Bounded Degree High Dimensional Expanders

Tali Kaufman

Bar-Ilan University, Israel

Joint work with David Kazhdan and Alex Lubotzky

Expander graphs and their Cheeger constant

Graph: $X=(V,E)$.

X is an ε-expander iff its Cheeger constant, $h(x) \geq \varepsilon$.

$$h(x) = \min_{\phi \neq A \subseteq V} \frac{|E(A,\overline{A})|}{\min(|A|,|\overline{A}|)} = \frac{|\delta(A)|}{|[A]_{B0}|}$$

$\delta(A)$: Edges whose vertices in A sum to 1 (mod 2);

Cocycles - {**V**,**ϕ**} : $|\delta(V)|=0$; $|\delta(\phi)|=0$,

$|[A]_{B0}|$ - the distance of A from cocycles.

What is the generalization of Cheeger to higher dimensional simplicial complexes?

Generalizing Cheeger to 2-dimensions

[Linial Meshulam, Gromov]

2 dimensional simplicial complex: X=(V,E,T)

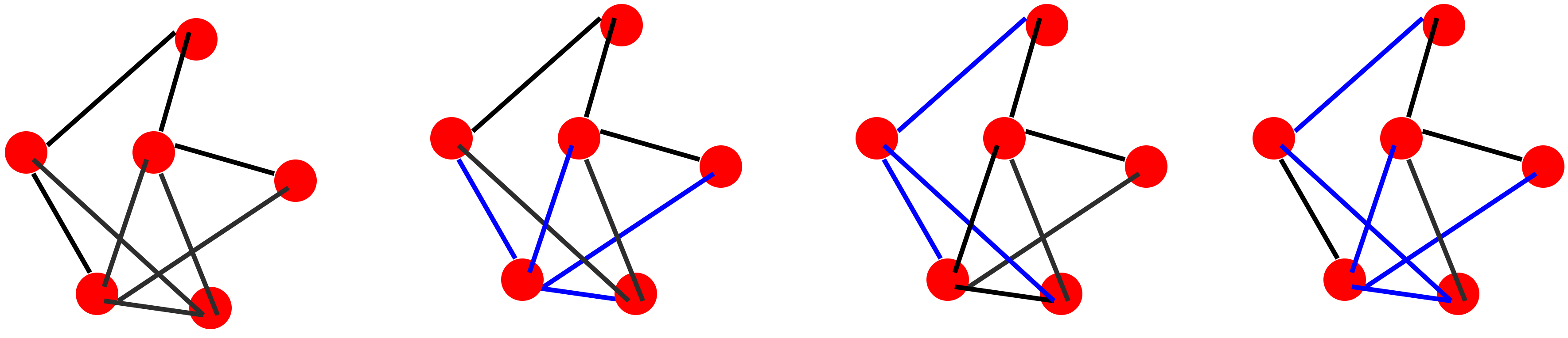
$$\varepsilon_1^{LM}(X) = \min_{\phi \neq A \subseteq E, A \notin B^1(X)} = \frac{|\delta(A)|/|T|}{|[A]_{B1}|/|E|}$$

δ(A): Δ's whose sum of edges in A is 1 (mod 2);

Cocycles :|δ(A)|=0: cuts

$|[A]_{B1}|$ - the distance of A from the cuts - $B^1(X)$.

Star: Star: Sum of stars: cuts



Brief introduction to F_2-cohomology

Let X be a d-dimensional simplicial complex on V vertices.

$F \subseteq V$, $F \in X$, $F' \subseteq F$ implies $F' \in X$.

$X(i) = \{F \in X \mid |F| = i+1\}$.

$X(-1) = \{\Phi\}$, X(0) the vertices, X(1) the edges, X(2) the triangles, etc.

$C^i = C^i(X, F_2)$ is the F_2 vector space of functions from X(i) to F_2.

The coboundary map

δ_i: $C^i(X,F_2) \rightarrow C^{i+1}(X,F_2)$

$\delta_i(f)(G)=\Sigma_{F \subseteq G;\ |F|=|G|-1} f(F)$ where $f \in C^i$ and $G \in X(i+1)$.

$B^i(X,F_2)$ =Image(δ_{i-1}) the space of i-coboundaries.

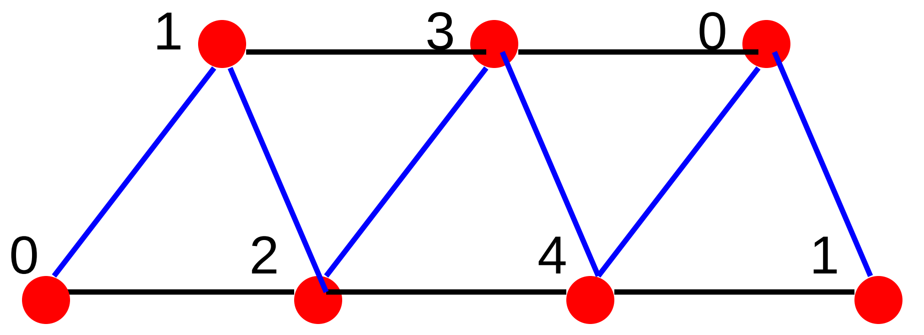
E.g., $B^0(X,F_2) = \{\mathbf{0},\mathbf{1}\}$, $B^1(X,F_2)$ is the "cut-space" of X.

$Z^i(X,F_2)$ =Ker(δ_i) the space of i-cocycles.

$\delta_i \circ \delta_{i-1} = 0$, Thus, $B^i \subseteq Z^i$, $H^i = Z^i / B^i$ - i-cohomology.

Example of non trivial cocycle ($H^1(X) \neq 0$)

2 dimensional simplicial complex: X=(V,E,T)



Definition of high dimensional expansion

X - d dimensional complex. For i=0,…, d-1, $f \in C^i(X,F_2)$

Normalized support: $||f|| = |\{F \in X(i) \mid f(F) \neq 0\}| / |X(i)|$.

Normalized distance from coboundaries/cocycles

$||[f]_B|| = \mathrm{dist}(f, B^i(X,F_2))$; $||[f]_Z|| = \mathrm{dist}(f, Z^i(X,F_2))$

[Gromov]

$$\varepsilon_i^G(X) = \min_{f \in C^i(X,F_2) \setminus Z^i(X,F_2)} \frac{||\delta_i f||}{||[f]_z||}$$

X is a ε-cocycle expander if for all i, $\varepsilon_i^G(X) > \varepsilon > 0$

[Linial Meshulam]:

$$\varepsilon_i^{LM}(X) = \min_{f \in C^i(X,F_2) \setminus B^i(X,F_2)} \frac{||\delta_i f||}{||[f]_B||}$$

X is a ε-coboundary expander if for all i, $\varepsilon_i^{LM}(X) > \varepsilon > 0$

On the definitions of high dimensional expanders

X - d dimensional complex. For i=0,…, d-1, $f \in C^i(X,F_2)$

$$\varepsilon_i^G(X) = \min_{f \in C^i(X,F_2) \setminus Z^i(X,F_2)} \frac{\| \delta_i f \|}{\| [f]_z \|} \qquad \varepsilon_i^{LM}(X) = \min_{f \in C^i(X,F_2) \setminus B^i(X,F_2)} \frac{\| \delta_i f \|}{\| [f]_B \|}$$

- Coboundary expander is a cocycle expander: $\varepsilon_i^G(X) \geq \varepsilon_i^{LM}(X)$
- If no non-trivial cocycles, i.e., $H^i(X) = 0$, then: $\varepsilon_i^{LM}(X) = \varepsilon_i^G(X)$
- Generalize Cheeger : $\varepsilon_0^{LM}(X) = h(X) |X(0)|/|X(1)|$
- If X is connected $\varepsilon_0^{LM}(X) = \varepsilon_0^G(X)$

Linial Meshulam: study theory of random graphs in higher dims

Gromov: study topological overlapping.

Background - Topological overlapping

[Boros, Furedi]: For any set P of n points in R^2 there exists a point $z \in R^2$ contained in at least 2/9 of the triangles determined by P.

[Barany]: For any set P of n points in R^d there exists a point $z \in R^d$ contained in at least $\varepsilon(>0)$-fraction of the d-simplicies determined by P.

[Gromov]: The above results hold, with any drawing of the triangles (simplicies) (i.e., any continues triangle through 3 points).

Topological overlapping

[Gromov]: A d-dimensional complex X has ε-topological (geometric) overlapping if for every $f:X(0) \to R^d$ and every continuous (affine) extension of it to $f':X \to R^d$ there exists $z \in R^d$ covered by at least ε-fraction of the images of the facets in X(d) under f'.

[Boros, Furedi] and **[Barany]:** The complete d-dimensional complex has a geometric overlapping property.

[Gromov]: The complete d-dimensional complex has a topological overlapping property.

The overlapping is a property of complex, not of R^d .

Gromov's Question

Question [Gromov] : Is there a bounded degree d-dimensional complex with a ε-geometric/topological overlapping property for ε>0?

[Fox,Gromov,Lafforgue, Naor, Pach]: There are bounded degree d-dimensional complexes with geometric overlapping.

The question about bounded degree d-dimensional complexes with the topological overlapping property remained open.

High dim expanders and topological overlapping

Thm [Gromov]: A d-dimensional complex X that is an ε-coboundary expander has ε'-topological overlapping property.

Trivially holds for d=1.

Gromov result showing that the complete d-dimensional complex has the topological overlapping property is obtained by showing that this complex is a coboundary expander.

Major questions

Is there a bounded degree d-dimension complex that is an ε-coboundary expander ? (known only for d=1)

Is there a bounded degree d-dimension complex that has an ε-topological overlapping property?

Is there a bounded degree d-dimension complex that is an ε-cocycle expander?

Our results

Explicit **2-dimensional bounded degree** complex with the **topological overlapping property.**

Explicit **2-dimensional bounded degree** complex which is a **cocycle expander.**

Assuming Serre's conjecture on the congruence subgroup property our complex is also a **coboundary expander.**

Ramanujan complexes

[Lubotzky, Samuels, Vishne] constructed d-dimensional bounded degree complexes that generalize the Ramanujan graph construction of **[Lubotzky,Philips, Sarnak].**

[LSV] complexes obtained as a **quotient of the Bruhat-Tits building** associated with $PGL_{d+1}(F)$ where F is a local field,

Our 2-dimensional complex is the **2-skeleton of the 3-dimensional Ramanujan complex**.

Systoles and topological overlapping

Some of our complexes have $H^i(X) \neq 0$, i.e., not coboundary expanders!

How to prove topological overlapping for them?

Systole: minimal size of a non-trivial cocycle

$$\mathrm{syst}_i(X) = \min_{f \in Z^i(X,F_2) \setminus B^i(X,F_2)} \frac{|f|}{|X(i)|}$$

A d-dim complex X has an η-systole if $\mathrm{syst}_i(X) > \eta$ for i=0, ..., d-1.

[Kaufman,Wagner] (Generalizing Gromov to the case $H^i(X) \neq 0$):

d-dimensional complex X with η-systole that is an ε-cocycle expander has ε'-overlapping property, ε'= ε'(ε, η).

Topological overlapping from Isoparametric inequalities

Thm :Let X be 3-dim complex.

Assume for every i=0,1,2 every locally minimal $f \in C^i(X,F_2)$ with $||f|| \leq \eta_i$: $||\delta_i(f)||=c||f||$ for c>0.

Then the 2-skekelon of X has **ε'-overlapping property,** $\varepsilon' = \varepsilon'(\eta_0,\eta_1,\eta_2)$

Conclusion: A locally minimal cocycle $f \in Z^i(X,F_2)$ in X sat $||f|| = 0$ or $||f|| > \eta_i > 0$

locally minimal is a relaxation of minimal. $f \in C^i(X,F_2)$ that is locally minimal satisfies for every $f' \in C^i(X,F_2)$ s.t. $\delta_i(f) = \delta_i(f')$. $||f|| \leq ||f'||$.

Topological overlapping from Isoparametric inequalities

Let X be 3-dim complex satisfying the iso-inequalities, Y - its 2 skeleton

η-systole : $syst_i(Y) > \eta$: immediate, since a minimal systole is locally min

ε-cocycle expander: $f \in C^i(Y,F_2)\backslash Z^i(Y,F_2)=C^i(X,F_2)\backslash Z^i(X,F_2)$ $||\delta_1(f)||/||[f]_Z|| > \varepsilon$.

if $||\delta_1(f)|| > \eta_2$ done.

o.w. if $\delta_1(f)$ is locally minimal then by iso-inequalities $||\delta_1(f)||=0$ done

Remain: if $||\delta_1(f)|| \leq \eta_2$, not locally minimal,

We show: $\delta_1(f) = \delta_1(h) + \delta_1(g)$, $\delta_1(h)$ is locally min $||[g]_Z|| < c||\delta_1(f)||$

Note: $\delta(\delta(h))=0$, $\delta_1(h)$ - loc min 2-cocycle, thus $||\delta_1(h)||=0$ or $||\delta_1(h)|| > \eta_2$

Thus: $||[h]_Z|| \leq 1/\eta_2 ||\delta_1(h)|| \leq 1/\eta_2 ||\delta_1(f)||$

$||\delta_1(f)||/||[f]_Z|| \geq 1/(c+1/\eta_2)$

Isoparametric inequalities

Thm (main technical theorem) :Let X be 3-dimensional Ramanujan. complex. For every i=0,1,2 every locally minimal $f \in C^i(X,F_2)$ with $||f|| \leq \eta_i$: $||\delta_i(f)||=c||f||$ for c>0.

Baby version: Iso-inequalities for 2-dim complexes

Let X = (V,E,T) be a 2-dimensional Ramanujan complex:

Their skeleton graph is a degree Q almost Ramanujan expander.

The link of every vertex is a degree q ~ $\sqrt{Q}$ Ramanujan expander.

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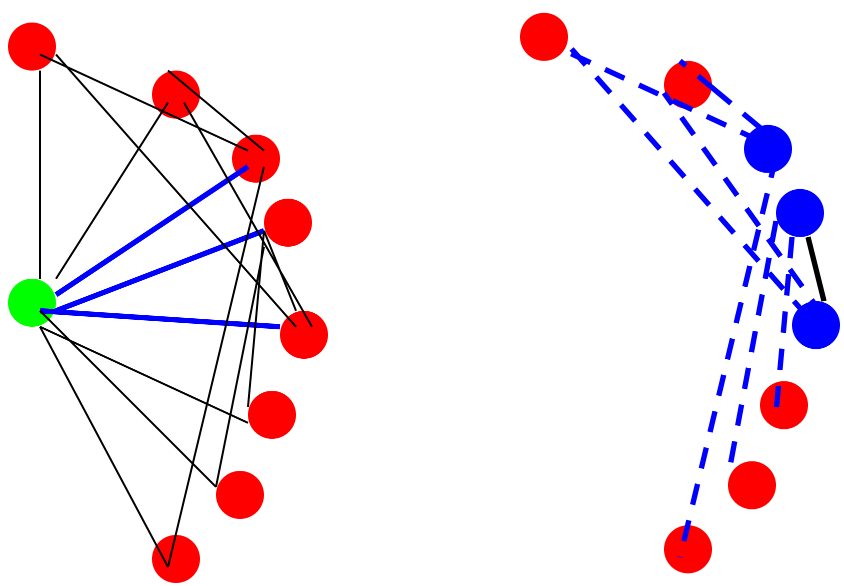


the link graph of

Proof idea: every locally minimal 1-cocycle is large

1-cochain (a collection of edges) is a subgraph in the skeleton graph of X.

A sub-linear sub-graph of the expander graph X has an average degree < $\lambda(X) \ll Q$ (as X is almost Ramanujan).

In a sub-linear sub-graph of the expander graph X most vertices are **thin** (with degree $\ll Q$). Rest are **thick**

Proof idea: every locally minimal 1-cocycle is large

Link of a vertex is a q regular graph.

Every vertex in link corresponds to an edge.

Every edge in link corresponds to a Δ.

A vertex v defines a set S_v in its link,

By local minimality, for every v, $|S_v| \leq Q/2$,

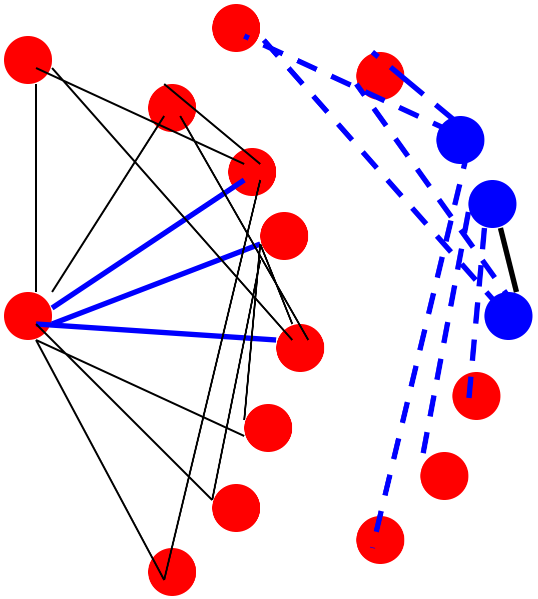
A **thin** vertex v defines a small set S_v in its link.

If v is thin, S_v expands extremely well in the link graph

If v is thick, S_v expands well, since $|S_v| \leq Q/2$,

Many Δ's are colored once (as most vertices are thin)

Every sublinear locally minimal 1-cochain has many 1-colored Δ's.



the link graph

Conditional Coboundary expanders

Our 2-dim complex is a **cocycle expander** and it has **large systole**, thus it has the **topological overlapping property** (by Gromov, Kaufman & Wagner thms).

If our complex had $H^i(Y) = 0$ then it would be also a coboundary expander.

According to Serre's conjecture, one can choose the 3 dimensional Ramanujan complex X, s.t., its 2 skeleton Y has $H^i(Y) = 0$.

Open

Unconditional bounded degree coboundary expanders

Are the 2-dim Ramanujan complexes cocycle expanders?

Generalize the result to d-dimensional bounded degree complexes.

Random constructions of bounded degree high dim expanders.

Thank You!!!