Polynomials on Products

Or: Elekes-Rónyai-Szabó theorems and their recent improvements

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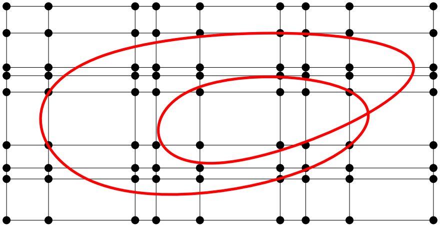


Elekes-Rónyai

Disclaimer: In the theorems I will be sloppy with constants, quantifiers and thresholds.

Theorem (Elekes-Rónyai (2000) / Raz-Sharir-Solymosi (2014))

The number of values of a constant-degree polynomial $f(x,y) \in \mathbb{R}[x,y]$ on an $n \times n$ product $A \times B$ is

$$\Omega(n^{4/3}),$$

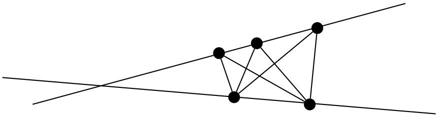
unless f has the form

$$g(h(x) + k(y)) \quad \text{or} \quad g(h(x) \cdot k(y)),$$

for $g, h, k \in \mathbb{R}[x,y]$.

- ER proved $\omega(n)$; RSS improved this to $\Omega(n^{4/3})$.
- RSS also proved an "unbalanced" version (where $|A| \neq |B|$).

Application: Distinct distances between lines



Corollary

Given two lines in $\mathbb{R}^2$ with n points each, the number of distances between them is $\Omega(n^{4/3})$, unless the lines are parallel or orthogonal.

Proof: We can assume the lines are $y = 0$ and $y = mx$.
The squared distance between $(s, 0)$ and (t, mt) is

$$f(s,t) = (s-t)^2 + m^2t^2 = s^2 - 2st + (m^2+1)t^2.$$

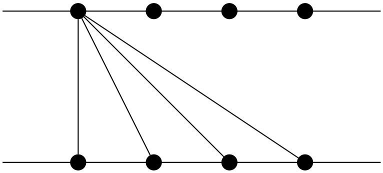
$f = g(h(s) + k(t)) \Rightarrow \deg(g) = 2, \deg(h) = \deg(k) = 1 \Rightarrow$ no.
$f = g(h(s) \cdot k(t)) \Rightarrow \deg(g) = 1, \deg(h) = \deg(k) = 1 \Rightarrow$ no.
RSS $\Rightarrow \Omega(n^{4/3})$. $\square$

Originally proved without RSS by Sharir-Sheffer-Solymosi (2013).
In fact, their proof was the basis for RSS.

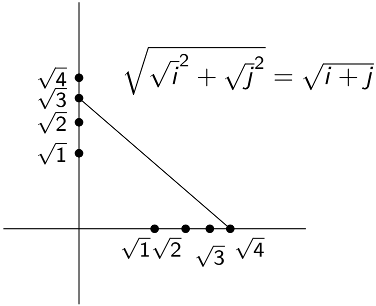
Distances between lines

The exceptions are really exceptions:

cn distances on two parallel lines:



cn on two orthogonal lines:



Sketch of outline of proof of Raz-Sharir-Solymosi

We count the quadruples

$$Q = \{(a, b, a', b') \in A \times B \times A \times B : f(a, b) = f(a', b')\}.$$

Then

$$\frac{n^4}{|f(A, B)|} \le |Q| \le cn^{8/3} \quad \Rightarrow \quad |f(A, B)| = \Omega(n^{4/3}).$$

The lower bound for $|Q|$ is easy with Cauchy-Schwarz; the upper bound is all the work. The main tool is the Pach-Sharir incidence theorem, applied to the curves

$$\gamma_{aa'} = \left\{(x, y) \in \mathbb{R}^2 : f(a, x) = f(a', y)\right\}.$$

This works unless many γ_{ij} have many high-multiplicity common components, in which case f will have one of the special forms.

Variants of Elekes-Rónyai

- Schwartz-Solymosi-De Zeeuw (2013): One can extend ER to **unbalanced** situations, where A and B have different sizes. RSS14 actually take this much further.
- SSZ13: One can also extend ER to functions of **three variables**, with special forms $g(h(x)+k(y)+l(z))$ and $g(h(x)\cdot k(y)\cdot l(z))$. Or more variables...
- Solymosi also conjectured, and partly proved, an **arithmetic** version, with $A, B \subset \mathbb{Q}$ and $f \in \mathbb{Q}[x,y]$, and special forms

$$f(x,y) = g(ax+by), \qquad f(x,y) = g((x+a)^{\alpha}(y+b)^{\beta}).$$

- Bukh-Tsimerman (2011) and Tao (2013) show under various restrictions that a polynomial over a **finite field** is an "expander" unless it has the additive or multiplicative form.

The Elekes-Szabó Theorem

Theorem (Elekes-Rónyai reformulated)

Let $f \in \mathbb{R}[x, y]$, $V = \{(x, y, z) \in \mathbb{R}^3 : f(x, y) = z\}$, *and* $A, B, C \subset \mathbb{R}$ *of size* n. *Then* $|V \cap (A \times B \times C)| = o(n^2)$ *unless* f *is additive or multiplicative.*

Theorem (Elekes-Szabó (2012), "How to find groups?")

Let $A, B, C \subset \mathbb{R}$ *of size* n, *let* $F \in \mathbb{R}[x, y, z]$ *with* $F_x F_y F_z \not\equiv 0$ *and* $\deg F = d$, *and define* $V = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0\}$. *There is an* $\eta = \eta(\deg(F)) > 0$

$$|V \cap (A \times B \times C)| = O(n^{2-\eta}),$$

unless there are an open ball $D \subset \mathbb{R}^3$ *and three analytic functions* $\varphi_i : \mathbb{R} \to \mathbb{R}$ *such that*

$$V \cap D = \{(x, y, z) \in D : \varphi_1(x) + \varphi_2(y) + \varphi_3(z) = 0\}.$$

In work in progress at IPAM, we have improved this to:

Theorem (Raz-Sharir-De Zeeuw (2014))

Let $A, B, C \subset \mathbb{R}$ of size n, let $F \in \mathbb{R}[x, y, z]$ with $F_x F_y F_z \not\equiv 0$ and $\deg F = d$, *and define* $V = \{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0\}$. *Then*

$$|V \cap (A \times B \times C)| = O_d(n^{11/6}),$$

unless there are an open ball $D \subset \mathbb{R}^3$ and three analytic functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that

$$V \cap D = \{(x, y, z) \in D : \varphi_1(x) + \varphi_2(y) + \varphi_3(z) = 0\}.$$

We also obtain an unbalanced version, with the bound

$$O\left(|A|^{2/3}|B|^{2/3}|C|^{1/2} + |A|^{1/2}|B| + |A|^{1/2}|C|\right).$$

Application: Distinct distances from three points

Theorem (ES12, Sharir-Solymosi (2013))

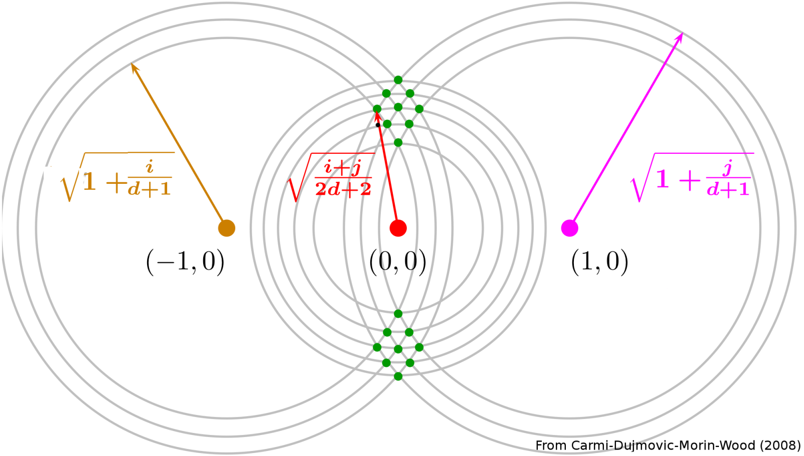
The number of distinct distances between three noncollinear points and n other points is $\Omega(n^{6/11})$.

Proof sketch: Assume the points are $(\pm 1, 0)$ and (a, b).
Let $X = (u-1)^2 + v^2$, $Y = (u+1)^2 + v^2$, $Z = (u-a)^2 + (v-b)^2$.
Then (u, v) is at distance X, Y, Z from resp. $(1,0), (-1,0), (0,0)$ iff (X, Y, Z) satisfies

$$0 = F(X,Y,Z) = ((a+1)^2+b^2)X^2 + ((a-1)^2+b^2)Y^2 + 4Z^2 - 2(a^2+b^2-1)XY - 4(a+1)XZ + 4(a-1)YZ$$
$$+4(a-1)((a+1)^2+b^2)X - 4(a+1)((a-1)^2+b^2)Y - 8(a^2+b^2-1)Z + 4(b^4+2a^2b^2+(a^2-1)^2+2b^2)$$

Check that F does not have the special form, unless $b = 0$.
Hint: partial derivatives and logs. $\square$

From 3 collinear points, the number of distances can be $\Theta(\sqrt{n})$.



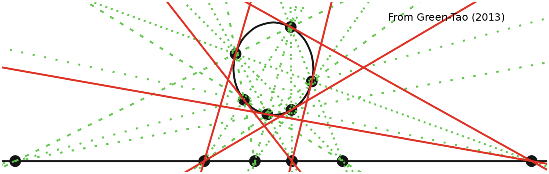
Here $d = \sqrt{n}$ and $i, j \in [d]$. Note that

$$(x-1)^2+y^2 = 1+\frac{j}{d+1},\ (x+1)^2+y^2 = 1+\frac{i}{d+1} \Rightarrow x^2+y^2 = \frac{i+j}{2d+2}.$$

Applications

Theorem (Elekes-Szabó (2013) / Raz-Sharir-De Zeeuw (2014))

A set of n points on an algebraic curve of degree d in $\mathbb{R}^2$ *spans* $O_d(n^{11/6})$ *lines with* ≥ 3 *points, unless the curve is a cubic.*



Theorem (Elekes-Szabó (2013) / Raz-Sharir-De Zeeuw (2014))

A set of n points on an irreducible algebraic curve C of degree d in $\mathbb{R}^2$ *determines* $\Omega_d(n^{4/3})$ *distinct directions, unless C is a conic.*

Conjecture (Elekes): If n non-collinear points determine $O(n)$ directions, then at least 6 of the points are on a conic.

Theorem

A set A of n points on an irreducible algebraic curve C of degree d in $\mathbb{R}^2$ *determines* $\Omega_d(n^{4/3})$ *distinct directions, unless C is a conic.*

Proof sketch: Let C be defined by $f(x, y) = 0$, and let S be the set of slopes. Two points $(x_i, y_i) \in C$ determine a slope s iff $(x_1, y_1, x_2, y_2, s) \in \mathbb{R}^5$ satisfies

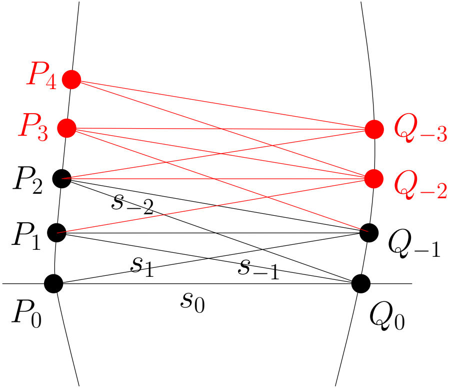
$$s(y_1 - x_1) = y_2 - x_2, \quad f(x_1, y_1) = 0, \quad f(x_2, y_2).$$

This is a 2-dimensional variety.
Do a generic rotation in $\mathbb{R}^2$, and project from $\mathbb{R}^5$ to (x_1, x_2, s).
Then the $|A|^2$ points (x_1, y_1, x_2, y_2, s) map to $|A|^2$ points of an $|A| \times |A| \times |S|$ product in $\mathbb{R}^3$, that lie on a surface $F(x_1, x_2, x_3) = 0$.
By unbalanced RSZ14, we get

$$|A|^2 = O_d(|A|^{2/3}|A|^{2/3}|S|^{1/2}) \;\;\Rightarrow\;\; |S| = \Omega_d(n^{4/3}),$$

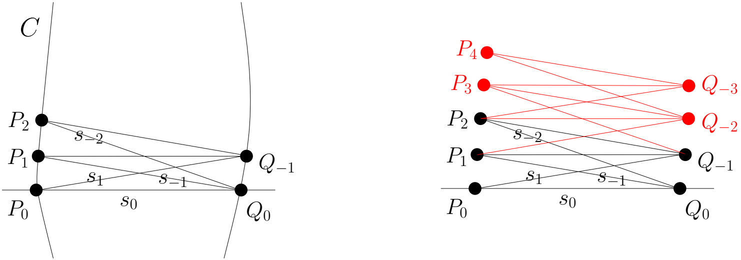
unless F is special, i.e. locally there are φ_i like in RSZ14.

Suppose F is special. Then we can draw the following local picture:



$$\varphi_1(P_0) + \varphi_2(Q_0) + \varphi_3(s_0) = 0$$
$$\varphi_1(P_1) + \varphi_2(Q_0) + \varphi_3(s_{-1}) = 0$$
$$\varphi_1(P_0) + \varphi_2(Q_{-1}) + \varphi_3(s_1) = 0$$
$$\varphi_1(P_2) + \varphi_2(Q_0) + \varphi_3(s_{-2}) = 0$$
$$\vdots$$

Here we rescaled the φ_i and chose the points so that $\varphi_1(P_j) = j$, $\varphi_2(Q_k) = k$, and $\varphi(s_l) = l$.
Then the line through P_j and Q_k has slope s_l iff $j + k + l = 0$.



The points $P_0, Q_0, P_1, Q_{-1}, P_2$ are determined by C, but every subsequent point is determined by these 5, without needing C.

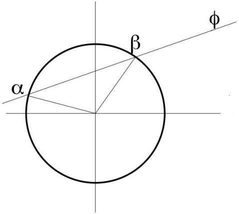
Take a conic C^* through these first five points.

Then all subsequent points must also lie on C^*, because all conics can be parametrized so that slopes satisfy this rule (see next slide).

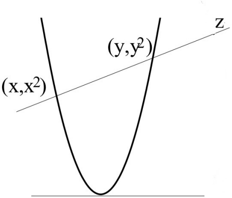
Do this until $|C \cap C^*| = 2d + 1$. Bézout then gives $C = C^*$. $\square$

Directions on conics

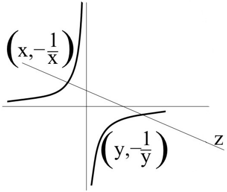
For any conic C, there exists a parametrization and functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ such that P and Q give slope s if and only if $\varphi_1(P) + \varphi_2(Q) + \varphi_3(s) = 0$.



$\alpha+\beta-2\phi=0 \pmod{2\pi}$



$x+y-z=0$



$xyz=1$

From Elekes-Szabo (2013)

Elekes-Szabó's "Main Theorem"

ES12 contains a more general theorem, which however seems harder to apply. Here is my attempt to formulate it.

Theorem (Elekes-Szabó (2012))

Let U, V, W be varieties of the same dimension D, and $A_1 \subset U, A_2 \subset V, A_3 \subset W$ subsets of size n in very general position. Let $F \subset U \times V \times W$ be an irreducible $2D$-dimensional subvariety with surjective and generically finite projections onto any two factors. Then

$$|F \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\eta}),$$

unless there there are multi-functions $\varphi_i : A_i \to G$ to some algebraic group $\mathcal{G}$ of dimension D such that F equals

$$\{(p, q, r) \in U \times V \times W : \varphi_1(p) \oplus \varphi_2(q) \oplus \varphi_2(r) = \mathcal{O}\}.$$

Here "very general position" means that no A_i contains more than a constant number of points from a constant-degree curve.

Main open problems

- Extend to **higher-dimensional** objects, like in Elekes-Szabó's Main Theorem.

- Extend to **more variables**, i.e. for intersections like

$$|\{F(x_1, \dots, x_k) = 0\} \cap (A_1 \times \cdots \times A_k)|.$$

  For instance, show that n points on a curve determine many distinct triangle areas, unless...

- Improve the upper bound **exponent** $11/6$ (a.k.a. the lower bound exponent $4/3$).
  We seem to need an improved incidence bound for algebraic families of points and curves.
  For instance, let the points be a Cartesian product $A \times B$ and the curves $y = (x - c)^2 + d$, for $(c, d) \in C \times D$.

Main references (in order of appearance)

- György Elekes and Lajos Rónyai, *A combinatorial problem on polynomials and rational functions*, JCTA, 89 (2000), 1–20.
- Orit Raz, Micha Sharir, and József Solymosi, *Polynomials vanishing on grids: The Elekes-Rónyai problem revisited*, arXiv:1401.7419.
- Micha Sharir, Adam Sheffer, and József Solymosi, *Distinct distances on two lines*, JCTA, 120 (2013), 1732–1736.
- György Elekes and Endre Szabó, *How to find groups? (and how to use them in Erdős geometry?)*, Combinatorica, 32 (2012), 537–571.
- György Elekes and Endre Szabó, *On triple lines and cubic curves: The Orchard Problem revisited*, arXiv:1302.5777.
- Micha Sharir and József Solymosi, *Distinct distances from three points*, arXiv:1308.0814.