Arcs in the Projective Plane

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Idea

Take the $\mathbb{F}_q$-points of a smooth conic. All smooth conics are equivalent under change of coordinates and have $q+1$ rational points.
A line that intersects a curve of degree d in more than d points must be contained in that curve, so the points of a conic form an arc.

Ovals and Hyperovals

Theorem (Segre, 1955)

Suppose q is odd. The largest arc in $\mathbb{P}^2(\mathbb{F}_q)$ is of size $q+1$ and every such arc consists of the $\mathbb{F}_q$-rational points of a smooth conic.

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When q is even the maximum size of an arc is $q+2$ and any $(q+2)$-arc is called a hyperoval. Classifying the hyperovals in $\mathbb{P}^2(\mathbb{F}_{2^h})$ is an open problem.

Ovals and Hyperovals

An important fact related to Segre's Theorem

Let $p_1, \ldots, p_{q+1}$ be an arc. There are $q+1$ $\mathbb{F}_q$-rational lines through p_1 and q of them contain another point of this arc. The remaining line is called the tangent line at p_1.

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This is false in characteristic 2. For example, consider the zero set of $f(x,y,z) = x^2 + yz$ in $\mathbb{P}^2(\mathbb{F}_4)$. There are 5 rational points of this conic

$$\{[0:0:1],[0:1:0],[1:1:1],[a:a+1:1],[a+1:a:1]\},$$

and the five rational tangent lines

$$\ell_1 = \{y=0\},\ \ell_2 = \{z=0\},\ \ell_3 = \{y+z=0\},$$
$$\ell_4 = \{y+(a+1)z=0\},\ \ell_5 = \{y+az=0\},$$

all intersect at a point, $[1:0:0]$, called the *nucleus* of the conic.

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all intersect at a point, $[1:0:0]$, called the *nucleus* of the conic.

Such a hyperoval is called *regular*. For $h \geq 4$, not every hyperoval in $\mathbb{P}^2(\mathbb{F}_{2^h})$ is regular.

Counting Arcs

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The number of 5-arcs is $(q^5-q^2)\binom{q+1}{5}$.
This follows from the fact that any 5-arc lies on a unique conic. We can first choose the conic and then choose the five points.

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The number of 6-arcs is

$$\frac{1}{6!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q-2)(q-3)(q^2-9q+21).$$

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Given a 5-arc how many points can we choose to extend it to a 6-arc?

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Given a 5-arc how many points can we choose to extend it to a 6-arc? There are two cases to consider:

1. Choose another point on the conic: $q-4$ choices.
2. Another point not on the conic: $(q-5)^2$ choices.

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The number of ways of extending a 6-arc to a 7-arc depends on which 6-arc you start with. The first question to consider is whether the 6-arc lies on a conic.

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If not, we still need to consider whether the last point added was on 0 or 2 rational tangent lines.

The Formula for 7-arcs

Theorem (Glynn)

Suppose q is odd. Then the number of 7-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ is given by

$$\begin{aligned} A_7(q) &= \frac{1}{7!}(q^2+q+1)(q+1)q^3(q-1)^2(q-3)(q-5) \\ &\times (q^4-20q^3+148q^2-468q+498). \end{aligned}$$

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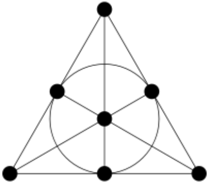
Theorem (Glynn)

The number of 7-arcs in $\mathbb{P}^2(\mathbb{F}_{2^h})$ is given by $A_7(2^h)$ minus the number of copies of the Fano plane in $\mathbb{P}^2(\mathbb{F}_{2^h})$. Let $q = 2^h$ and $n = (q^2+q+1)(q+1)q^3(q-1)^2$. The number of such planes is $\frac{n}{168}$.

7-arcs and Fano Planes



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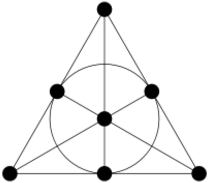

Consider $\{(p_1, \ldots, p_7) \mid p_i \in \mathbb{P}^2(\mathbb{F}_q)\}$. Then (p_1, p_2, p_3) lie on a line if and only if $\det(p_1, p_2, p_3) = 0$.

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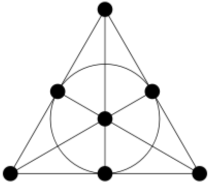

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We have $\binom{7}{3} = 35$ determinants and want to count the number of points where none of them vanish. This leads to a kind of inclusion/exclusion.

Counting 8-arcs

Theorem (Glynn)

The number of 8-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ *is*

$$\begin{aligned} & \frac{1}{8!}(q^2+q+1)(q+1)q^3(q-1)^2(q-5)(q^7-43q^6+788q^5 \\ & -7937q^4+47097q^3-162834q^2+299280q-222960) \\ + \quad & (8_3)-(q^2-20q+78)(7_3), \end{aligned}$$

where (7_3) *denotes the number of Fano subplanes of* $\mathbb{P}^2(\mathbb{F}_q)$ *and* (8_3) *denotes the number of copies of the affine plane of order* 3 *minus a point.*

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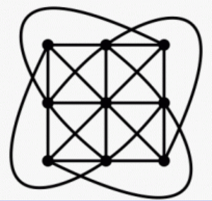

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We have

$$(8_3) = \begin{cases} \frac{n}{48} \text{ if } q \equiv 0 \pmod 3, \\ \frac{n}{24} \text{ if } q \equiv 1 \pmod 3, \\ 0 \text{ if } q \equiv 2 \pmod 3. \end{cases}$$

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For the number of 9-arcs we will get another polynomial formula that involves a number of other special configurations (collection of points, at least three lines through each one).

General Position: One Generalization of an Arc

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Proposition

The number of 6-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ that do not lie on a conic is

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Proposition

Suppose q is odd. The number of 7-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ with no six points lying on a conic is

$$\frac{1}{7!}(q^2+q+1)(q^2+q)q^2(q-1)^2(q-3)(q-5)(q-7)(q^3-20q^2+119q-175).$$

An Application of Counting Arcs: del Pezzo Surfaces

Definition

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Examples:

1. $d = 4$ intersection of two quadrics in $\mathbb{P}^4$.
2. $d = 3$ cubic surface in $\mathbb{P}^3$.
3. $d = 2$ double cover of $\mathbb{P}^2$ branched over a plane quartic.

Counting Points on del Pezzo Surfaces

Theorem (Weil)

Let S be a surface defined over $\mathbb{F}_q$. If $S \otimes \overline{\mathbb{F}_q}$ is birationally trivial, then

$$\#S(\mathbb{F}_q) = q^2 + q\,\mathrm{Tr}(\varphi^*) + 1,$$

where φ denotes the Frobenius endomorphism and Tr denotes the trace of φ in the representation of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $\mathrm{Pic}(S \otimes \overline{\mathbb{F}_q})$.

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The Picard group of S has a very concrete generating set:
$\langle H, e_1, \ldots, e_{9-d}\rangle$.

'Maximal' del Pezzo Surfaces

Theorem (Elkies)

The number of homogeneous cubic polynomials $f_3(w,x,y,z)$ *such that* $\{f_3 = 0\}$ *is a smooth cubic surface with* q^2+7q+1 $\mathbb{F}_q$*-points, the maximum possible, is*

$$\frac{|\operatorname{GL}_4(\mathbb{F}_q)|(q-2)(q-3)(q-5)^2}{51840}.$$

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Theorem (K.)

1. *The number of smooth quartics* $f_4(x,y,z)$ *such that* $w^2 = f_4(x,y,z)$ *has* q^2+8q+1 $\mathbb{F}_q$*-rational points, the maximum possible, is*

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$$\frac{|\operatorname{GL}_3(\mathbb{F}_q)|(q-3)(q-5)(q-7)(q^3 - 20q^2 + 119q - 175)}{2903040}.$$

2. *Every such quartic over* $\mathbb{F}_9$ *is isomorphic to the Fermat quartic* $\{x^4 + y^4 + z^4 = 0\}$.

'Maximal' del Pezzo Surfaces

How many 8-arcs have no six points on a conic and do not lie on a singular cubic curve with one of the points being the singular point?

This would play a key role in adapting these results to del Pezzo surfaces of degree 1.

Higher Dimensions: Arcs in $\mathbb{P}^n(\mathbb{F}_q)$

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For $n \geq 3$, $q \geq n+1$ the maximum size of an arc in $\mathbb{P}^n(\mathbb{F}_q)$ is $q+1$.

A rational normal curve in $\mathbb{P}^n(\mathbb{F}_q)$ is projectively equivalent to

$$\left\{(1, t, t^2, \ldots, t^n) \mid t \in \mathbb{F}_q\right\} \cup \{(0, \ldots, 0, 1)\}.$$

It is easy to check that this gives an arc.

Arcs in $\mathbb{P}^n(\mathbb{F}_q)$

For which n, q is it true that every $(q+1)$-arc in $\mathbb{P}^n(\mathbb{F}_q)$ consists of the $\mathbb{F}_q$-points of a rational normal curve?

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This is true for q odd when $n = 3$, but is false for $n = 4$ and $q = 9$.

Example (Glynn)
A 10-arc in $\mathbb{P}^4(\mathbb{F}_9)$ is either equivalent to a rational normal curve or is equivalent to

$$\left\{(1, t, t^2 + \eta t^6, t^3, t^4) \middle|\ t \in \mathbb{F}_9\right\} \cup \{(0, \ldots, 0, 1)\},$$

where $\eta^4 = -1$ in $\mathbb{F}_9$.
This is the only known example not equivalent to a rational normal curve when q is odd.

Glynn's Example

If we project twice from a 10-arc from a rational normal curve in $\mathbb{P}^4(\mathbb{F}_9)$, we get an 8-arc contained in a conic in $\mathbb{P}^2(\mathbb{F}_9)$.

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For example, for $q = 25$ there is a complete arc of size 21. For $q = 29$ the answer is 24 and these points lie on an irreducible quartic curve.

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What is the largest $(3, n)$-arc in $\mathbb{P}^2(\mathbb{F}_q)$? Denote this quantity by $m_3(2, q)$.

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What is the largest $(3, n)$-arc in $\mathbb{P}^2(\mathbb{F}_q)$? Denote this quantity by $m_3(2, q)$.

Idea

Take the $\mathbb{F}_q$-rational points of a smooth cubic curve.

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Let C be a smooth cubic curve in $\mathbb{P}^2(\mathbb{F}_q)$. Then $\#C(\mathbb{F}_q) = q + 1 - t$, where $|t| \leq 2\sqrt{q}$.

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Theorem (Deuring)

Suppose $|t| < 2\sqrt{q}$ and $\mathrm{char}(\mathbb{F}_q) \nmid t$. Then there exists a smooth cubic C with $\#C(\mathbb{F}_q) = q + 1 - t$.

Complete Arcs from Cubics

Theorem (Hirschfeld, Voloch)

If $q \geq 79$ and q is not a power of 2 or 3, then a non-singular cubic C with k points is a complete $(k, 3)$-arc unless the j-invariant of C is zero, in which case the completion of this arc has at most $k + 3$ points.

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q	2	3	4	5	7	8	9	11	13
$m_3(2, q)$	7	9	9	11	15	15	17	21	23
$q + 1 + \lfloor 2\sqrt{q} \rfloor$	5	7	9	10	13	14	16	18	21

Conjecture (Hirschfeld)

For $q \neq 4$, $m_3(2, q) > q + 1 + \lfloor 2\sqrt{q} \rfloor$.

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Proposition

We have $m_3(2,q) \leq 2q+1$.

$(3, n)$-arcs from Quartic Curves

Idea

1. Take the zero set of a quartic curve with many rational points.
2. There will usually be some lines that intersect the curve in 4 points.
3. Keep removing points from the curve until there are no lines with four points left.

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Example

1. Over $\mathbb{F}_5$ the curve $x^4 + y^4 + 3z^4 = 0$ has 16 $\mathbb{F}_5$-points.
2. There are 12 lines that intersect it in exactly 4 rational points.
3. There is a set of 5 points such that every one of these lines contains at least one of them. This gives an $(3, 11)$-arc, the largest possible.

What's the largest $(3, n)$-arc contained in a quartic curve?

1. Suppose C is a curve with rational points $\{p_1, \ldots, p_k\}$.
2. Let $\{\ell_1, \ldots, \ell_r\}$ be the set of all lines intersecting C in exactly four of these points.
3. We get a bipartite graph where there is an edge from p_i to ℓ_j if and only if p_i is on ℓ_j.
4. We want to find a set of points of minimum size such that every line is adjacent to at least one of these points.

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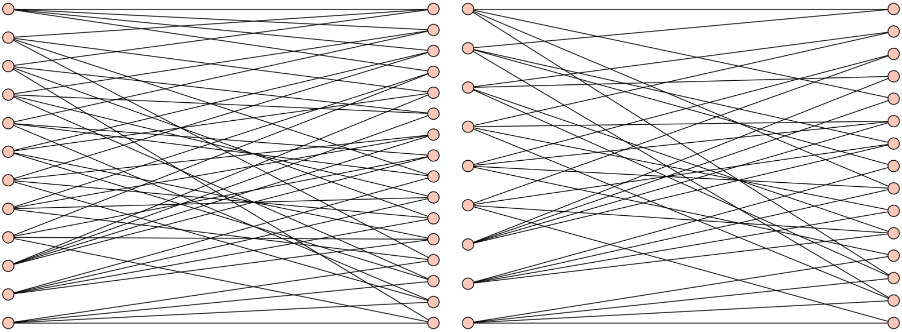
A lazier method of finding large arcs contained in these curves is to remove points in a greedy way, at each step taking away the point contained in the maximum number of 4-secants.

An Example

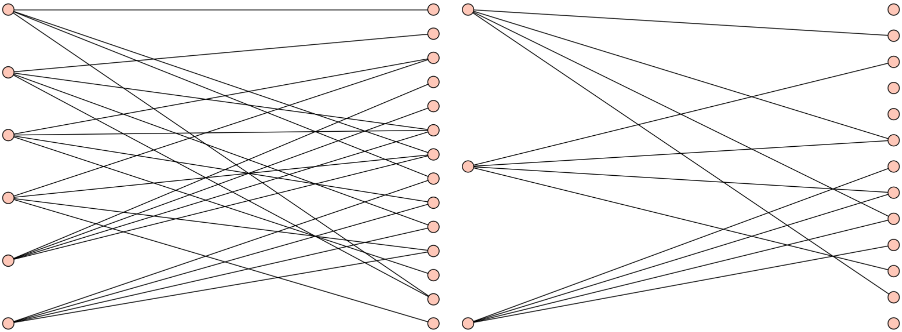
Let $C = \{x^4 + y^4 + 3z^4 = 0\} \subset \mathbb{P}^2(\mathbb{F}_5)$. We get a bipartite graph with a partition of 12 and 16 vertices.

First we remove any point and delete the lines containing it.



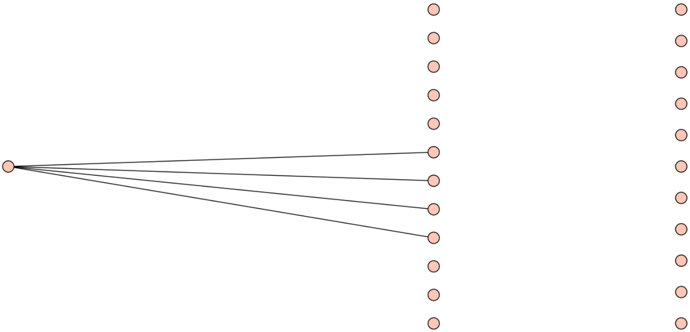
An Example

Let $C = \{x^4 + y^4 + 3z^4 = 0\} \subset \mathbb{P}^2(\mathbb{F}_5)$. We then remove another vertex of degree 3 and then another.



An Example

Let $C = \{x^4 + y^4 + 3z^4 = 0\} \subset \mathbb{P}^2(\mathbb{F}_5)$. We remove a vertex of degree 2 and are left with a single line. Removing a point gives a $(3, 11)$-arc.



$(3, n)$-arcs from quartic curves

Theorem (Weil bound)

Let C be a smooth plane quartic in $\mathbb{P}^2(\mathbb{F}_q)$. Then $\#C(\mathbb{F}_q) = q + 1 - t$, where $|t| \leq 3\lfloor 2\sqrt{q} \rfloor$.

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Theorem (Lauter)

There exists a smooth quartic in $\mathbb{P}^2(\mathbb{F}_q)$ with $\#C(\mathbb{F}_q) \geq q + 1 + 3\lfloor 2\sqrt{q} \rfloor - 3$.

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Theorem (Oyono, Ritzenthaler)

Let C be a smooth plane quartic over $\mathbb{F}_q$. If $q \geq 127$ then there exists a line ℓ which intersects C at rational points only.

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The rational points of C form an abelian group. If $q + 1 + t$ is even then we can take the coset of the subgroup of index 2. This is called a *Zirilli arc*. This is (almost) the largest arc contained in a cubic.