Sharp square function estimates for the Bochner-Riesz means

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Bochner-Riesz means

- Bochner-Riesz operator of order α ($\alpha > 0$)

$$\mathcal{R}_t^\alpha f(x) = \int_{|\xi| \le t} e^{2\pi i x\cdot\xi} \left(1 - \frac{|\xi|^2}{t^2}\right)^\alpha \widehat{f}(\xi)\, d\xi.$$

- **Question.** Given p, for which α

$$\mathcal{R}_t^\alpha f \;\to f \;\text{ in } L^p \;\text{ as } t \to \infty\,?$$

- (Fefferman) Except for $p = 2$, the condition $\alpha > 0$ is necessary.
- Relation between smoothing order α and L^p convergence: The smoothing of order α reduces the effect of new part into the integral as t increases. The better the integral behaves, the larger α is.
- The problem has its origin in the study of convergence of Fourier series in $L^p(\mathrm{T}^d)$ and the problem is equivalent with that of its discrete counterpart

$$\sum_{|\mathbf{n}| \le t} \left(1 - \frac{|\mathbf{n}|^2}{t^2}\right)^\alpha \widehat{f}(\mathbf{n}) e^{2\pi i \mathbf{n}\cdot x} \to f \;\text{ in } L^p(\mathrm{T}^d) \;\text{ as } t \to \infty?$$

• By the uniform boundedness principle the question is equivalent with the uniform estimate

$$\|\mathcal{R}_t^\alpha f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}.$$

• **Bochner-Riesz Conjecture:** For $0 < \alpha$ and $1 \le p \le \infty$,

$$\|\mathcal{R}_t^\alpha f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$$

if and only if

$$\alpha > \alpha(p) = \max\big(d\big|\frac{1}{2} - \frac{1}{p}\big| - \frac{1}{2}, 0\big).$$

• The conjecture is true for $d = 2$ but open for $d \ge 3$.
• For $\max(p, p') > \frac{2d}{d-1}$, the necessity part follows from a Schwartz function and the explicit kernel estimate for $\mathcal{R}_t^\alpha$.
• For $\frac{2d}{d-1} \ge \max(p, p') > 2$ Fefferman's counterexample of the disk multiplier gives the necessary condition.

Brief history

• Bochner (1930): L^p boundedness for $1 \leq p \leq \infty$ if $\alpha > \frac{d-1}{2}$.
• Carleson and Sjölin (1972): When $d = 2$, the conjecture is true.

In higher dimensions

• Stein-Tomas (1979): The sharp L^2 restriction estimate for the sphere and Stein's argument which deduces L^p boundedness from L^2 restriction estimate,

$$\max(p, p') \geq 2(d+1)/(d-1).$$

• Bourgain (1990): In $\mathbb{R}^3$, L^p boundedness beyond the range of Stein-Tomas $((\max(p, p') \geq 4 - \epsilon_0)$.
• Tao-Vargas-Vega (1998), Tao-Vargas (2002): Further progress by a systematic bilinear approach in $\mathbb{R}^3$.
• L. (2004): For $d \geq 3$, by making use of the sharp bilinear restriction estimate for the elliptic surfaces due to Tao,

$$\max(p, p') > 2(d+2)/d.$$

• Bourgain-Guth, (2012): Using Bennett-Carbery-Tao's multilinear oscillatory estimates, the conjecture is now verified for

$$\max(p, p') \geq p_\circ,$$

where

$$p_\circ = 2 + \frac{12}{4d - 3 - k} \qquad \text{if } d \equiv k \ (mod\ 3),\ k = -1, 0, 1$$

• For large $d \gg 1$

$$\max(p, p') \geq \text{(bilinear) } 2 + \frac{4}{d}$$

$$\text{(multilinear) } p_\circ \approx 2 + \frac{3}{d}$$

$$\text{(conjecture) } \frac{2d}{d-1} \approx 2 + \frac{2}{d}.$$

Square function

- Consider the square function

$$G^\alpha f(x) = \Big(\int_0^\infty \Big| K_t^\alpha * f(x) \Big|^2 \frac{dt}{t} \Big)^{1/2},$$

where

$$\widehat{K_t^\alpha}(\xi) = 2(\alpha+1) \frac{|\xi|^2}{t^2} \Big(1 - \frac{|\xi|^2}{t^2} \Big)_+^\alpha.$$

- Naturally, G^α may be considered as vector valued operator of order α. This was introduced by Stein, in a slightly different form, to control maximal Bochner-Riesz operators for everywhere convergence of Bochner-Riesz means.
- Note that

$$t \frac{\partial}{\partial_t} \widehat{\mathcal{R}_t^{\alpha+1} f}(\xi) = \widehat{K_t^\alpha}(\xi) \widehat{f}(\xi).$$

- L^2 bound, Stein (1958): For $\alpha > -1/2$, $\|G^\alpha f\|_2 \le C\|f\|_2$. The proof relies on Plancherel's theorem.

L^p boundedness of G^α

- For $1 \le p < 2$, it is known that

$$\|G^\alpha f\|_p \le C\|f\|_p.$$

if and only if

$$\alpha > d\big(1/p - 1/2\big) - 1/2.$$

The oscillation for large $|x|$ plays no role here.

- In the range $2 < p < \infty$, the oscillation of the kernel K_t^α is important and the problem is closely related to the Fourier restriction and Bochner–Riesz problems. It is known that the condition

$$\alpha > \alpha(p) - 1/2$$

is necessary.

Square function conjecture

- **Conjecture.** For $2 < p \le \infty$

$$\|G^\alpha f\|_p \le C\|f\|_p$$

if and only if $\alpha > \alpha(p) - 1/2$.

- When $d = 2$, the conjecture was verified by Carbery (1983) via an adaptation of Fefferman and Cordoba arguments which prove Bochner-Riesz conjecture in $\mathbb{R}^2$.

- For $p \ge \frac{2(d+1)}{d-1}$, Christ (1985): using the Tomas-Stein restriction theorem.

• $p \geq \frac{2(d+2)}{d}$, L.-Rogers-Seeger (2010): relying on the sharp bilinear restriction theorem for the elliptic surfaces,

$$\int (G^\alpha f(x))^2 w(x)\, dx \lesssim \int |f(x)|^2 \mathfrak{W}_q w(x)\, dx,$$

for $1 \leq q < (d+2)/2$, and $\alpha > d/2q - 1$, where $w \to \mathfrak{W}_q w$ is bounded on L^r with $q < r \leq \infty$.

Theorem (L.-Seeger, 2013)

Let $p_s = p_s(d)$ be defined by

$$p_s = 2 + \frac{12}{4d-6-k}, \text{ if } d \equiv k \ (mod\ 3),\ k = 0, 1, 2.$$

If $p \geq p_s$ and $\alpha > \alpha(p) - \frac{1}{2}$, $\|G^\alpha f\|_p \leq C\|f\|_p$.

• $p_s > p_\circ$ due to inefficiency in exploiting orthogonality for the square funciton) and this is new when $d \geq 9$.

Applications

- **Maximal Bochner-Riesz.** If $\alpha > \alpha(p)$, for $p \geq p_s$

$$\|\sup_{t>0} |\mathcal{R}_t^\alpha f|\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

- **Radial multiplier.** Let m be a bounded function on $(0,\infty)$ and define a multiplier operator by

$$\mathcal{F}(T_m f)(\xi) = m(|\xi|)\widehat{f}(\xi).$$

- Hörmander (1960): Let $\varphi \in C_c^\infty[1/2,2]$. T_m is bounded on L^p, $1 < p < \infty$, if $\sup_{t>0} \|\varphi\, m(t\,\cdot\,)\|_{L^2_\alpha(\mathbb{R})} < \infty$, $\alpha > d/2$.
- (Carbery-Gasper-Trebels (1983))

$$g[T_m f](x) \leq C \sup_{t>0} \|\varphi m(t\cdot)\|_{L^2_\alpha(\mathbb{R})}\, G^{\alpha-1} f(x),$$

where g is a g-function satisfying $\|g[f]\|_p \sim \|f\|_p$ for $1 < p < \infty$.
- If $\max(p,p') \geq \min(p_s, 2(d+2)/d)$, then for $\alpha > d\left|\frac{1}{p} - \frac{1}{2}\right|$

$$\left\|T_m f\right\|_{L^p \to L^p} \lesssim \sup_{t>0} \|\varphi\, m(t\,\cdot\,)\|_{L^2_\alpha(\mathbb{R})}.$$

Frequency localization

- Let $\phi \in \mathcal{S}$ supported in $[1/2, 2]$. For $0 < \delta < 1/2$ set

$$\widehat{S_\delta^t f}(\xi) = \phi\big(\delta^{-1}\big(1 - \tfrac{|\xi|^2}{t^2}\big)\big)\widehat{f}(\xi).$$

- By dyadic decomposition of the multiplier $|\xi|^2(1-|\xi|^2)_+^\alpha$ away from the singularity and Littlewood-Paley decomposition (and discarding harmless factors), the matter reduces to showing

$$\Big\|\Big(\int_1^2 |S_\delta^t f(x)|^2 dt\Big)^{1/2}\Big\|_p \lesssim \delta^{\frac{1}{2}}\delta^{-\alpha(p)-\epsilon}\|f\|_p.$$

- If $p = 2$, by Plancherel's theorem (because $||\xi| - t| \lesssim \delta$ if the integrand is nonzero)

$$\Big\|\Big(\int_1^2 |S_\delta^t f(x)|^2 dt\Big)^{1/2}\Big\|_2^2 = \iint_1^2 \big|\phi\big(\delta^{-1}\big(1 - \tfrac{|\xi|^2}{t^2}\big)\big)\big|^2 dt\big|\widehat{f}(\xi)\big|^2 d\xi$$
$$\lesssim \delta \int \big|\widehat{f}(\xi)\big|^2 d\xi.$$

From L^2-restriction to square function estimates

• Tomas-Stein restriction theorem for the sphere, Minkowski's inequality and Plancherel's theorem gives

$$\|S_\delta^t f\|_{\frac{2(d+1)}{d-1}} \lesssim \delta^{\frac{1}{2}} \|f\|_2 \text{ for } t \sim 1.$$

• Let J be essentially disjoint intervals of length δ such that $\cup J = [1,2]$ and $\widetilde{J}$ be the interval $\{t : \mathit{dist}\,(t, J) \le 2\delta\}$. Then, set

$$\widehat{f}_{\widetilde{J}}(\xi) = \widehat{f}(\xi)\chi_{\{|\xi| \in \widetilde{J}\}}.$$

• For each fixed t,$\widehat{S_\delta^t f}$ is supported in $\{\xi : t - 2\delta \le |\xi| \le t + 2\delta\}$. For $t \in J$,

$$S_\delta^t f = S_\delta^t f_{\widetilde{J}}.$$

• With $p = \frac{2(d+1)}{d-1}$

$$\Big\| \Big(\int_1^2 |S_\delta^t f|^2 dt \Big)^{1/2} \Big\|_p = \Big\| \Big(\sum_I \int_I |S_\delta^t f_{\widetilde{I}}|^2 dt \Big)^{1/2} \Big\|_p$$

$$\le \Big(\sum_I \int_I \|S_\delta^t f_{\widetilde{I}}\|_p^2 \, dt \Big)^{\frac{1}{2}} \lesssim \delta^{\frac{1}{2}} \Big(\sum_I \int_I \|f_{\widetilde{I}}\|_2^2 \, dt \Big)^{\frac{1}{2}} \lesssim \delta \|f\|_2.$$

• Using the fact that the kernel of S_δ^t rapidly decaying outside of $B(0, C\delta^{-1})$, one may assume f is supported in a ball of radius $\sim \delta^{-1-\epsilon}$. By Hölder's inequality

$$\Big\| \Big(\int_1^2 |S_\delta^t f|^2 dt \Big)^{1/2} \Big\|_p \lesssim \delta \delta^{-d(\frac{1}{2} - \frac{1}{p}) - c\epsilon} \|f\|_p.$$

• The disjointness of Fourier support of S_δ^t as t varies is important for the extra gain of $\delta^{\frac{1}{2}}$.

Oscillatory integral operators

• Bourgain-Guth's result on Bochner-Riesz bounds were obtained from the sharp oscillatory integral estimate

$$\|T_\lambda g\|_p \lesssim \lambda^{-\frac{d}{p}} \|g\|_p \,,$$

where

$$T_\lambda g(x) = \int_{\mathbb{R}^{d-1}} e^{i\lambda\phi(x,z)} a(x,z) g(z) dz, \quad x \in \mathbb{R}^d.$$

• This has obvious advantage over handling multiplier operators because it is similar to the adjoint restriction operators. In fact, one may consider $g \in L^\infty$.
• However, this approach doesn't seem adequate for the study of the square function. Especially, the disjointness of Fourier transform (as t varies) is difficult to exploit.

Fourier transform side approach

• Denote by $m(D)f$ the multiplier operator given by

$$\mathcal{F}(m(D)f)(\xi) = m(\xi)\widehat{f}(\xi), \text{ and } D = (D', D_d).$$

• (General class of square functions) For $0 < \delta$ define $S_\delta = S_\delta(\psi, \eta)$ by

$$S_\delta f(x) = \Big(\int_{I=[-1,1]} \Big| \phi\Big(\frac{\eta(D,t)(D_d - \psi(D',t))}{\delta} \Big) f(x) \Big|^2 dt \Big)^{\frac{1}{2}}.$$

• We need to consider the square functions given by classes of ψ and η which are suitable the induction argument.
• In fact, this makes it possible to absorb the perturbed terms which arise from translation and rescaling after decomposition.

Multilinear approach: From multilinear to linear

- Mutilinear estimates for the square function
  - ▶ k-linear estimate with transversality
  - ▶ k-linear estimate with transversality but confined Fourier support ($k+1$ transverality fails)
- Stability of these estimates under smooth perturbation of the associates surfaces.
- Improvement of bounds due to smaller Fourier supports
- **Multi-scale decomposition.** Control of given operator $S_\delta f$ a sum of products $(\prod S_\delta f_i)^{\frac{1}{k}}$ while remaining parts are controlled by functions of smaller Fourier supports: With implicit constants (over simplified!)

$$S_\delta f \lesssim \sum S_\delta b_i + \sum_{k=2}^{l-1} \sum_{\substack{k-trans \text{ but} \\ non-(k+1)-trans}} (\prod_{i=1}^{k} S_\delta g_i)^{\frac{1}{k}} + \sum_{l-trans} (\prod_{i=1}^{l} S_\delta g_i)^{\frac{1}{l}}$$

The induction quantity

- (One parameter family of elliptic functions) Denote by $\mathfrak{G}(\epsilon_\circ, N)$ the class of smooth function ψ defined on $I^{d-1} \times I$ which satisfy

$$\Big\|\psi(\xi', t) - \frac{1}{2}|\xi'|^2 - t\Big\|_{C^N(I^{d-1}\times I)} \le \epsilon_\circ.$$

- Define a class of smooth functions $\mathcal{E}(N)$ by

$$\mathcal{E}(N) = \big\{\eta \in C^\infty(I^d \times I) : \|\eta\|_{C^N(I^d\times I)} \le 2, \eta \sim 1\big\}.$$

- Define $B(\delta) = B_p(\delta)$ by

$$B(\delta) = \sup\Big\{\|S_\delta(\psi,\eta)f\|_{L^p} : \psi \in \mathfrak{G}(\epsilon_\circ, N), \\ \eta \in \mathcal{E}(N),\ \|f\|_p \le 1, \mathrm{supp}\widehat{f} \subset \frac{1}{2}I^d\Big\}.$$

- Since $\mathrm{supp}\widehat{f} \subset \frac{1}{2}I^d$, by estimating the kernel

$$B(\delta) \le C \text{ if } \delta \ge 1, \text{ and } B(\delta) \le C\delta^{-(d-1)/2+1/2} \text{ if } \delta < 1.$$

- We need
$$B(\delta) \leq C\delta^{-\frac{d-2}{2}+\frac{d}{p}-\epsilon}.$$
- For monotonicity we slightly modify it. Define for $\beta > 0$
$$\mathcal{B}^\beta(\delta) = \mathcal{B}_p^\beta(\delta) \equiv \sup_{\delta \leq s \leq 1} s^{\frac{d-2}{2}-\frac{d}{p}+\beta} B_p(s).$$

We show $\mathcal{B}^\beta(\delta) \leq C$ for any $\beta > 0$.

Proposition (Improvement by small Fourier supports)

Let $0 < \delta \ll 1$, $\psi \in \mathfrak{C}(\epsilon_\circ, N)$, and $\eta \in \mathcal{E}(N)$. Suppose that $\widehat{f}$ is supported in $\mathfrak{q}(a, \varepsilon)$, $\sqrt{\delta} \leq \varepsilon \leq 1/2$, and $a \in \frac{1}{2}I^d$. Then, if $\epsilon_\circ > 0$ is small enough, there is an $\kappa = \kappa(\epsilon_\circ, N)$ such that
$$\|S_\delta(\psi, \eta)f\|_p \leq C\varepsilon^{\frac{1}{p}+\frac{1}{2}} B_p(\varepsilon^{-2}\delta)\|f\|_p$$
holds with C, independent of ψ, η and ε, whenever $\sqrt{\delta} \leq \varepsilon \leq \kappa$.

Multilinear estimates for square function

Let $\psi \in \mathfrak{G}(\epsilon_\circ, N)$ and set

$$\Gamma^t = \Gamma^t(\psi) := \{(\xi', \psi(\xi', t)) : \xi' \in [-1, 1]^{d-1}\}.$$

- **Normal vector field $\mathbf{n} = \mathbf{n}(\psi)$** For each $\xi = (\zeta, \tau) \in I^d$ there is a unique t such that $\xi = (\zeta, \psi(\zeta, t))$ since $\partial_t \psi \sim 1$. Then we define $\mathbf{n}(\xi)$ to be the unit normal vector to Γ^t at ξ which forms a vector field on I^d.

Theorem

Let $\psi \in \mathfrak{G}(\epsilon_\circ, N)$, $\eta \in \mathcal{E}(N)$ *and* $\sigma > 0$. *Suppose that, for* $0 < \delta \ll \sigma$,

$$|\mathbf{n}(\xi_1) \wedge \mathbf{n}(\xi_2) \wedge \cdots \wedge \mathbf{n}(\xi_k)| \gtrsim \sigma$$

holds whenever $\xi_i \in \operatorname{supp}\widehat{f_i} + O(\delta)$, $i = 1, 2, \ldots, k$. *Then, if* $p \geq 2k/(k-1)$ *and* $\epsilon_\circ$ *is small enough, for* $\epsilon > 0$, *there is an* $N(\epsilon)$ *such that the following estimates hold with* C, C_ϵ, *independent of* ψ, η, *if* $N \geq N(\epsilon)$:

$$\Big\| \prod_{i=1}^{k} S_\delta(\psi, \eta) f_i \Big\|_{\frac{p}{k}} \leq C\sigma^{-C_\epsilon}\delta^{-\epsilon} \prod_{i=1}^{k} \Big(\delta \|f_i\|_2\Big),$$

$$\Big\| \prod_{i=1}^{k} S_\delta(\psi, \eta) f_i \Big\|_{\frac{p}{k}} \leq C\sigma^{-C_\epsilon}\delta^{-\epsilon} \prod_{i=1}^{k} \Big(\delta^{\frac{d}{p} - \frac{d-2}{2}} \|f_i\|_p\Big).$$

- The second inequality follows from the first by spatial localization and Hölder's inequality.

• Currently, we have $C_\epsilon \sim C \log \frac{C}{\epsilon}$.
• When the degree of transverality gets smaller, the bound becomes really bad. In particular, the bound becomes inefficient when $\sigma \sim \delta^c$.

• This cannot be obtained by direct application of multilinear estimates because $p/k \leq 1$ in general. Instead we redo the proof of the multilinear restriction estimate using the multilinear Kakeya estimate.
• It is not easy to run induction argument with the estimate of this form. We need to find a suitable setting which allows the induction argument.

Proposition

Let $\psi \in \mathfrak{G}(\epsilon_\circ, N)$, *and* $\sigma > 0$. *Suppose that, for* $0 < \delta \ll \sigma$, $i = 1, \ldots, k$,

$$\operatorname{supp} \mathcal{F}(G_i(\cdot, t)) \subset \Gamma^t + O(\delta), \quad t \in I,$$

and

$$|\mathbf{n}(\xi_1) \wedge \mathbf{n}(\xi_2) \wedge \cdots \wedge \mathbf{n}(\xi_k)| \gtrsim \sigma,$$

whenever $\xi_i \in \operatorname{supp} \mathcal{F}(G_i(\cdot, t)) + O(\delta)$ *for some* $t \in I$. *Then, if* $p \geq 2k/(k-1)$ *and* $\epsilon_\circ > 0$ *is small enough, for* $\epsilon > 0$ *there are constants* $N = N(\epsilon)$ *such that*

$$\Big\| \prod_{i=1}^{k} \|G_i\|_{L^2_t(I)} \Big\|_{\frac{p}{k}} \leq C\sigma^{-C_\epsilon}\delta^{-\epsilon} \prod_{i=1}^{k} \big(\delta^{\frac{1}{2}} \|G_i\|_{L^2_{x,t}}\big)$$

holds with C, C_ϵ, *independent of* ψ, η, *if* $N \geq N(\epsilon)$.

- Applying this with $G_i = \phi\Big(\frac{\eta(D,t)(D_d - \psi(D',t))}{\delta}\Big) f_i$ gives the theorem.

Proposition (with confined Fourier support)

Let $\Pi \subset \mathbb{R}^d$ be a k-plane containing the origin. Under the same assumption as in the above proposition and suppose, additionally, that, for $i = 1, \ldots, k$,

$$\mathbf{n}\big(\operatorname{supp} \mathcal{F}(G_i(\cdot, t)\big) \subset \mathbb{S}^{d-1} \cap (\Pi + O(\delta)).$$

Then, if $2 \le p \le 2k/(k-1)$ and ϵ_0 is sufficiently small, for $\epsilon > 0$ there is an $N(\epsilon)$ such that the following estimate holds uniformly for ψ, η, if $N \ge N(\epsilon)$:

$$\Big\| \prod_{i=1}^{k} \|G_i\|_{L_t^2(I)} \Big\|_{\frac{p}{k}} \lesssim \sigma^{-C_\epsilon} \delta^{dk(\frac{1}{2} - \frac{1}{p}) - \epsilon} \prod_{i=1}^{k} \|G_i\|_{L^2_{x,t}}.$$

• At the critical $p = 2k/(k-1)$ this gives a better bound $\delta^{\frac{d}{2} - \epsilon}$, which is better than $\delta^{\frac{k}{2} - \epsilon}$.

• This lemma covers the case $(k+1)$-transversality in dimension fails while k-transversality is satisfied. So this will be used with a relatively large δ.

• Let $\{\mathfrak{q}\}$ be the collection of dyadic cubes of side length l, $\widetilde{\sigma} < l \leq 2\widetilde{\sigma}$. Define $f_{\mathfrak{q}}$ by

$$\mathcal{F}(f_{\mathfrak{q}}) = \chi_{\mathfrak{q}} \mathcal{F}(f), \text{ and set } R = 1/\widetilde{\sigma}.$$

Theorem

Let $\psi \in \mathfrak{G}(\epsilon_\circ, N)$, $\eta \in \mathcal{E}(N)$, Π be a k-plane containing the origin, and $\sigma > 0$. Suppose that, for $0 < \widetilde{\sigma} \ll \sigma$,

$$|\mathbf{n}(\xi_1) \wedge \mathbf{n}(\xi_2) \wedge \cdots \wedge \mathbf{n}(\xi_k)| \gtrsim \sigma$$

whenever $\xi_i \in \operatorname{supp} \widehat{f_i} + O(\widetilde{\sigma}), \quad i = 1, 2, \ldots, k$. Suppose

$$\mathbf{n}\big(\operatorname{supp} \widehat{f_i}\big) \subset \Pi + O(\widetilde{\sigma}), \quad i = 1, 2, \ldots, k.$$

Then, if $2k/(k-1) \leq p \leq 2$ and $0 < \delta \leq \widetilde{\sigma}$, for $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$\Big\| \prod_{i=1}^{k} S_\delta f_i \Big\|_{L^{\frac{p}{k}}(B(x,R))} \lesssim \sigma^{-C_\epsilon} \widetilde{\sigma}^{-\epsilon} \times \prod_{i=1}^{k} \Big\| \Big(\sum_{\mathfrak{q}} |S_\delta f_{i,\mathfrak{q}}|^2 \Big)^{\frac{1}{2}} \rho\Big(\frac{\cdot - x}{R}\Big) \Big\|_{L^p}$$

holds uniformly for $\psi \in \mathfrak{G}(\epsilon_\circ, N)$ if $N \geq N(\epsilon)$.

• Increasing multi-linearity gives better bounds but the size of non-transversal fourier support also get larger.

• The proof is basically to compromise the estimates in two theorems.

• Non-transversal is handled by the ℓ^2-estimate and the smallness of Fourier support.

Multi-scale decomposition

• Let $\sigma_1, \dots, \sigma_d$ be dyadic numbers to be chosen later such that

$$\delta \ll \sigma_{d-1} \ll \cdots \ll \sigma_1 \ll \sigma_0 = 1.$$

We call σ_i i-th scale.
• $\{\mathfrak{q}^i\}$: the collection of all (closed) dyadic cubes $\mathfrak{q}^i$ of sidelength $2\sigma_i$, $\mathfrak{q}^i \subset I^d$. $i = 1, \dots, d-1$,

$$f = \sum_{\mathfrak{q}^i} f_{\mathfrak{q}^i}, \text{ for } i = 1, \dots, d-1.$$

• Let $2 \leq p < \infty$. Then, there is a constant C, independent of $\{\mathfrak{q}\}$, such that

$$\Big(\sum_{\mathfrak{q}^i} \| f_{\mathfrak{q}^i} \|_p^p \Big)^{\frac{1}{p}} \leq C \| f \|_p.$$

- Let $\mathfrak{q}_1^k, \dots, \mathfrak{q}_{k+1}^k \in \{\mathfrak{q}^k\}$. We write $\mathfrak{q}_1^k$, $\mathfrak{q}_2^k$, $\dots$, $\mathfrak{q}_{k+1}^k$: $trans$ (transversal) if

$$|\mathbf{n}(\xi_1) \wedge \mathbf{n}(\xi_2) \wedge \cdots \wedge \mathbf{n}(\xi_{k+1})| \gtrsim \sigma_1 \sigma_2 \dots \sigma_k$$

whenever $\xi_i \in \mathfrak{q}_i^k$, $i = 1, \dots, k+1$.

- **Dual cubes of i-th scale** $\{\mathfrak{Q}^i\}$: the collection of dyadic intervals of sidelength $2/\sigma_i$, which covers $\mathbb{R}^d$ so that

$$\bigcup_{\mathfrak{Q}^i} \mathfrak{Q}^i = \mathbb{R}^d$$

Decomposition increasing the degree of multilinearity

- After l-th scale decomposition

$$\|S_\delta f\|_p \lesssim \sum_{k=1}^{l} \sigma_{k-1}^{-C} \sigma_k^{\frac{2}{p}} B_p(\sigma_k^{-2}\delta)\|f\|_p + \sum_{k=2}^{l} \sigma_{k-1}^{-C} \widetilde{\Lambda}^k f$$

$$+ \sigma_l^{-C} \sup_{\tau_1,\dots,\tau_{l+1}} \max_{\mathfrak{q}_1^l,\dots\mathfrak{q}_{l+1}^l : trans} \Big\| \prod_{i=1}^{l+1} S_\delta \tau_i f_{\mathfrak{q}_i^l} \Big\|_{L^{\frac{p}{l+1}}}^{\frac{1}{l+1}},$$

where

$$\widetilde{\Lambda}^k f = \sup_{\tau_1,\dots,\tau_k} \max_{\mathfrak{q}_1^{k-1},\dots,\mathfrak{q}_k^{k-1} : trans} \times$$

$$\Big(\sum_{\mathfrak{Q}^k} \Big\| \prod_{i=1}^{k} S_\delta \Big(\sum_{\substack{\mathfrak{q}_i^k \subset \mathfrak{q}_i^{k-1} : \\ \mathbf{n}(\mathfrak{q}_i^k) \in \Pi^k + O(\sigma_k)}} \tau_i f_{\mathfrak{q}_i^k} \Big) \Big\|_{L^{\frac{p}{k}}(\mathfrak{Q}^k)}^{\frac{p}{k}} \Big)^{\frac{1}{p}}.$$

Here Π^k is a k-plane containing the origin which depends on $\mathfrak{q}_1^{k-1}, \dots, \mathfrak{q}_k^{k-1}, \mathfrak{Q}^k$, and $\tau_1, \dots, \tau_k$.

Two estimates for $\widetilde{\Lambda}^k f$

- Since the mother cubes $\mathfrak{q}_1^{k-1}, \mathfrak{q}_2^{k-1}, \dots, \mathfrak{q}_k^{k-1}$ are already transversal, for $p \geq 2k/(k-1)$

$$\|\widetilde{\Lambda}^k f\|_p \leq C\sigma_k^{-C}\delta^{-\frac{d-2}{2}+\frac{d}{p}-\beta}\|f\|_p.$$

- Using square function estimates for confined Frequency support and $\#\Big(\{\mathfrak{q}_i^k : \mathbf{n}(\mathfrak{q}_i^k) \subset \Pi + O(\sigma_k)\}\Big) \lesssim \sigma_k^{-k}$, for $2 \leq p \leq 2k/(k-1)$

$$\|\widetilde{\Lambda}^k f\|_p \leq C\sigma_k^{-\epsilon-(k+1)(\frac{1}{2}-\frac{1}{p})}\sigma_k B(\sigma_k^{-2}\delta)\|f\|_p.$$

- Combining two cases, if $\delta_\circ \leq \delta$, for some $\alpha > 0$

$$\|\widetilde{\Lambda}^k f\|_{L^p} \leq C\delta^{-\frac{d-2}{2}+\frac{d}{p}-\beta}\Big(\sigma_k^{-C} + \sigma_k^{\alpha}\,\mathcal{B}^{\beta}(\delta_\circ)\Big)\|f\|_p$$

if $p > \min(\frac{2(2d-k-1)}{2d-k-3}, \frac{2k}{k-1})$.

- If

$$p \geq p(l) = \max_{k=1,\ldots,l} \min(\frac{2(2d-k-1)}{2d-k-3}, \frac{2k}{k-1}) \vee \frac{2(l+1)}{l} \vee \frac{2(d-1)}{d-2},$$

$$\|S_\delta f\|_p \lesssim \Big(\sum_{k=1}^{l} (\sigma_{k-1}^{-C} + \sigma_{k-1}^{-C} \sigma_k^{\alpha} \mathcal{B}^{\beta}(\delta_\circ)) + \sigma_l^{-C} \Big) \delta^{-\frac{d-2}{2}+\frac{d}{p}-\beta} \|f\|_p.$$

- By stability of the estimates along ψ, η and f, taking sup we have, for $\delta_\circ \leq \delta \leq 1$ and some $\alpha > 0$,

$$\delta^{-\frac{d-2}{2}+\frac{d}{p}-\beta} B(\delta) \lesssim \sum_{k=1}^{l} \Big(\sigma_{k-1}^{-C} + \sigma_{k-1}^{-C}\, \sigma_k^{\alpha} \mathcal{B}^{\beta}(\delta_\circ) \Big) + \sigma_l^{-C}.$$

- Hence taking sup along δ, $\delta_\circ \leq \delta \leq 1$, for $p \geq p(l)$

$$\mathcal{B}^{\beta}(\delta_\circ) \lesssim \Big(\sum_{k=1}^{l} \sigma_{k-1}^{-C} \sigma_k^{\alpha} \Big) \mathcal{B}^{\beta}(\delta_\circ) + \sum_{k=1}^{l} \sigma_k^{-C}.$$

- The minimum of $p(l)$ is p_s.

Remarks

• $p(l)$ minimizes with $l \sim \frac{2d}{3}$. The roles of multilinear estimates of the higher degrees seem unclear.

• A different proof of the currently best known result for the Bochner-Riesz problem in Fourier transform side.

• The same result for a little more general (non-symmetric) multiplier

$$\mathcal{F}(T^\alpha f)(\xi) = \big(\tau - \psi(\zeta)\big)_+^\alpha \chi_\circ(\xi)\widehat{f}(\xi)$$

if ψ is elliptic.