Restricted families of projections

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Outline of the talk

- Review of the classical results, in so much as they are relevant for the...
- ...review of the more recent results.

The questions

- What is the relation between the size of a set, and the size of its projections (e.g. onto hyperplanes)?
- One can think of many ways to ask more specific questions. Here's a purely incidence geometric formulation:

Question

Suppose $P \subset \mathbb{R}^2$ consists of n points. How many 1-dim subspaces L can there be, at most, such that

$$\operatorname{card} \pi_L(P) \leq n^s, \qquad s < 1?$$

The questions

- Szemerédi-Trotter is very efficient here. Assume that there are k subspaces $\mathcal{L}$ such that $\operatorname{card}\pi_L(P) \leq n^s$ for $L \in \mathcal{L}$.
- Consider the line family

$$\mathcal{L}^{\perp} := \{\pi_L^{-1}\{t\} : L \in \mathcal{L} \text{ and } t \in \pi_L(P)\}.$$

  Then $\operatorname{card}\mathcal{L}^{\perp} \leq kn^s$, and there are exactly kn point-line incidences between P and $\mathcal{L}^{\perp}$.
- Szemerédi-Trotter[1] gives $k \lesssim \max\{1, n^{2s-1}\}$.

[1] $|I(P, \mathcal{L}^{\perp})| \lesssim |P|^{2/3}|\mathcal{L}^{\perp}|^{2/3} + |P| + |\mathcal{L}^{\perp}|$

The Questions

- Here's the "geometric measure theory formulation":

Question

Let $K \subset \mathbb{R}^d$ be compact with $\dim K = s$. How many m-dimensional subspaces V can there be such that

$$\dim \pi_V(K) < \dim K? \tag{$*$}$$

Here $\pi_V : \mathbb{R}^d \to V$ is the orthogonal projection.

- If $\dim K > m$, $(*)$ holds for all V, so assume that $\dim K \leq m$.

The classical answer

- Even though Szemerédi-Trotter is no longer available, we still have quite satisfactory answers:

Theorem (Marstrand-Mattila)

If $K \subset \mathbb{R}^d$ and $\dim K \leq m$, then $\dim \pi_V(K) = \dim K$ for almost all m-dim subspaces V.[a]

[a] Here "almost all" refers to the natural Haar measure $\gamma_{d,m}$ living on the manifold $G(d, m)$ of m-dim subspaces

- Marstrand proved the case $d = 2$, $m = 1$ in 1954. General case by Mattila in 1975.
- Between 1954 and 1975, Kaufman re-proved the case $d = 2$, $m = 1$ using a technique now known as the "potential theoretic method". Marstrand's argument was more geometric.

The potential theoretic method

- Let's recall Kaufman's argument quickly: understanding the proof is useful in understanding (the issues in) the "restricted families" framework a bit later.
- In the discrete case, a projection is smaller than the set, if and only if the projection is non-injective. Analogously, the enemy here is the event that

$$|\pi_V(x) - \pi_V(y)| \ll |x - y|.$$

- The enemy is overcome by simply noting that this cannot happen for too many V's.

The potential theoretic method

- More precisely, fix $\delta \geq 0$ and $z := x - y \in \mathbb{R}^d \setminus \{0\}$, and consider the "sub-level set"

$$\{V \in G(d,m) : |\pi_V(z)| \leq \delta |z|\}.$$

- The volume of this set decays uniformly in z as $\delta \to 0$:

$$\gamma_{d,m}(\{V \in G(d,m) : |\pi_V(z)| \leq \delta |z|\}) \lesssim \delta^m.$$

- Note that the decay exponent is the same as in the assumption "$\dim K \leq m$" in Marstrand-Mattila's theorem.

The potential-theoretic method

- Now, we could prove the Marstrand-Mattila projection theorem. For simplicity, I'll stick to a discretised version.
- We say that a union of δ-balls is s-dimensional, if there are $\sim \delta^{-s}$ balls, the set C of their centres is δ-separated, and C also satisfies

$$\text{card}[C \cap B(x, r)] \lesssim \left(\frac{r}{\delta}\right)^s, \qquad x \in \mathbb{R}^d, \ r \geq \delta.^2$$

[2] If $\mathcal{H}^s(K) > 0$, one can find an s-dimensional collection of δ-balls with centres in K, for any $\delta > 0$.

Discrete Marstrand-Mattila

Theorem

Assume that $K \subset B(0,1)$ is an s-dimensional union of δ-balls. Then, for $t < s$, the set of V's such that $\pi_V(K)$ can be covered by $\leq \delta^{-t}$ δ-balls has $\gamma_{d,m}$-measure $\lessapprox \delta^{s-t}$.

- The proof is just double-counting.
- Whenever $\pi_V(K)$ can be covered by $\leq \delta^{-t}$ δ-balls, then there are $\gtrsim \delta^{t-2s}$ pairs (c_i, c_j), $c_i \neq c_j$ such that $|\pi_V(c_i - c_j)| < \delta$.

The details

- So, if

$$\gamma_{d,m}(\{V : \pi_V(K) \text{ can be covered by } \leq \delta^{-t} \text{ balls}\}) =: \Gamma,$$

we find

$$\gamma_{d,m} \times \sharp \times \sharp(\{(V, c_i, c_j) : |\pi_V(c_i - c_j)| \leq \delta\}) \gtrsim \Gamma \cdot \delta^{t-2s}.$$

- On the other hand, by the sub-level set estimate,

$$\begin{aligned}
\gamma_{d,m} \times \sharp \times \sharp(\{(V, c_i, c_j) &: |\pi_V(c_i - c_j)| \leq \delta\}) \\
&= \sum_{c_i \neq c_j} \gamma_{d,m}(\{V : |\pi_V(c_i - c_j)| \leq \delta\}) \\
&\lesssim \sum_{c_i \neq c_j} \left(\frac{\delta}{|c_i - c_j|}\right)^m \leq \sum_{c_i \neq c_j} \left(\frac{\delta}{|c_i - c_j|}\right)^s \lessapprox \delta^{-s}.
\end{aligned}$$

An abstraction

- For later use, I record a more general form of the result, which follows by the same proof.
- Let (Λ, γ) be probability space, and let $(\pi_\lambda)_\lambda : \mathbb{R}^d \to \mathbb{R}^m$ be a collection of 1-Lipschitz linear mappings satisfying

$$\gamma(\{\lambda : |\pi_\lambda(x)| \leq \delta|x|\}) \lesssim \delta^r.$$

Theorem (Abstract projection theorem (APT))

Assume that $K \subset \mathbb{R}^d$ is a compact set with $\dim K \leq r$. Then $\dim \pi_\lambda(K) = \dim K$ for γ-a.e. λ.

Restricted families of projections

- What are restricted families of projections? A vague formulation of the problem could be the following:
- You have a family of projections, and you'd like to prove a.s. dimension conservation, but it doesn't follow from the APT. Then it's likely that you're dealing with a *restricted family of projections*.
- At the moment, I don't know any examples of "restricted families" in the strict sense; only conjectures and partial results.

Restricted families of projections: non-examples

- For (almost) the rest of the talk, I'll concentrate on families of projections onto lines and planes in $\mathbb{R}^3$.
- First, consider the subfamily $\mathcal{G} \subset G(3,1)$ of all lines contained in the xy-plane. If $x = (0,0,1)$, one has

$$\{L \in \mathcal{G} : \pi_L(x) = 0\} = \mathcal{G}.$$

- So, there's no possibility of decay like

$$\gamma(\{L \in \mathcal{G} : |\pi_L(x)| \leq \delta|x|\}) \lesssim \delta^r, \quad r > 0,$$

  and the APT gives nothing useful. There's also nothing to be had: the 1-dimensional set $K = z$-axis π_L-projects to $\{0\}$ for all $L \in \mathcal{G}$.

Restricted families: non-examples

- Now something a bit more complicated: let $\mathcal{G} \subset G(3,2)$ be the "vertical" planes containing the z-axis.
- This is a 1-dimensional submanifold, and the best possible decay for any probability measure $\gamma_{\mathcal{G}}$ on $\mathcal{G}$ is

$$\gamma_{\mathcal{G}}(\{V \in \mathcal{G} : |\pi_V(x)| \leq \delta |x|\}) \lesssim \delta.$$

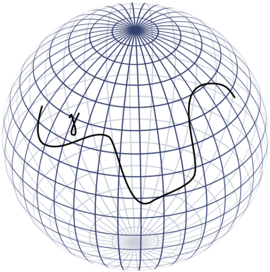
  The uniformly distributed measure achieves that.
- Hence, the APT promises dimension conservation for up to 1-dimensional sets. Again, that's the best you can get, because any subset K of the xy-plane projects inside the line $V \cap \{xy - \text{plane}\}$ for all $V \in \mathcal{G}$.

Restricted families of projections

- These examples are simple – and rather uninteresting – because they lack curvature.
- Let's add some curvature. Consider a smooth curve $\eta\colon (0,1) \to S^2$, satisfying

$$\text{span}\{\eta(t), \dot{\eta}(t), \ddot{\eta}(t)\} = \mathbb{R}^3, \qquad t \in (0,1).$$

- Something like this:



Restricted families of projections

- Then, we get a family of lines and planes by setting

$$\mathcal{G}_1(\eta) := \{\text{span}\{\eta(t)\} : t \in (0,1)\}$$

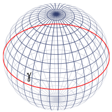
and

$$\mathcal{G}_2(\eta) := \{\text{span}\{\eta(t)\}^\perp : t \in (0,1)\}.$$

- The examples above were $\mathcal{G}_1(\eta)$ and $\mathcal{G}_2(\eta)$, corresponding to the curve η parametrising the unit circle on the *xy*-plane. But of course

$$\text{span}\{\eta(t), \dot{\eta}(t), \ddot{\eta}(t)\} = xy - \text{plane}$$

for $t \in (0,1)$, so the curvature requirement excludes these examples.



Restricted families of projections

- Assuming the curvature condition, counterexamples become very evasive, so I may as well conjecture:

Conjecture

The projections π_L, $L \in \mathcal{G}_1(\eta)$, should conserve dimension for up to 1-dimensional sets, and the projections π_V, $V \in \mathcal{G}_2(\eta)$ should conserve dimension for up to 2-dimensional sets.

Kaufman's method

- Sanity check: does it follows from the APT?
- The projection families $\mathcal{G}_1(\eta)$ and $\mathcal{G}_2(\eta)$ are parametrised by $(0,1)$, so the most natural choice for $\gamma_{\mathcal{G}}$ on both manifolds is essentially $\mathcal{L}^1|_{(0,1)}$. These bounds are easy and sharp:

$$\mathcal{L}^1(\{L \in \mathcal{G}_1(\eta) : |\pi_L(x)| \leq \delta\}) \lesssim \delta^{1/2},$$

and

$$\mathcal{L}^1(\{V \in \mathcal{G}_2(\eta) : |\pi_V(x)| \leq \delta\}) \lesssim \delta.$$

Kaufman's method

- The bounds have the following corollary:

Corollary (to the APT)

The projections π_L, $L \in \mathcal{G}_1(\eta)$, conserve dimension for up to $1/2$-dimensional sets, and the projections π_V, $V \in \mathcal{G}_2(\eta)$, conserve dimension for up to 1-dimensional sets.

- It's worth observing that the "$1/2$" already improves on the non-curved case (where no positive result was to be had), but the "1" doesn't.

Small improvements

- Nevertheless, the "1" is not the end of the story here:

Theorem (Fässler, O. (2013))

For every $s > 1$, there is $\sigma(s) > 1$ such that the following holds. If $\dim K = s$, then $\dim \pi_V(K) \geq \sigma(s)$ for almost every $V \in \mathcal{G}_2(\eta)$.

- For $\mathcal{G}_1$, we obtained the same result for the *packing dimension* of projections, but the Hausdorff dimension narrowly escaped. Except for this special curve:

$$\eta(t) = (\cos(t), \sin(t), 1).$$

Theorem (O. (2013))

For this special curve η, the previous theorem holds with $\mathcal{G}_2$ replaced by $\mathcal{G}_1$ and "1" replaced by "$1/2$".

The proof in four slides (1)

- Recall: we're interested in the one-dimensional family of 2-dim subspaces given by

$$V_t := \text{span}\{\eta(t)\}^{\perp}, \qquad t \in (0,1).$$

- In order for the proof of the APT to work directly, we should get a good upper bound for

$$\sum_{c_i \neq c_j} |\{t : |\pi_{V_t}(c_i - c_j)| \leq \delta\}|$$

  for a well-distributed finite set $\{c_1, \ldots, c_N\}$ inside K.
- But for general $c_i - c_j$, the best we can get is

$$|\{t : |\pi_{V_t}(c_i - c_j)| \leq \delta\}| \lesssim \frac{\delta}{|c_i - c_j|}.$$

The proof in four slides (2)

- The key words are "general $c_i - c_j$".
- It's often the case that $c_i - c_j$ is not even close to perpendicular to **any** of the planes V_t, and then in fact

$$\{t : |\pi_{V_t}(c_i - c_j)| \leq \delta\} = \varnothing$$

for small $\delta > 0$.

- This leads us to consider a "counter-assumption": suppose that the sum

$$\sum_{c_i \neq c_j} |\{t : |\pi_{V_t}(c_i - c_j)| \leq \delta\}|$$

is roughly as large as the "general $c_i - c_j$" estimate allows. Can we describe the structure of $\{c_1, \ldots, c_N\}$?

The proof in four slides (3)

- Quite easily, in fact, and here's the answer:
- If the sum is almost as large as it can be (in view of the bound for general $c_i - c_j$), then there's a δ^κ-proportion of the points c_i such that a δ^κ-proportion of the set $\{c_1, \ldots, c_N\}$ is contained in a δ^κ-neighbourhood of

$$c_i + C := c_i + \bigcup_{t \in (0,1)} \mathrm{span}\{\eta(t)\}.$$

- Here $\kappa \searrow 0$, as the counter-assumption gets stronger.
- C is a conical surface of some sort, and $C(\delta^\kappa)$ will stand for its δ^κ-neighbourhood.

The proof in four slides (4)

- Almost done: since a large part of $\{c_1, \ldots, c_N\}$ is contained in many $(c_i + C(\delta^\kappa))$'s...
- ...a large part of $\{c_1, \ldots, c_N\}$ is actually contained in $(c_i + C(\delta^\kappa)) \cap (c_j + C_j(\delta^\kappa))$ for some $i \neq j$!
- We can also choose i, j so that the cone vertices c_i, c_j are pretty far apart.
- How does $(c_i + C(\delta^\kappa)) \cap (c_j + C(\delta^\kappa))$ look like? Since $(c_i + C) \cap (c_j + C)$ is the intersection of two conical surfaces, it's something essentially one-dimensional.
- This is a bit tedious to prove, but the upshot is that we've managed to cram a large part of a > 1-dimensional discrete set $\{c_1, \ldots, c_N\}$ inside an essentially one-dimensional set. But that's just not possible.

Further results

- The "restricted families of projections" problem in $\mathbb{R}^3$ is closely related to Fourier restriction questions.
- D. and R. Oberlin wrote a paper about this last year:

Theorem (D. and R. Oberlin, 2013)

Assuming the curvature condition,

$$\dim \pi_{V_t}(K) \geq \frac{3 \dim K}{4}$$

for almost all $t \in (0,1)$. *If* $\dim K \geq 2$, *the lower bound can be improved to* $\min\{\dim K - 1/2, 2\}$.

Something I know nothing about

- It would be nice to improve on these results in $\mathbb{R}^3$...
- ...but similar questions in $\mathbb{R}^2$ are even more baffling.
- Vaguely speaking:

Question

Are there/what are the restricted families of projections in $\mathbb{R}^2$?

- While, for those who prefer more precise questions:

Question

Does there exist a zero-dimensional family $\mathcal{L}$ of 1-dim subspaces in $\mathbb{R}^2$ with the following property: given any compact set K with $\dim K \leq 1$, there's $L \in \mathcal{L}$ with

$$\dim \pi_L(K) = \dim K?$$

A discrete version

- A good starting point would be to understand the following question about point-line incidences.

Question

A set of 1-dim subspaces $\mathcal{L}$ is called *n-good*, if for any set $P \subset \mathbb{R}^2$ with $\operatorname{card} P = n$ there exists $L \in \mathcal{L}$ such that

$$\operatorname{card} \pi_L(P) \geq n^{3/4}.$$

How small n-good sets are there?

Proposition

All sets $\mathcal{L}$ with $\operatorname{card} \mathcal{L} \gg n^{1/2}$ are n-good. No $\mathcal{L}$ with $\operatorname{card} \mathcal{L} = C$ is n-good for $n \geq n_C$.

Conjecture

A random choice of $\sim \log n$ lines should be n-good.

Proof of the proposition

- To see that no C-element set can be n-good, there's a direct construction available, but here I present a slicker one that András Máthé noticed.
- Given any finite set $K = \{k_1, \ldots, k_C\} \subset \mathbb{R}$, a construction of Elekes-Erdős says that there exists an n-point set $A \subset \mathbb{R}$ (for some large n) containing $\gtrapprox n^2$ nomothetic copies of K.
- In other words, there are $\gtrapprox n^2$ pairs $(x, y) \in \mathbb{R}^2$ such that $x + yK \subset A$.
- Now, let P be the set of these pairs, and note that $\pi_{L_j}(P) \subset A$ for all $L_j := \mathrm{span}\{(1, k_j)\}$, $1 \leq j \leq C$. In particular,

$$\mathrm{card}\, \pi_{L_j}(P) \leq n = (n^2)^{1/2} \lessapprox (\mathrm{card}\, P)^{1/2}, \qquad 1 \leq j \leq C.$$

- G. ELEKES AND P. ERDŐS: *Similar Configurations and Pseudo Grids*, Colloq. Math. Soc. János Bolyai **63** (1994)
- K. FÄSSLER AND T.O.: *On restricted families of projections in* $\mathbb{R}^3$, Proc. LMS (to appear), arXiv:1302.6550
- R. KAUFMAN: *On Hausdorff dimension of projections*, Mathematatika **15** (1968), pp. 153–155
- J.M. MARSTRAND: *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proc. LMS (3) **4** (1954), pp. 257-302
- P. MATTILA: *Hausdorff dimension, orthogonal projections, and intersections with planes*, Ann. Acad. Sci. Fenn. Ser. A I Math **1** (1975), pp. 227–244
- T.O.: *Hausdorff dimension estimates for restricted families of projections in* $\mathbb{R}^3$, arXiv:1304.4955