Arcs in the Projective Plane

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**Definition**

An $n$-arc is a collection of $n$ distinct points in $\mathbb{P}^2(\mathbb{F}_q)$ no three of which lie on a line.

An arc is complete if it is not contained in an arc of size $n + 1$. 

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**Kaplan (Yale University)**

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What is the largest size of an arc?
Arcs

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What is the largest size of an arc?

Idea

Take the \( \mathbb{F}_q \)-points of a smooth conic. All smooth conics are equivalent under change of coordinates and have \( q + 1 \) rational points. A line that intersects a curve of degree \( d \) in more than \( d \) points must be contained in that curve, so the points of a conic form an arc.
Ovals and Hyperovals

Theorem (Segre, 1955)

Suppose $q$ is odd. The largest arc in $\mathbb{P}^2(\mathbb{F}_q)$ is of size $q + 1$ and every such arc consists of the $\mathbb{F}_q$-rational points of a smooth conic.
Theorem (Segre, 1955)

Suppose $q$ is odd. The largest arc in $\mathbb{P}^2(F_q)$ is of size $q + 1$ and every such arc consists of the $F_q$-rational points of a smooth conic.

When $q$ is even the maximum size of an arc is $q + 2$ and any $(q + 2)$-arc is called a hyperoval. Classifying the hyperovals in $\mathbb{P}^2(F_{2h})$ is an open problem.
Ovals and Hyperovals

An important fact related to Segre’s Theorem

Let $p_1, \ldots, p_{q+1}$ be an arc. There are $q + 1 \mathbb{F}_q$-rational lines through $p_1$ and $q$ of them contain another point of this arc. The remaining line is called the tangent line at $p_1$. 

When $q$ is odd, every point not in our arc is on exactly 0 or 2 of these $q + 1$ rational tangent lines. This is false in characteristic 2. For example, consider the zero set of $f(x, y, z) = x^2 + yz$ in $\mathbb{P}^2(\mathbb{F}_4)$. There are 5 rational points of this conic $\{[0 : 0 : 1], [0 : 1 : 0], [1 : 1 : 1], [a : a + 1 : 1], [a + 1 : a : 1]\}$, and the five rational tangent lines $\ell_1 = \{y = 0\}, \ell_2 = \{z = 0\}, \ell_3 = \{y + z = 0\}, \ell_4 = \{y + (a + 1)z = 0\}, \ell_5 = \{y + az = 0\}$, all intersect at a point, $[1 : 0 : 0]$, called the nucleus of the conic.

Such a hyperoval is called regular. For $h \geq 4$, not every hyperoval in $\mathbb{P}^2(\mathbb{F}_h)$ is regular.
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\[
\begin{align*}
&\{[0 : 0 : 1], [0 : 1 : 0], [1 : 1 : 1], [a : a + 1 : 1], [a + 1 : a : 1]\},
\end{align*}
\]

and the five rational tangent lines

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\begin{align*}
\ell_1 &= \{y = 0\}, \quad \ell_2 = \{z = 0\}, \quad \ell_3 = \{y + z = 0\}, \\
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all intersect at a point, \( [1 : 0 : 0] \), called the \textit{nucleus} of the conic. Such a hyperoval is called \textit{regular}. For \( h \geq 4 \), not every hyperoval in \( \mathbb{P}^2(\mathbb{F}_{2^h}) \) is regular.
Counting Arcs

Instead of asking for the largest arc in $\mathbb{P}^2(F_q)$ we can ask for the number of arcs of a given size.

Example

The number of 4-arcs in $\mathbb{P}^2(F_q)$ is

$$
\frac{1}{4!} (q^2 + q + 1)(q^2 + q)(q^2 - 1)^2.
$$

This follows from the fact that there exists a unique change of coordinates sending $(p_1, p_2, p_3, p_4)$ to $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$.

The number of 5-arcs is

$$(q^5 - q^2)(q + 1)^5.$$

This follows from the fact that any 5-arc lies on a unique conic. We can first choose the conic and then choose the five points.
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The number of 5-arcs is $(q^5 - q^2)(q^5_5)$. This follows from the fact that any 5-arc lies on a unique conic. We can first choose the conic and then choose the five points.
The number of 6-arcs is

$$\frac{1}{6!}(q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q - 2)(q - 3)(q^2 - 9q + 21).$$

Given a 5-arc how many points can we choose to extend it to a 6-arc?

There are two cases to consider:

1. Choose another point on the conic: $q - 4$ choices.
2. Another point not on the conic: $(q - 5)^2$ choices.

The number of ways of extending a 6-arc to a 7-arc depends on which 6-arc you start with. The first question to consider is whether the 6-arc lies on a conic. If not, we still need to consider whether the last point added was on 0 or 2 rational tangent lines.
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The Formula for 7-arcs

Theorem (Glynn)

Suppose $q$ is odd. Then the number of 7-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ is given by

$$A_7(q) = \frac{1}{7!} (q^2 + q + 1)(q + 1)q^3(q - 1)^2(q - 3)(q - 5) \times (q^4 - 20q^3 + 148q^2 - 468q + 498).$$
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Theorem (Glynn)

The number of 7-arcs in $\mathbb{P}^2(\mathbb{F}_{2^h})$ is given by $A_7(2^h)$ minus the number of copies of the Fano plane in $\mathbb{P}^2(\mathbb{F}_{2^h})$. Let $q = 2^h$ and $n = (q^2 + q + 1)(q + 1)q^3(q - 1)^2$. The number of such planes is $\frac{n}{168}$.
Consider \( \{(p_1, \ldots, p_7) | p_i \in \mathbb{P}^2(F_q)\} \). Then \((p_1, p_2, p_3)\) lie on a line if and only if \(\det \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = 0\).

We have \(\binom{7}{3} = 35\) determinants and want to count the number of points where none of them vanish. This leads to a kind of inclusion/exclusion.

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We have \(\binom{7}{3} = 35\) determinants and want to count the number of points where none of them vanish. This leads to a kind of inclusion/exclusion.
The number of 8-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ is

\[
\frac{1}{8!} (q^2 + q + 1)(q + 1)q^3(q - 1)^2(q - 5)(q^7 - 43q^6 + 788q^5 - 7937q^4 + 47097q^3 - 162834q^2 + 299280q - 222960)
\]
\[
+ (8_3) - (q^2 - 20q + 78)(7_3),
\]

where $(7_3)$ denotes the number of Fano subplanes of $\mathbb{P}^2(\mathbb{F}_q)$ and $(8_3)$ denotes the number of copies of the affine plane of order 3 minus a point.
Counting 8-arcs

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Formulas for $n$-arcs

This formula is a quasi-polynomial in $q$. That is, for each nonzero residue class modulo $2 \cdot 3$ there is a different polynomial formula for the number of 8-arcs.

Proposition

We have

$$\binom{8}{3} = \begin{cases} 
n & q \equiv 0 \pmod{3}, 
n & q \equiv 1 \pmod{3}, 
0 & q \equiv 2 \pmod{3}. 
\end{cases}$$

For the number of 9-arcs we will get another polynomial formula that involves a number of other special configurations (collection of points, at least three lines through each one).
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$$(8_3) = \begin{cases} \frac{n}{48} & \text{if } q \equiv 0 \pmod{3}, \\ \frac{n}{24} & \text{if } q \equiv 1 \pmod{3}, \\ 0 & \text{if } q \equiv 2 \pmod{3}. \end{cases}$$
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For the number of 9-arcs we will get another polynomial formula that involves a number of other special configurations (collection of points, at least three lines through each one).
General Position: One Generalization of an Arc

If we choose 3 random points in $\mathbb{P}^2$ we do not expect them to lie on a line. We also do not expect 6 points to lie on a conic, or 10 points to lie on a cubic, etc.
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**Proposition**

The number of 6-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ that do not lie on a conic is

$$\frac{1}{6!}(q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q - 2)(q - 3)(q - 5)^2.$$
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**Proposition**

Suppose $q$ is odd. The number of 7-arcs in $\mathbb{P}^2(\mathbb{F}_q)$ with no six points lying on a conic is

$$\frac{1}{7!}(q^2 + q + 1)(q^2 + q)q^2(q - 1)^2(q - 3)(q - 5)(q - 7)(q^3 - 20q^2 + 119q - 175).$$
A del Pezzo surface is a smooth projective surface $S$ with ample anti-canonical bundle $-K_S$. The degree of $S$ is $K_S \cdot K_S = d$. 

Examples:
1. $d = 4$ intersection of two quadrics in $\mathbb{P}^4$.
2. $d = 3$ cubic surface in $\mathbb{P}^3$.
3. $d = 2$ double cover of $\mathbb{P}^2$ branched over a plane quartic.
A del Pezzo surface is a smooth projective surface $S$ with ample anti-canonical bundle $-K_S$. The degree of $S$ is $K_S \cdot K_S = d$. Over $\overline{\mathbb{F}_q}$ such a surface is given by the blow-up of $\mathbb{P}^2$ at $9 - d$ points.
Definition

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Counting Points on del Pezzo Surfaces

Theorem (Weil)

Let $S$ be a surface defined over $\mathbb{F}_q$. If $S \otimes \overline{\mathbb{F}_q}$ is birationally trivial, then

$$\#S(\mathbb{F}_q) = q^2 + q \text{ Tr}(\varphi^*) + 1,$$

where $\varphi$ denotes the Frobenius endomorphism and $\text{Tr}$ denotes the trace of $\varphi$ in the representation of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $\text{Pic}(S \otimes \overline{\mathbb{F}_q})$. 
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The Picard group of $S$ has a very concrete generating set:
$$
\langle H, e_1, \ldots, e_{9-d} \rangle.
$$
Theorem (Elkies)

The number of homogeneous cubic polynomials \( f_3(w, x, y, z) \) such that \( \{ f_3 = 0 \} \) is a smooth cubic surface with \( q^2 + 7q + 1 \) \( \mathbb{F}_q \)-points, the maximum possible, is

\[
\frac{|GL_4(\mathbb{F}_q)| (q - 2)(q - 3)(q - 5)^2}{51840}.
\]
‘Maximal’ del Pezzo Surfaces

Theorem (Elkies)

The number of homogeneous cubic polynomials $f_3(w, x, y, z)$ such that $\{f_3 = 0\}$ is a smooth cubic surface with $q^2 + 7q + 1$ $\mathbb{F}_q$-points, the maximum possible, is

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Theorem (K.)

The number of smooth quartics $f_4(x, y, z)$ such that $w^2 = f_4(x, y, z)$ has $q^2 + 8q + 1$ $\mathbb{F}_q$-rational points, the maximum possible, is

$$\frac{|\text{GL}_3(\mathbb{F}_q)|(q - 3)(q - 5)(q - 7)(q^3 - 20q^2 + 119q - 175)}{2903040}.$$
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2. Every such quartic over \( \mathbb{F}_9 \) is isomorphic to the Fermat quartic \( \{ x^4 + y^4 + z^4 = 0 \} \).
How many 8-arcs have no six points on a conic and do not lie on a singular cubic curve with one of the points being the singular point?

This would play a key role in adapting these results to del Pezzo surfaces of degree 1.
Higher Dimensions: Arcs in $\mathbb{P}^n(F_q)$

Definition

A $k$-arc in $\mathbb{P}^n(F_q)$ is a collection of $k$ distinct points, no $n + 1$ of which lie in a hyperplane.
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What is the largest $k$-arc in $\mathbb{P}^n(\mathbb{F}_q)$ when $n \geq 3$?
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**Conjecture (MDS Conjecture)**

For $n \geq 3$, $q \geq n + 1$ the maximum size of an arc in $\mathbb{P}^n(F_q)$ is $q + 1$. 
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**Conjecture (MDS Conjecture)**

For $n \geq 3$, $q \geq n+1$ the maximum size of an arc in $\mathbb{P}^n(\mathbb{F}_q)$ is $q + 1$.

A rational normal curve in $\mathbb{P}^n(\mathbb{F}_q)$ is projectively equivalent to

$$\{(1, t, t^2, \ldots, t^n) | t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\}.$$  

It is easy to check that this gives an arc.
For which $n, q$ is it true that every $(q + 1)$-arc in $\mathbb{P}^n(F_q)$ consists of the $F_q$-points of a rational normal curve?

This is true for $q$ odd when $n = 3$, but is false for $n = 4$ and $q = 9$. Example (Glynn) A 10-arc in $\mathbb{P}^4(F_9)$ is either equivalent to a rational normal curve or is equivalent to $\{(1, t, t^2 + \eta t^6, t^3, t^4) \mid t \in F_9\} \cup \{(0, \ldots, 0, 1)\}$, where $\eta^4 = -1$ in $F_9$. This is the only known example not equivalent to a rational normal curve when $q$ is odd.
For which \( n, q \) is it true that every \((q + 1)\)-arc in \( \mathbb{P}^n(\mathbb{F}_q) \) consists of the \( \mathbb{F}_q \)-points of a rational normal curve?

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For which $n, q$ is it true that every $(q + 1)$-arc in $\mathbb{P}^n(\mathbb{F}_q)$ consists of the $\mathbb{F}_q$-points of a rational normal curve?

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Example (Glynn)
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where $\eta^4 = -1$ in $\mathbb{F}_9$.

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Glynn’s Example

If we project twice from a 10-arc from a rational normal curve in $\mathbb{P}^4(\mathbb{F}_9)$, we get an 8-arc contained in a conic in $\mathbb{P}^2(\mathbb{F}_9)$. However, if we start with Glynn’s nonclassical arc there are many projections which give the unique complete 8-arc in $\mathbb{P}^2(\mathbb{F}_9)$. Small complete arcs can arise as the projections of interesting structures in higher dimensions.

What’s the second largest complete arc in $\mathbb{P}^2(\mathbb{F}_q)$? For example, for $q = 25$ there is a complete arc of size 21. For $q = 29$ the answer is 24 and these points lie on an irreducible quartic curve.
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(k, n)-arcs in $\mathbb{P}^2(F_q)$

**Definition**

A $(k, n)$-arc is a collection of $n$ points in $\mathbb{P}^2(F_q)$ no more than $k$ on a line.
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**Theorem (Hasse)**

Let \( C \) be a smooth cubic curve in \( \mathbb{P}^2(\mathbb{F}_q) \). Then \( \# C(\mathbb{F}_q) = q + 1 - t \), where \( |t| \leq 2\sqrt{q} \).
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**Theorem (Deuring)**

Suppose \( |t| < 2\sqrt{q} \) and \( \text{char}(\mathbb{F}_q) \nmid t \). Then there exists a smooth cubic \( C \) with \( \#C(\mathbb{F}_q) = q + 1 - t \).
Theorem (Hirschfeld, Voloch)

If $q \geq 79$ and $q$ is not a power of 2 or 3, then a non-singular cubic $C$ with $k$ points is a complete $(k,3)$-arc unless the j-invariant of $C$ is zero, in which case the completion of this arc has at most $k + 3$ points.
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For \( q \neq 4 \), \( m_3(2, q) > q + 1 + \lfloor 2\sqrt{q} \rfloor \).
Complete Arcs from Cubics

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Proposition

We have $m_3(2, q) \leq 2q + 1$. 
(3, n)-arcs from Quartic Curves

Idea

1. Take the zero set of a quartic curve with many rational points.
2. There will usually be some lines that intersect the curve in 4 points.
3. Keep removing points from the curve until there are no lines with four points left.

Example

1. Over $\mathbb{F}_5$ the curve $x^4 + y^4 + 3z^4 = 0$ has 16 $\mathbb{F}_5$-points.
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1. Suppose \(C\) is a curve with rational points \(\{p_1, \ldots, p_k\}\).

2. Let \(\{\ell_1, \ldots, \ell_r\}\) be the set of all lines intersecting \(C\) in exactly four of these points.

3. We get a bipartite graph where there is an edge from \(p_i\) to \(\ell_j\) if and only if \(p_i\) is on \(\ell_j\).

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A lazier method of finding large arcs contained in these curves is to remove points in a greedy way, at each step taking away the point contained in the maximum number of 4-secants.
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Let \( C = \{x^4 + y^4 + 3z^4 = 0\} \subset \mathbb{P}^2(\mathbb{F}_5) \). We remove a vertex of degree 2 and are left with a single line. Removing a point gives a \((3, 11)\)-arc.
(3, n)-arcs from quartic curves

**Theorem (Weil bound)**

Let $C$ be a smooth plane quartic in $\mathbb{P}^2(\mathbb{F}_q)$. Then $\#C(\mathbb{F}_q) = q + 1 - t$, where $|t| \leq 3 \lfloor 2\sqrt{q} \rfloor$. 

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There exists a smooth quartic in $\mathbb{P}^2(\mathbb{F}_q)$ with $\#C(\mathbb{F}_q) \geq q + 1 + 3 \lfloor 2\sqrt{q} \rfloor - 3$.

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Let $C$ a cubic with $q + 1 + t$ points. How many points do we have to remove to get an arc?

At least $t$, otherwise this would contradict Segre's theorem.

The rational points of $C$ form an abelian group. If $q + 1 + t$ is even then we can take the coset of the subgroup of index 2. This is called a Zirilli arc. This is (almost) the largest arc contained in a cubic.
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