Colouring multijoints

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IPAM
9th May 2014
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Joint work with Stefán Valdimarsson

Outline

Overview

We're going to discuss some developments related to the discrete multilinear Kakeya probem (DMK).

In contrast to "linear" Kakeya, some of the problems in the discrete setting seem harder than their continuous counterparts – at least in the sense that the continuous results are known and the discrete variants remain unknown at present.

Multijoints

Throughout this talk we will have n families of lines $\mathcal{L}_1, \ldots, \mathcal{L}_n$ in an n-dimensional vector space $\mathbb{F}^n$ over some field $\mathbb{F}$.

We'll assume that the families are **transversal** in the sense that if $l_1 \in \mathcal{L}_1, \ldots, l_n \in \mathcal{L}_n$ meet at some point $x \in \mathbb{F}^n$, then the directions of the lines span $\mathbb{F}^n$.

We call such a point x a **multijoint** of $\{\mathcal{L}_1, \ldots, \mathcal{L}_n\}$ and let J denote the set of multijoints.

Theorem (AC & S.I.Valdimarsson)

There exists an n-colouring $\kappa : J \to \{1, 2, \ldots, n\}$ *such that for each* j*, for each* $l \in L_j$*,*

$$|\{x \in J \cap l \,:\, \kappa(x) = j\}| \lesssim_n |J|^{1/n}.$$

Joints

The setting of multijoints is a multiparameter version of the more familiar joints setting of Sharir et al where we have a *single* family of lines $\mathcal{L}$ in $\mathbb{R}^n$ and we define a **joint** to be a point of intersection of n of the lines of $\mathcal{L}$ whose directions span $\mathbb{R}^n$. Let J be the set of joints.

Theorem (Guth–Katz, Kaplan–Sharir–Shustin, Quilodrán)

Let L denote $|\mathcal{L}|$. *Then*

$$|J| \lesssim_n L^{n/(n-1)}.$$

Folklore: Result continues to hold when $\mathbb{R}$ is replaced by any field $\mathbb{F}$.

Basic multijoints problem: Count the number of multijoints in terms of the cardinalities $L_1, \ldots, L_n$ of $\mathcal{L}_1, \ldots, \mathcal{L}_n$ respectively.

Linear vs. multilinear Kakeya

The multijoints problem bears the same relation to the joints problem as multilinear Kakeya bears to linear Kakeya.

So it is natural to ask whether, in the multijoints setting, we have

$$\sum_{x\in J} N_1(x)^{1/(n-1)} \dots N_n(x)^{1/(n-1)} \lesssim_n L_1^{1/(n-1)} \dots L_n^{1/(n-1)} \qquad \textbf{(DMK)}$$

where J is the set of multijoints and where

$$N_j(x) = |\{l_j \in \mathcal{L}_j \,:\, x \in l_j\}|.$$

In particular, do we have

$$|J| \lesssim_n L_1^{1/(n-1)} \dots L_n^{1/(n-1)}? \quad \textbf{(Multijoints Problem)}$$

If $L_j \sim L$ for all j this follows directly from the joints theorem.

Our result and the joints theorem

What does our result

Theorem

There exists an n-colouring $\kappa : J \to \{1, 2, \ldots, n\}$ such that for each j, for each $l \in L_j$,

$$|\{x \in J \cap l \;:\; \kappa(x) = j\}| \lesssim_n |J|^{1/n}$$

have to do with the joints theorem?

Firstly, it implies a (weak) version of the joints theorem:

$$|J| = \sum_{x \in J} 1 = \sum_j \sum_{\{x \in J : \kappa(x) = j\}} 1 \leq \sum_j \sum_{l \in \mathcal{L}_j} \sum_{\{x \in J \cap l : \kappa(x) = j\}} 1 \lesssim_n (\sum_j L_j)|J|^{1/n}$$

so that

$$|J| \lesssim_n L^{n/(n-1)}$$

where $L = \sum_j L_j$.

Our result and the joints theorem, cont'd

Secondly, unsurprisingly, it uses the technology of the proof of the joints theorem, in particular Quilodrán's lemma:

Lemma (Quilodrán)

Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^n$ and J some set of joints of $\mathcal{L}$ such that $|J \cap l| \geq m$ for all $l \in L$. Then $|J| \gtrsim_n m^n$.

Folklore: Continues to hold in any field.

Underlying Quilodrán's lemma is the polynomial method. So there is a polynomial lurking behind our colouring.

Remark. If we apply Quilodrán's argument directly, we obtain, for each $x \in J$, a choice of line such that for all j, for all $l_j \in \mathcal{L}_j$,

$$|\{x \in J \cap l \,:\, x \text{ chooses } l\}| \lesssim_n |J|^{1/n};$$

however there are potentially many more multijoints which have the *same colour as l* than those which simply *choose l*.

Our result and multilinear Kakeya

The analogous statement in the context of tubes is:

Let $\mathcal{T}_1, \dots, \mathcal{T}_n$ be families of 1-tubes in $\mathbb{R}^n$ with the directions of the tubes in $\mathcal{T}_j$ close to the standard basis vector e_j (transversality). Let

$$\mathcal{Q} = \{Q \text{ unit cubes } : \forall j \ \exists T \in \mathcal{T}_j \text{ with } Q \cap T \neq \emptyset.\}$$

Then there is an n-colouring κ of $\mathcal{Q}$ such that for all j, all $T \in \mathcal{T}_j$,

$$|\{Q \in \mathcal{Q} : \mathcal{Q} \cap T \neq \emptyset \text{ and } \kappa(Q) = j\}| \lesssim_n |\mathcal{Q}|^{1/n}.$$

Guth: this follows from Borsuk–Ulam: take a polynomial p of degree $\lesssim_n |\mathcal{Q}|^{1/n}$ bisecting each $Q \in \mathcal{Q}$. Then $\mathcal{H}_{n-1}(Z_p \cap Q) \gtrsim_n 1$ for each $Q \in \mathcal{Q}$, so we can choose the colour of Q to be j where j is such that $\mathrm{surf}_{e_j}(Z_p \cap Q) \gtrsim_n 1$. [$\mathcal{H}_{n-1}(Z_p \cap Q) \sim \sum_j \mathrm{surf}_{e_j}(Z_p \cap Q)$.]

It is not so clear how to create quantities analogous to $\mathrm{surf}_{e_j}(Z_p \cap Q)$ in the discrete setting.

Remarks

What we have is that there exist $S_j : J \to \mathbb{R}$ $(1 \leq j \leq n)$ such that

$$\sum_j S_j(x) \gtrsim 1 \text{ for all } x \in J$$

and, for all j, for all $l \in \mathcal{L}_j$,

$$\sum_{x \in l} S_j(x) \lesssim_n |J|^{1/n}.$$

Our proof is constructive – we actually *build* the S_j (unlike in some approaches) – but not really geometric.

This is perhaps the first time that such quantities S_j have been exhibited in a discrete context (??)

The precise relationship between the quantities S_j and zero sets of appropriate polynomials is at this stage still unclear.

There is no topology available to us in this context.

Relation to results in discrete geometry

??????

Isn't it obvious?

Theorem

There exists an n-colouring $\kappa : J \to \{1, 2, \ldots, n\}$ *such that for each* j*, for each* $l \in L_j$*,*

$$|\{x \in J \cap l \ : \ \kappa(x) = j\}| \lesssim_n |J|^{1/n}.$$

Take a polynomial p of degree $\lesssim_n |J|^{1/n}$ whose zero set Z_p contains J. For those points x where $\nabla p(x) \neq 0$ we shall have that for some j, all lines l in $\mathcal{L}_j$, $l \nsubseteq Z_p$. (Otherwise, if for all j there is an $l \in \mathcal{L}_j$ with $l \subseteq Z_p$, we'll have $\nabla p(x) = 0$.) Give x colour j and note that for any j and $l \in \mathcal{L}_j$ we have $\{x \in J \cap l \ : \ \kappa(x) = j\} \subseteq Z_p \cap l$ which has cardinality at most deg p as $l \nsubseteq Z_p$.

But this doesn't deal with the singular points at which $\nabla p(x) = 0$...

When $n = 2$ this argument can be modified to work; in any case there is a simple *ad hoc* argument covering that case.

DMK – partial results

- When all the lines in $\mathcal{L}_j$ are parallel, then DMK holds. This is Loomis–Whitney.
- When we are in $\mathbb{R}^n$ and the lines in $\mathcal{L}_j$ have directions close to e_j, then DMK holds. This follows from Guth's endpoint multilinear Kakeya theorem.

More recent history

M. Iliopoulou has made recent progress on the multijoints problem

$$|J| \lesssim_n L_1^{1/(n-1)} \dots L_n^{1/(n-1)}.$$

In her PhD thesis she proved this in $\mathbb{R}^3$ (without the transversality hypothesis) using the partitioning technique of Guth and Katz and an analysis of the critical lines (which limited the to euclidean space and $n = 3$).

She also proved the stronger result

$$\sum_{x \in J} N_1(x)^{1/2} N_2(x)^{1/2} N_3(x)^{1/2} \lesssim L_1^{1/2} L_2^{1/2} L_3^{1/2}$$

subject to a mild but annoying technical hypothesis.

Constants and extremals

In the continuous case, Bennett, Tao and I had shown via a monotonicity argument that multibushes are quasi-extremals for Multilinear Kakeya in the regime $q > 1/(n-1)$. It's natural to ask about best constants and extremal configurations in the DMK problem.

In dimension $n = 2$ of course the best constant is 1 and the DMK inequality is an identity if we assume that no line in $\mathcal{L}_1$ is parallel to any line in $\mathcal{L}_2$.

In higher dimensions, the natural "extremals" for the DMK problem – multibushes – are not in fact extremals (AC & SIV).

More precisely, in $\mathbb{F}_3^3$ we can find 3 families of lines, each of cardinality 5, for which the constant in DMK is $\frac{6+4\sqrt{2}}{\sqrt{125}} \sim \frac{11.66}{11.18}$.

There are computer-generated examples which are slightly worse.

Overview of proof

The argument is rather combinatorial and formal, with geometry entering in only one key place.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be transversal families of lines in $\mathbb{F}^n$ and let J be a subset of their multijoints. Let $m \in \mathbb{N}$.

Definition. A colouring $\kappa : J \to \{1, \ldots, n\}$ is ***m*-bounded** if for all j, all $l_j \in \mathcal{L}_j$, $|\{x \in J \cap l_j : \kappa(x) = j\}| \leq m$. A subset $K \subseteq J$ is ***m*-good** if it is colourable with an m-bounded colouring.

Theorem. There exists an $m \lesssim_n |J|^{1/n}$ such that J is m-good.

Main claim. Fix m. Suppose that $K \subseteq J$ is m-good, and suppose $x_0 \in K \setminus J$. If $m \gtrsim |K|^{1/n}$, then $K \cup \{x_0\}$ is also m-good.

The claim establishes the theorem: singletons are m-good; we add points one at a time via the claim, the process continuing unhindered until we've added $\sim m^n$ points and exhausted J.

Coloured trees

Main claim. Fix m. Suppose that J is m-good, and suppose $x_0 \notin J$ is a multijoint. If $m \gtrsim |J|^{1/n}$, then $J \cup \{x_0\}$ is also m-good.

We give J an arbitrary ordering $J = \{x_1, \ldots, x_\nu\}$.

We let $\mathcal{K}(\neq \emptyset)$ be the collection of m-bounded colourings of J.

We will define a strict partial order on $\mathcal{K}$ by constructing some *coloured trees*. Indeed, for each $\kappa \in \mathcal{K}$ we'll define a coloured tree $T(\kappa)$ which is rooted at x_0 and whose other vertices will be members of J.

$T(\kappa)$ will itself be constructed iteratively from an sequence of coloured trees $T_i(\kappa)$. We begin with $T_0(\kappa) = \{x_0\}$.

The following set plays a key role in the construction of $T_i(\kappa)$: for the i'th element [if it exists] y_i of $T_{i-1}(\kappa)$ and $j \neq \kappa(y_i)$ let

$$\mathcal{L}_j^i = \{l_j \in \mathcal{L}_j \,:\, y_i \in l_j \text{ and } (*) \text{ holds}\}$$

where

$$(*)\ |\{x \in l_j \cap J \,:\, \kappa(x) = j\}| + |\{x \in l_j \cap T_{i-1} \cap J \,:\, \kappa(x) \neq j\}| \geq m.$$

Strict partial order and a dichotomy

Let $\kappa_1, \kappa_2 \in \mathcal{K}$. We say that κ_1 is **more advanced** than κ_2 if there is some i_0 such that $T_i(\kappa_1) = T_i(\kappa_2)$ for $i < i_0$ but $T_{i_0}(\kappa_1) \subsetneq T_{i_0}(\kappa_2)$ as coloured trees.

Then for all $\kappa \in \mathcal{K}$, either κ is **advanceable** or it is **non-advanceable**.

If κ is advanceable, the construction ultimately leads to a more advanced colouring $\tilde{\kappa}$ of $J \cup \{x_0\}$ which is m-bounded. The process comes to an end as everything is finite. (This part of the argument is combinatorial and formal.)

If κ is non-advanceable, then we will conlude that $|T(\kappa) \cap J| \gtrsim_n m^n$. But the hypothesis of the main claim is that $m \gtrsim |J|^{1/n}$, contradiction.

The colouring κ is non-advanceable if for all i, the collection $\mathcal{L}_j^i$ is nonmepty for all $j \neq \kappa(y_i)$.

The heart of the argument

Assume: for all i, for all $j \neq \kappa(y_i)$

$$\mathcal{L}_j^i = \{l_j \in \mathcal{L}_j \,:\, y_i \in l_j \text{ and } (*) \text{ holds}\} \neq \emptyset, \quad \text{where}$$

$$(*)\ |\{x \in l_j \cap J \,:\, \kappa(x) = j\}| + |\{x \in l_j \cap T_{i-1} \cap J \,:\, \kappa(x) \neq j\}| \geq m.$$

Want to conclude:

$$|T(\kappa) \cap J| \gtrsim_n m^n.$$

The construction ensures that the members of $T(\kappa) \cap J$ have the property that for each j there exists an $l \in \cup_{\{i\,:\,\kappa(y_i)\neq j\}} \mathcal{L}_j^i$ which passes through it.

Thus each member of $T(\kappa) \cap J$ is a joint for $\overline{\mathcal{L}} := \cup_j \cup_{\{i\,:\,\kappa(y_i)\neq j\}} \mathcal{L}_j^i$.

Furthermore the two sets appearing in $(*)$ are disjoint subsets of $l_j \cap T(\kappa) \cap J$. So for any line $l \in \overline{\mathcal{L}}$, say $l_j \in \mathcal{L}_j^i$, we have $|l_j \cap T(\kappa) \cap J| \geq m$. Quilodrán's lemma therefore tells us that $|T(\kappa) \cap J| \gtrsim_n m^n$, as needed.

Geometric means?

What we have is that there exist $S_j : J \to \mathbb{R}$ $(1 \le j \le n)$ such that

$$\sum_j S_j(x) \gtrsim 1 \text{ for all } x \in J \tag{1}$$

and, for all j, for all $I \in \mathcal{L}_j$,

$$\sum_{x \in I} S_j(x) \lesssim_n |J|^{1/n}.$$

What we would really like is that instead of (1) we'd have

$$\prod_j S_j(x)^{1/n} \gtrsim 1 \text{ for all } x \in J. \tag{2}$$

or even

$$\sum_j \beta_j S_j(x) \gtrsim 1 \text{ for all } x \in J \tag{3}$$

where β_j are suitable positive weights.

So?

So we're led to entertain the possibility that the colouring theorem

Theorem

There exists an n-colouring $\kappa : J \to \{1, 2, \ldots, n\}$ such that for each j, for each $l \in \mathcal{L}_j$,

$$|\{x \in J \cap l \; : \; \kappa(x) = j\}| \lesssim_n |J|^{1/n}$$

might be just one of a family of such results.

A glance at the proof presented indicates that a suitable multiparameter Quilodrán lemma would be useful in such an approach.

An alternative approach might be to try to associate a vector of colours to a multijoint. This leads to a similar questions concerning Quilodrán's lemma.

Very recent news

Theorem (M.Iliopoulou)

Suppose we are in $\mathbb{R}^n$. Then the multijoints estimate

$$|J| \lesssim_n L_1^{1/(n-1)} \dots L_n^{1/(n-1)}$$

holds for all n (without the transversality hypothesis), and moreover, for $\lambda_j \geq 1$,

$$|\{x \in J \, : \, N_j(x) > \lambda_j \text{ for all } j\}| \lesssim_n \frac{(L_1 \dots L_n)^{1/(n-1)}}{(\lambda_1 \dots \lambda_n)^{1/(n-1)}}.$$

This extends the case $n = 3$ she had previously established. It still uses the Guth–Katz polynomial partitioning technique plus the idea the as the multijoints are arranged on lines one can hope to find a polynomial of smaller-than-expected degree which vanishes on them.

Even more recent news

Theorem (M.Iliopoulou)

Suppose we are in $\mathbb{F}^3$. Then the multijoints estimate

$$|J| \lesssim L_1^{1/2} L_2^{1/2} L_3^{1/2}$$

holds, and moreover, for $\lambda_j \geq 1$,

$$|\{x \in J \,:\, N_j(x) > \lambda_j, j = 1, 2, 3\}| \lesssim \frac{(L_1 L_2 L_3)^{1/2}}{(\lambda_1 \lambda_2 \lambda_3)^{1/2}}.$$

No transversality hypothesis is needed here for estimate on $|J|$.

The proof uses a probabilistic construction and again the idea that multijoints are arranged on lines with special structures, so one can hope to find a polynomial of smaller-than-expected degree which vanishes on them.

It is not yet clear what the S_j's are in these contexts.