Sharp square function estimates for the Bochner-Riesz means

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Bochner-Riesz means

- Bochner-Riesz operator of order $\alpha$ ($\alpha > 0$)

\[
\mathcal{R}_t^\alpha f(x) = \int_{|\xi| \leq t} e^{2\pi i x \cdot \xi} \left(1 - \frac{|\xi|^2}{t^2}\right)^\alpha \hat{f}(\xi) \, d\xi.
\]

- **Question.** Given $p$, for which $\alpha$

\[
\mathcal{R}_t^\alpha f \to f \text{ in } L^p \text{ as } t \to \infty?
\]

- (Fefferman) Except for $p = 2$, the condition $\alpha > 0$ is necessary.
- Relation between smoothing order $\alpha$ and $L^p$ convergence: The smoothing of order $\alpha$ reduces the effect of new part into the integral as $t$ increases. The better the integral behaves, the larger $\alpha$ is.
- The problem has its origin in the study of convergence of Fourier series in $L^p(\mathbb{T}^d)$ and the problem is equivalent with that of its discrete counterpart

\[
\sum_{|n| \leq t} \left(1 - \frac{|n|^2}{t^2}\right)^\alpha \hat{f}(n) e^{2\pi i n \cdot x} \to f \text{ in } L^p(\mathbb{T}^d) \text{ as } t \to \infty?
\]
• By the uniform boundedness principle the question is equivalent with the uniform estimate

\[ \| R_\alpha^t f \|_{L^p(\mathbb{R}^d)} \leq C \| f \|_{L^p(\mathbb{R}^d)}. \]

• **Bochner-Riesz Conjecture:** For \( 0 < \alpha \) and \( 1 \leq p \leq \infty \),

\[ \| R_\alpha^t f \|_{L^p(\mathbb{R}^d)} \leq C \| f \|_{L^p(\mathbb{R}^d)} \]

if and only if

\[ \alpha > \alpha(p) = \max \left( d \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right). \]

• The conjecture is true for \( d = 2 \) but open for \( d \geq 3 \).
• For \( \max(p, p') > \frac{2d}{d-1} \), the necessity part follows from a Schwartz function and the explicit kernel estimate for \( R_\alpha^t \).
• For \( \frac{2d}{d-1} \geq \max(p, p') > 2 \) Fefferman’s counterexample of the disk multiplier gives the necessary condition.
Brief history

- Bochner (1930): $L^p$ boundedness for $1 \leq p \leq \infty$ if $\alpha > \frac{d-1}{2}$.
- Carleson and Sjölin (1972): When $d = 2$, the conjecture is true.

In higher dimensions

- Stein-Tomas (1979): The sharp $L^2$ restriction estimate for the sphere and Stein’s argument which deduces $L^p$ boundedness from $L^2$ restriction estimate,

$$\max(p, p') \geq 2(d + 1)/(d - 1).$$

- Bourgain (1990): In $\mathbb{R}^3$, $L^p$ boundedness beyond the range of Stein-Tomas $(\max(p, p') \geq 4 - \epsilon_0)$.
- L. (2004): For $d \geq 3$, by making use of the sharp bilinear restriction estimate for the elliptic surfaces due to Tao,

$$\max(p, p') > 2(d + 2)/d.$$
• Bourgain-Guth, (2012): Using Bennett-Carbery-Tao’s multilinear oscillatory estimates, the conjecture is now verified for

\[ \max(p, p') \geq p_\circ, \]

where

\[ p_\circ = 2 + \frac{12}{4d - 3 - k} \quad \text{if } d \equiv k \, (mod \, 3), \quad k = -1, 0, 1 \]

• For large \( d \gg 1 \)

\[ \max(p, p') \geq (\text{bilinear}) \, 2 + \frac{4}{d} \]

\[ (\text{multilinear}) \, p_\circ \approx 2 + \frac{3}{d} \]

\[ (\text{conjecture}) \, \frac{2d}{d - 1} \approx 2 + \frac{2}{d}. \]
Square function

• Consider the square function

\[ G^\alpha f(x) = \left( \int_0^\infty \left| K_t^\alpha * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \]

where

\[ \hat{K}_t^\alpha(\xi) = 2(\alpha + 1)\frac{|\xi|^2}{t^2} \left( 1 - \frac{|\xi|^2}{t^2} \right)^\alpha. \]

• Naturally, \( G^\alpha \) may be considered as vector valued operator of order \( \alpha \). This was introduced by Stein, in a slightly different form, to control maximal Bochner-Riesz operators for everywhere convergence of Bochner-Riesz means.

• Note that

\[ t \frac{\partial}{\partial t} \hat{R}_t^{\alpha+1} f(\xi) = \hat{K}_t^\alpha(\xi) \hat{f}(\xi). \]

• \( L^2 \) bound, Stein (1958): For \( \alpha > -1/2 \), \( \|G^\alpha f\|_2 \leq C\|f\|_2 \). The proof relies on Plancherel’s theorem.
$L^p$ boundedness of $G^\alpha$

- For $1 \leq p < 2$, it is known that

\[ \|G^\alpha f\|_p \leq C\|f\|_p. \]

if and only if

\[ \alpha > d\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}. \]

The oscillation for large $|x|$ plays no role here.

- In the range $2 < p < \infty$, the oscillation of the kernel $K_t^\alpha$ is important and the problem is closely related to the Fourier restriction and Bochner–Riesz problems. It is known that the condition

\[ \alpha > \alpha(p) - 1/2 \]

is necessary.
Square function conjecture

- **Conjecture.** For $2 < p \leq \infty$

  $$\|G^\alpha f\|_p \leq C\|f\|_p$$

  if and only if $\alpha > \alpha(p) - 1/2$.

- When $d = 2$, the conjecture was verified by Carbery (1983) via an adaptation of Fefferman and Cordoba arguments which prove Bochner-Riesz conjecture in $\mathbb{R}^2$.

- For $p \geq \frac{2(d+1)}{d-1}$, Christ (1985): using the Tomas-Stein restriction theorem.
• \( p \geq \frac{2(d+2)}{d} \), L.-Rogers-Seeger (2010): relying on the sharp bilinear restriction theorem for the elliptic surfaces,

\[
\int (G^\alpha f(x))^2 w(x) \, dx \lesssim \int |f(x)|^2 \mathcal{M}_q w(x) \, dx,
\]

for \( 1 \leq q < (d + 2)/2 \), and \( \alpha > d/2q - 1 \), where \( w \to \mathcal{M}_q w \) is bounded on \( L^r \) with \( q < r \leq \infty \).

**Theorem (L.-Seeger, 2013)**

Let \( p_s = p_s(d) \) be defined by

\[
p_s = 2 + \frac{12}{4d - 6 - k}, \quad \text{if } d \equiv k \pmod{3}, \; k = 0, 1, 2.
\]

If \( p \geq p_s \) and \( \alpha > \alpha(p) - \frac{1}{2} \), \( \|G^\alpha f\|_p \leq C\|f\|_p \).

• \( p_s > p_\circ \) due to inefficiency in exploiting orthogonality for the square function) and this is new when \( d \geq 9 \).
Applications

- **Maximal Bochner-Riesz.** If $\alpha > \alpha(p)$, for $p \geq p_s$
  \[
  \left\| \sup_{t>0} |R_t^\alpha f| \right\|_{L^p(\mathbb{R}^d)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^d)}.
  \]

- **Radial multiplier.** Let $m$ be a bounded function on $(0, \infty)$ and define a multiplier operator by
  \[
  \mathcal{F}(T_m f)(\xi) = m(|\xi|) \hat{f}(\xi).
  \]

- **Hörmander (1960):** Let $\varphi \in C_0^\infty[1/2, 2]$. $T_m$ is bounded on $L^p$, $1 < p < \infty$, if $\sup_{t>0} \left\| \varphi m(t\cdot) \right\|_{L^2_\alpha(\mathbb{R})} < \infty$, $\alpha > d/2$.

- **(Carbery-Gasper-Trebels (1983))**
  \[
  g[T_m f](x) \leq C \sup_{t>0} \left\| \varphi m(t\cdot) \right\|_{L^2_\alpha(\mathbb{R})} G^{\alpha-1} f(x),
  \]
  where $g$ is a $g$-function satisfying $\left\| g[f] \right\|_p \sim \left\| f \right\|_p$ for $1 < p < \infty$.

- **If** $\max(p, p') \geq \min(p_s, 2(d+2)/d)$, **then for** $\alpha > d \left| \frac{1}{p} - \frac{1}{2} \right|$
  \[
  \left\| T_m f \right\|_{L^p \to L^p} \lesssim \sup_{t>0} \left\| \varphi m(t\cdot) \right\|_{L^2_\alpha(\mathbb{R})}.
  \]
Frequency localization

- Let $\phi \in S$ supported in $[1/2, 2]$. For $0 < \delta < 1/2$ set
  \[
  \widehat{S_{\delta}^t f}(\xi) = \phi\left(\delta^{-1}\left(1 - \frac{|\xi|^2}{t^2}\right)\right)\hat{f}(\xi).
  \]

- By dyadic decomposition of the multiplier $|\xi|^2(1 - |\xi|^2)_{\pm}$ away from the singularity and Littlewood-Paley decomposition (and discarding harmless factors), the matter reduces to showing
  \[
  \left\| \left( \int_{1}^{2} |S_{\delta}^t f(x)|^2 \, dt \right)^{1/2} \right\|_p \lesssim \delta^{1/2} \delta^{-\alpha(p) - \epsilon} \| f \|_p.
  \]

- If $p = 2$, by Plancherel’s theorem (because $||\xi| - t| \lesssim \delta$ if the integrand is nonzero)
  \[
  \left\| \left( \int_{1}^{2} |S_{\delta}^t f(x)|^2 \, dt \right)^{1/2} \right\|_2^2 = \int_{1}^{2} \left| \phi\left(\delta^{-1}\left(1 - \frac{|\xi|^2}{t^2}\right)\right)\right|^2 \, dt \left| \hat{f}(\xi) \right|^2 \, d\xi
  \lesssim \delta \int \left| \hat{f}(\xi) \right|^2 \, d\xi.
  \]
From $L^2$-restriction to square function estimates

- Tomas-Stein restriction theorem for the sphere, Minkowski’s inequality and Plancherel’s theorem gives

$$\|S^t_\delta f\|_{2(d+1)/(d-1)} \lesssim \delta^{1/2} \|f\|_2 \text{ for } t \sim 1.$$ 

- Let $J$ be essentially disjoint intervals of length $\delta$ such that $\bigcup J = [1, 2]$ and $\tilde{J}$ be the interval $\{t : \text{dist} (t, J) \leq 2\delta\}$. Then, set

$$\hat{f}_{\tilde{J}}(\xi) = \hat{f}(\xi) \chi_{\{\xi \in \tilde{J}\}}.$$ 

- For each fixed $t$, $S^t_\delta f$ is supported in $\{\xi : t - 2\delta \leq |\xi| \leq t + 2\delta\}$. For $t \in J$,

$$S^t_\delta f = S^t_\delta f_{\tilde{J}}.$$
• With $p = \frac{2(d+1)}{d-1}$

$$\left\| \left( \int_1^2 |S_\delta^t f|^2 dt \right)^{1/2} \right\|_p = \left\| \left( \sum_I \int_I |S_\delta^t f_I|^2 dt \right)^{1/2} \right\|_p$$

$$\leq \left( \sum_I \int_I \|S_\delta^t f_I\|_p^2 dt \right)^{1/2} \lesssim \delta^{1/2} \left( \sum_I \int_I \|f_I\|_2^2 dt \right)^{1/2} \lesssim \delta \|f\|_2.$$

• Using the fact that the kernel of $S_\delta^t$ rapidly decaying outside of $B(0, C\delta^{-1})$, one may assume $f$ is supported in a ball of radius $\sim \delta^{-1-\epsilon}$. By Hölder’s inequality

$$\left\| \left( \int_1^2 |S_\delta^t f|^2 dt \right)^{1/2} \right\|_p \lesssim \delta \delta^{-d(\frac{1}{2} - \frac{1}{p}) - c\epsilon} \|f\|_p.$$

• The disjointness of Fourier support of $S_\delta^t$ as $t$ varies is important for the extra gain of $\delta^{1/2}$. 


Oscillatory integral operators

• Bourgain-Guth’s result on Bochner-Riesz bounds were obtained from the sharp oscillatory integral estimate

\[ \|T_\lambda g\|_p \lesssim \lambda^{-\frac{d}{p}} \|g\|_p, \]

where

\[ T_\lambda g(x) = \int_{\mathbb{R}^{d-1}} e^{i\lambda \phi(x,z)} a(x,z) g(z) dz, \quad x \in \mathbb{R}^d. \]

• This has obvious advantage over handling multiplier operators because it is similar to the adjoint restriction operators. In fact, one may consider \( g \in L^\infty \).

• However, this approach doesn’t seem adequate for the study of the square function. Especially, the disjointness of Fourier transform (as \( t \) varies) is difficult to exploit.
Fourier transform side approach

• Denote by $m(D)f$ the multiplier operator given by

$$\mathcal{F}(m(D)f)(\xi) = m(\xi)\hat{f}(\xi), \text{ and } D = (D', D_d).$$

• (General class of square functions) For $0 < \delta$ define $S_\delta = S_\delta(\psi, \eta)$ by

$$S_\delta f(x) = \left( \int_{I=[-1,1]} \left| \phi\left( \frac{\eta(D, t)(D_d - \psi(D', t))}{\delta} \right) f(x) \right|^2 dt \right)^{\frac{1}{2}}.$$

• We need to consider the square functions given by classes of $\psi$ and $\eta$ which are suitable the induction argument.
• In fact, this makes it possible to absorb the perturbed terms which arise from translation and rescaling after decomposition.
Multilinear approach: From multilinear to linear

- Mutilinear estimates for the square function
  - \( k \)-linear estimate with transversality
  - \( k \)-linear estimate with transversality but confined Fourier support (\( k + 1 \) transverality fails)
- Stability of these estimates under smooth perturbation of the associates surfaces.
- Improvement of bounds due to smaller Fourier supports
- **Multi-scale decomposition.** Control of given operator \( S_\delta f \) a sum of products \( \left( \prod S_\delta f_i \right)^{\frac{1}{k}} \) while remaining parts are controlled by functions of smaller Fourier supports: With implicit constants (over simplified!)

\[
S_\delta f \lesssim \sum S_\delta b_i + \sum_{k=2}^{l-1} \sum_{k-trans but non-(k+1)-trans} (\prod_{i=1}^k S_\delta g_i)^{\frac{1}{k}} + \sum_{l-trans} (\prod_{i=1}^l S_\delta g_i)^{\frac{1}{l}}
\]
The induction quantity

- (One parameter family of elliptic functions) Denote by $\mathcal{G}(\epsilon_\circ, N)$ the class of smooth function $\psi$ defined on $I^{d-1} \times I$ which satisfy
  \[ \| \psi(\xi', t) - \frac{1}{2} |\xi'|^2 - t \|_{C^N(I^{d-1} \times I)} \leq \epsilon_\circ. \]

- Define a class of smooth functions $\mathcal{E}(N)$ by
  \[ \mathcal{E}(N) = \{ \eta \in C^\infty(I^d \times I) : \| \eta \|_{C^N(I^d \times I)} \leq 2, \eta \sim 1 \}. \]

- Define $B(\delta) = B_p(\delta)$ by
  \[ B(\delta) = \sup \left\{ \| S_\delta(\psi, \eta) f \|_{L^p} : \psi \in \mathcal{G}(\epsilon_\circ, N), \eta \in \mathcal{E}(N), \| f \|_p \leq 1, \text{supp} \hat{f} \subset \frac{1}{2} I^d \right\}. \]

- Since $\text{supp} \hat{f} \subset \frac{1}{2} I^d$, by estimating the kernel
  \[ B(\delta) \leq C \text{ if } \delta \geq 1, \text{ and } B(\delta) \leq C \delta^{-(d-1)/2 + 1/2} \text{ if } \delta < 1. \]
• We need
\[ B(\delta) \leq C\delta^{-\frac{d-2}{2} + \frac{d}{p} - \epsilon}. \]

• For monotonicity we slightly modify it. Define for \( \beta > 0 \)
\[ B^\beta(\delta) = B^\beta_p(\delta) \equiv \sup_{\delta \leq s \leq 1} s^{\frac{d-2}{2} - \frac{d}{p} + \beta} B_p(s). \]

We show \( B^\beta(\delta) \leq C \) for any \( \beta > 0 \).

**Proposition (Improvement by small Fourier supports)**

Let \( 0 < \delta \ll 1 \), \( \psi \in \mathcal{C}(\epsilon_0, N) \), and \( \eta \in \mathcal{E}(N) \). Suppose that \( \hat{f} \) is supported in \( q(a, \varepsilon) \), \( \sqrt{\delta} \leq \varepsilon \leq 1/2 \), and \( a \in \frac{1}{2} I^d \). Then, if \( \epsilon_0 > 0 \) is small enough, there is an \( \kappa = \kappa(\epsilon_0, N) \) such that
\[ \| S_\delta(\psi, \eta) f \|_p \leq C \varepsilon^{\frac{1}{p} + \frac{1}{2}} B_p(\varepsilon^{-2} \delta) \| f \|_p \]
holds with \( C \), independent of \( \psi, \eta \) and \( \varepsilon \), whenever \( \sqrt{\delta} \leq \varepsilon \leq \kappa \).
Let \( \psi \in \mathcal{G}(\epsilon_\circ, \mathcal{N}) \) and set

\[
\Gamma^t = \Gamma^t(\psi) := \{ (\xi', \psi(\xi', t)) : \xi' \in [-1, 1]^{d-1} \}.
\]

- **Normal vector field** \( \mathbf{n} = \mathbf{n}(\psi) \) For each \( \xi = (\zeta, \tau) \in I^d \) there is a unique \( t \) such that \( \xi = (\zeta, \psi(\zeta, t)) \) since \( \partial_t \psi \sim 1 \). Then we define \( \mathbf{n}(\xi) \) to be the unit normal vector to \( \Gamma^t \) at \( \xi \) which forms a vector field on \( I^d \).
Theorem

Let $\psi \in \mathcal{G}(\epsilon_0, N)$, $\eta \in \mathcal{E}(N)$ and $\sigma > 0$. Suppose that, for $0 < \delta \ll \sigma$,

$$|n(\xi_1) \wedge n(\xi_2) \wedge \cdots \wedge n(\xi_k)| \gtrsim \sigma$$

holds whenever $\xi_i \in \text{supp} \hat{f}_i + O(\delta), i = 1, 2, \ldots, k$. Then, if $p \geq 2k/(k - 1)$ and $\epsilon_0$ is small enough, for $\epsilon > 0$, there is an $N(\epsilon)$ such that the following estimates hold with $C, C_\epsilon$, independent of $\psi, \eta$, if $N \geq N(\epsilon)$:

$$\left\| \prod_{i=1}^{k} S_\delta(\psi, \eta) f_i \right\|_{p \frac{k}{p}} \leq C \sigma^{-C_\epsilon} \delta^{-\epsilon} \prod_{i=1}^{k} \left( \delta \| f_i \|_2 \right),$$

$$\left\| \prod_{i=1}^{k} S_\delta(\psi, \eta) f_i \right\|_{p \frac{k}{p}} \leq C \sigma^{-C_\epsilon} \delta^{-\epsilon} \prod_{i=1}^{k} \left( \delta^{\frac{d}{p} - \frac{d-2}{2}} \| f_i \|_p \right).$$

- The second inequality follows from the first by spatial localization and Hölder’s inequality.
Currently, we have $C_\epsilon \sim C \log \frac{C}{\epsilon}$.  

When the degree of transversality gets smaller, the bound becomes really bad. In particular, the bound becomes inefficient when $\sigma \sim \delta^c$.

This cannot be obtained by direct application of multilinear estimates because $p/k \leq 1$ in general. Instead we redo the proof of the multilinear restriction estimate using the multilinear Kakeya estimate.

It is not easy to run induction argument with the estimate of this form. We need to find a suitable setting which allows the induction argument.
Proposition

Let $\psi \in \mathcal{G}(\epsilon_0, N)$, and $\sigma > 0$. Suppose that, for $0 < \delta \ll \sigma$, $i = 1, \ldots, k,$

$$\text{supp } \mathcal{F}(G_i(\cdot, t)) \subset \Gamma^t + O(\delta), \quad t \in I,$$

and

$$|n(\xi_1) \wedge n(\xi_2) \wedge \cdots \wedge n(\xi_k)| \gtrsim \sigma,$$

whenever $\xi_i \in \text{supp } \mathcal{F}(G_i(\cdot, t)) + O(\delta)$ for some $t \in I$. Then, if $p \geq 2k/(k - 1)$ and $\epsilon_0 > 0$ is small enough, for $\epsilon > 0$ there are constants $N = N(\epsilon)$ such that

$$\left\| \prod_{i=1}^{k} \|G_i\|_{L^2_x(I)} \right\|^{\frac{p}{k}} \leq C\sigma^{-C_\epsilon} \delta^{-\epsilon} \prod_{i=1}^{k} \left( \delta^{\frac{1}{2}} \|G_i\|_{L^2_{x,t}} \right)$$

holds with $C, C_\epsilon$, independent of $\psi, \eta$, if $N \geq N(\epsilon)$.

- Applying this with $G_i = \phi\left( \frac{\eta(D,t)(D_d-\psi(D',t))}{\delta} \right)f_i$ gives the theorem.
Proposition (with confined Fourier support)

Let $\Pi \subset \mathbb{R}^d$ be a $k$-plane containing the origin. Under the same assumption as in the above proposition and suppose, additionally, that, for $i = 1, \ldots, k$,

$$n \left( \text{supp} \mathcal{F}(G_i(\cdot, t)) \right) \subset \mathbb{S}^{d-1} \cap (\Pi + O(\delta)).$$

Then, if $2 \leq p \leq 2k/(k - 1)$ and $\epsilon_0$ is sufficiently small, for $\epsilon > 0$ there is an $N(\epsilon)$ such that the following estimate holds uniformly for $\psi, \eta$, if $N \geq N(\epsilon)$:

$$\left\| \prod_{i=1}^{k} \| G_i \|_{L^2_t(I)} \right\|_{\frac{p}{k}} \lesssim \sigma^{-C_\epsilon} \delta^{d-\frac{k}{2}} \delta^{\frac{1}{2} - \frac{1}{p} - \epsilon} \prod_{i=1}^{k} \| G_i \|_{L^2_{x,t}}.$$

- At the critical $p = 2k/(k - 1)$ this gives a better bound $\delta^{\frac{d}{2} - \epsilon}$, which is better than $\delta^{\frac{k}{2} - \epsilon}$.
- This lemma covers the case $(k + 1)$-transversality in dimension fails while $k$-transversality is satisfied. So this will be used with a relatively large $\delta$. 
• Let \( \{ q \} \) be the collection of dyadic cubes of side length \( l \), \( \tilde{\sigma} < l \leq 2\tilde{\sigma} \). Define \( f_q \) by

\[
\mathcal{F}(f_q) = \chi_q \mathcal{F}(f), \text{ and set } R = 1/\tilde{\sigma}.
\]

**Theorem**

Let \( \psi \in \mathcal{G}(\epsilon_\circ, N) \), \( \eta \in \mathcal{E}(N) \), \( \Pi \) be a \( k \)-plane containing the origin, and \( \sigma > 0 \). Suppose that, for \( 0 < \tilde{\sigma} \ll \sigma \),

\[
|n(\xi_1) \wedge n(\xi_2) \wedge \cdots \wedge n(\xi_k)| \gtrsim \sigma
\]

whenever \( \xi_i \in \text{supp} \hat{f}_i + O(\tilde{\sigma}) \), \( i = 1, 2, \ldots, k \). Suppose

\[
n(\text{supp} \hat{f}_i) \subset \Pi + O(\tilde{\sigma}), \quad i = 1, 2, \ldots, k.
\]

Then, if \( 2k/(k - 1) \leq p \leq 2 \) and \( 0 < \delta \leq \tilde{\sigma} \), for \( \epsilon > 0 \) there is an \( N(\epsilon) \) such that

\[
\left\| \prod_{i=1}^{k} S_\delta f_i \right\|_{L^p_k(B(x,R))} \lesssim \sigma^{-C \epsilon \tilde{\sigma}^{-\epsilon}} \times \prod_{i=1}^{k} \left\| \left( \sum_q |S_\delta f_{i,q}|^2 \right)^{1/2} \rho\left( \frac{x}{\tilde{\sigma}} \right) \right\|_{L^p}
\]

holds uniformly for \( \psi \in \mathcal{G}(\epsilon_\circ, N) \) if \( N \geq N(\epsilon) \).
• Increasing multi-linearity gives better bounds but the size of non-transversal fourier support also get larger.

• The proof is basically to compromise the estimates in two theorems.

• Non-transversal is handled by the $\ell^2$-estimate and the smallness of Fourier support.
Multi-scale decomposition

• Let $\sigma_1, \ldots, \sigma_d$ be dyadic numbers to be chosen later such that

$$\delta \ll \sigma_{d-1} \ll \cdots \ll \sigma_1 \ll \sigma_0 = 1.$$ 

We call $\sigma_i$ $i$-th scale.

• $\{q^i\}$: the collection of all (closed) dyadic cubes $q^i$ of sidelength 2$\sigma_i$, $q^i \subset I^d$. $i = 1, \ldots, d - 1$,

$$f = \sum_{q^i} f_{q^i}, \text{ for } i = 1, \ldots, d - 1.$$ 

• Let $2 \leq p < \infty$. Then, there is a constant $C$, independent of $\{q\}$, such that

$$\left( \sum_{q^i} \|f_{q^i}\|_p^p \right)^{\frac{1}{p}} \leq C\|f\|_p.$$
• Let $q^k_1, \ldots, q^k_{k+1} \in \{q^k\}$. We write $q^k_1, q^k_2, \ldots, q^k_{k+1} : \text{trans (transversal)}$ if

$$|n(\xi_1) \wedge n(\xi_2) \wedge \cdots \wedge n(\xi_{k+1})| \gtrsim \sigma_1 \sigma_2 \cdots \sigma_k$$

whenever $\xi_i \in q^k_i$, $i = 1, \ldots, k + 1$.

• **Dual cubes of $i$-th scale** $\{Q^i\}$: the collection of dyadic intervals of sidelenath $2/\sigma_i$, which covers $\mathbb{R}^d$ so that

$$\bigcup_{Q^i} Q^i = \mathbb{R}^d$$
Decomposition increasing the degree of multilinearity

- After \( l \)-th scale decomposition

\[
\| S_\delta f \|_p \lesssim \sum_{k=1}^l \sigma_{k-1}^{C/2} \sigma_k^{2p} B_p(\sigma_k^{-2} \delta) \| f \|_p + \sum_{k=2}^l \sigma_{k-1}^{C} \tilde{\Lambda}^k f
\]

\[
+ \sigma_l^{-C} \sup_{\tau_1, \ldots, \tau_{l+1}} \max_{q_1^l, \ldots, q_{l+1}^l:\text{trans}} \left\| \prod_{i=1}^{l+1} S_{\delta \tau_i} f_{q_i^l} \right\|_p^{1/(l+1)}
\]

where

\[
\tilde{\Lambda}^k f = \sup_{\tau_1, \ldots, \tau_k} \max_{q_1^{k-1}, \ldots, q_k^{k-1}:\text{trans}} \times
\]

\[
\left( \sum_{\Omega^k} \left\| \prod_{i=1}^k S_{\delta} \left( \sum_{q_i^k \subset q_i^{k-1}: n(q_i^k) \in \Pi^k + O(\sigma_k)} \tau_i f_{q_i^k} \right) \right\|_k^p \right)^{1/p}.
\]

Here \( \Pi^k \) is a \( k \)-plane containing the origin which depends on \( q_1^{k-1}, \ldots, q_k^{k-1}, \Omega^k \), and \( \tau_1, \ldots, \tau_k \).
Two estimates for $\tilde{\Lambda}^k f$

- Since the mother cubes $q_1^{k-1}, q_2^{k-1}, \ldots, q_k^{k-1}$ are already transversal, for $p \geq 2k/(k - 1)$

$$\|\tilde{\Lambda}^k f\|_p \leq C\sigma_k^{-C} \delta^{-\frac{d-2}{2}} + \frac{d}{p} - \beta \|f\|_p.$$ 

- Using square function estimates for confined Frequency support and $\#\left(\left\{q_i^k : n(q_i^k) \subset \Pi + O(\sigma_k)\right\}\right) \lesssim \sigma_k^{-k}$, for $2 \leq p \leq 2k/(k - 1)$

$$\|\tilde{\Lambda}^k f\|_p \leq C\sigma_k^{-\epsilon - (k+1)\left(\frac{1}{2} - \frac{1}{p}\right)} \sigma_k B(\sigma_k^{-2}\delta)\|f\|_p.$$ 

- Combining two cases, if $\delta_\circ \leq \delta$, for some $\alpha > 0$

$$\|\tilde{\Lambda}^k f\|_{L^p} \leq C\delta^{-\frac{d-2}{2}} + \frac{d}{p} - \beta \left(\sigma_k^{-C} + \sigma_k^{\alpha} B^\beta(\delta_\circ)\right)\|f\|_p$$

if $p > \min\left(\frac{2(2d-k-1)}{2d-k-3}, \frac{2k}{k-1}\right)$. 
• If

\[ p \geq p(l) = \max_{k=1,\ldots,l} \min \left( \frac{2(2d - k - 1)}{2d - k - 3}, \frac{2k}{k - 1} \right) \vee \frac{2(l + 1)}{l} \vee \frac{2(d - 1)}{d - 2}, \]

\[ \| S_\delta f \|_p \lesssim \left( \sum_{k=1}^{l} \left( \sigma_{k-1}^{-C} + \sigma_{k-1}^{-C} \sigma_k^\alpha \mathcal{B}^\beta(\delta_\circ) \right) + \sigma_l^{-C} \right) \delta^{-\frac{d-2}{2} + \frac{d}{p} - \beta} \| f \|_p. \]

• By stability of the estimates along \( \psi, \eta \) and \( f \), taking sup we have, for \( \delta_\circ \leq \delta \leq 1 \) and some \( \alpha > 0 \),

\[ \delta^{-\frac{d-2}{2} + \frac{d}{p} - \beta} \mathcal{B}(\delta) \lesssim \sum_{k=1}^{l} \left( \sigma_{k-1}^{-C} + \sigma_{k-1}^{-C} \sigma_k^\alpha \mathcal{B}^\beta(\delta_\circ) \right) + \sigma_l^{-C}. \]

• Hence taking sup along \( \delta, \delta_\circ \leq \delta \leq 1 \), for \( p \geq p(l) \)

\[ \mathcal{B}^\beta(\delta_\circ) \lesssim \left( \sum_{k=1}^{l} \sigma_{k-1}^{-C} \sigma_k^\alpha \right) \mathcal{B}^\beta(\delta_\circ) + \sum_{k=1}^{l} \sigma_k^{-C}. \]

• The minimum of \( p(l) \) is \( p_s \).
Remarks

• $p(l)$ minimizes with $l \sim \frac{2d}{3}$. The roles of multilinear estimates of the higher degrees seem unclear.
• A different proof of the currently best known result for the Bochner-Riesz problem in Fourier transform side.
• The same result for a little more general (non-symmetric) multiplier

$$\mathcal{F}(T^\alpha f)(\xi) = (\tau - \psi(\zeta))^{\alpha} \chi_{\circ}(\xi) \hat{f}(\xi)$$

if $\psi$ is elliptic.