, \ldots, d-1$.

[Kaufman, Wagner] (Generalizing Gromov to the case $H^i(X) \neq 0$):
d-dimensional complex $X$ with $\eta$-systole that is an $\varepsilon$-cocycle expander has $\varepsilon'$-overlapping property, $\varepsilon' = \varepsilon'(\varepsilon, \eta)$. 
**Thm**: Let $X$ be a 3-dim complex.

Assume for every $i=0,1,2$ every locally minimal $f \in C^i(X, F_2)$ with $\|f\| \leq \eta_i$: $\|\delta_i(f)\| = c\|f\|$ for $c > 0$.

Then the 2-skekelon of $X$ has **$\epsilon'$-overlapping property**, $\epsilon' = \epsilon'(\eta_0, \eta_1, \eta_2)$

**Conclusion**: A locally minimal cocycle $f \in Z^i(X, F_2)$ in $X$ sat $\|f\| = 0$ or $\|f\| > \eta_i > 0$

locally minimal is a relaxation of minimal. $f \in C^i(X, F_2)$ that is locally minimal satisfies for every $f' \in C^i(X, F_2)$ s.t. $\delta_i(f) = \delta_i(f')$. $\|f\| \leq \|f'\|$.
Topological overlapping from Isoparametric inequalities

Let $X$ be 3-dim complex satisfying the iso-inequalities, $Y$ - its 2 skeleton

$\eta$-systole : $\text{syst}_i(Y) > \eta$ : immediate, since a minimal systole is locally min

$\epsilon$-cocycle expander: $f \in C^i(Y,F_2) \backslash Z^i(Y,F_2) = C^i(X,F_2) \backslash Z^i(X,F_2)$ \[||\delta_1(f)||/ ||[f]_Z|| > \epsilon.\]

If $||\delta_1(f)|| > \eta_2$ done.

Otherwise if $\delta_1(f)$ is locally minimal then by iso-inequalities $||\delta_1(f)||=0$ done

Remain: if $||\delta_1(f)|| \leq \eta_2$ , not locally minimal,

We show: $\delta_1(f) = \delta_1(h) + \delta_1(g)$, $\delta_1(h)$ is locally min $||[g]_Z|| < c\|\delta_1(f)\|$

Note: $\delta(\delta(h)) = 0$, $\delta_1(h)$ - loc min 2-cocycle, thus $||\delta_1(h)||=0$ or $||\delta_1(h)|| > \eta_2$

Thus: $||[h]_Z|| \leq 1/\eta_2 \|\delta_1(h)|| \leq 1/\eta_2 \|\delta_1(f)||$

$||\delta_1(f)||/ ||[f]_Z|| \geq 1/(c+1/\eta_2)$
Isoparametric inequalities

**Thm (main technical theorem):** Let $X$ be 3-dimensional Ramanujan complex. For every $i=0,1,2$ every locally minimal $f \in C^i(X,F_2)$ with $\|f\| \leq \eta_i$: $\|\delta_i(f)\|=c\|f\|$ for $c>0$. 
Baby version: Iso-inequalities for 2-dim complexes

Let $X = (V,E,T)$ be a 2-dimensional Ramanujan complex:

Their skeleton graph is a degree $Q$ almost Ramanujan expander. The link of every vertex is a degree $q \sim \sqrt{Q}$ Ramanujan expander.
Proof idea: every locally minimal 1-cocycle is large

1-cochain (a collection of edges) is a subgraph in the skeleton graph of X.

A sub-linear sub-graph of the expander graph X has an average degree $< \lambda(X) << Q$ (as X is almost Ramanujan).

In a sub-linear sub-graph of the expander graph X most vertices are thin (with degree $<<Q$). Rest are thick.
Proof idea: every locally minimal 1-cocycle is large

Link of a vertex is a q regular graph.
Every vertex in link corresponds to an edge.
Every edge in link corresponds to a $\Delta$.
A vertex $v$ defines a set $S_v$ in its link,
By local minimality, for every $v$, $|S_v| \leq Q/2$,
A thin vertex $v$ defines a small set $S_v$ in its link.
If $v$ is thin, $S_v$ expands extremely well in the link graph
If $v$ is thick, $S_v$ expands well, since $|S_v| \leq Q/2$,
Many $\Delta$’s are colored once (as most vertices are thin)
Every sublinear locally minimal 1-cochain has many 1-colored $\Delta$’s.
Conditional Coboundary expanders

Our 2-dim complex is a cocycle expander and it has large systole, thus it has the topological overlapping property (by Gromov, Kaufman & Wagner thms).

If our complex had $H^i(Y) = 0$ then it would be also a coboundary expander.

According to Serre’s conjecture, one can choose the 3 dimensional Ramanujan complex $X$, s.t., its 2 skeleton $Y$ has $H^i(Y) = 0$. 
Open

Unconditional bounded degree coboundary expanders

Are the 2-dim Ramanujan complexes cocycle expanders?

Generalize the result to d-dimensional bounded degree complexes.

Random constructions of bounded degree high dim expanders.
Thank You!!!