Algebraic Techniques for Incidence Geometry: Consumer Report

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The founding father

György Elekes (passed away in September 2008)

A very brief history of the new era

- Circa 2000, Elekes was thinking of

Erdős's distinct distances problem:

Estimate the smallest possible number $D(n)$ of distinct distances determined by any set of n points in the plane

[Erdős, 1946] conjectured:
$D(n) = \Omega(n/\sqrt{\log n})$
(Cannot be improved: Tight for the integer grid)

A 1946 classic
A hard nut; Slow steady progress

A very brief history of the new era

- Elekes transformed the distinct distances problem to an incidence problem between points and curves (lines) in 3D

- Formulated a couple of deep conjectures on the new setup (If proven, they yield the almost tight lower bound $\Omega(n/\log n)$)

A very brief history of the new era

• 10 years later, [Guth, Katz, 2010] settled Elekes's conjectures (in a more general setup)
And solved (almost) completely the distinct distances problem

• This followed an earlier breakthrough by the same couple:
[Guth, Katz, 2008]:
Algebraic Methods in Discrete Analogs of the Kakeya Problem
Showed: The number of joints in a set of n lines in 3D is $O(n^{3/2})$

A very brief history of the new era

In both cases:

- New algebraic machinery applied to an incidence problem between points and lines in 3D

- A hard problem, resisting decades of “conventional” geometric and combinatorial attacks

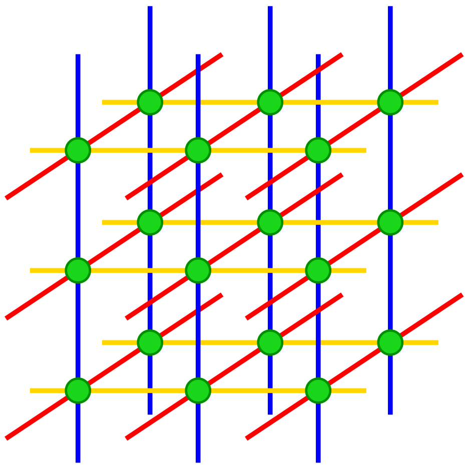
The joints problem

L – Set of n lines in $\mathbb{R}^3$

Joint: Point incident to (at least) three non-coplanar lines of L

[Chazelle et al., 1992]. Show:
The number of joints in L is $O(n^{3/2})$

Worst-case tight:



Joints

Solved completely in [Guth, Katz, 2008]

In retrospect, a "trivial" problem

In d dimensions
(Joint = point incident to at least d lines, not all on a hyperplane)
Max number of joints is $\Theta(n^{d/(d-1)})$
[Kaplan, Sharir, Shustin 10],
[Quilodrán 10]
(Similar, and very simple proofs)

Distinct distances: Summary

(Details coming up soon)

[Elekes, ≈ 2000]: Setup and conjectures $\Longrightarrow$

[Elekes, Sharir, 2010]: "Elekes-Sharir program" (exposition of Elekes's ideas) $\Longrightarrow$

[Guth, Katz, 2010]: Settling the conjectures via new algebraic tools $\Longrightarrow$

New era: Algebraic techniques for combinatorial and computational geometry

A new era has dawned

- Besides the excitement of solving Erdős's problem (after 65 years of frustration)

- A new and powerful algebraic methodology that many (Agarwal, Kaplan, Matoušek, Safernová, me, Solymosi, Tao, Zahl, Iosevich et al., Pach, Schwartz, de Zeeuw, Charalambides, Sheffer, Raz, Solomon, ???)
have already managed to apply to other problems

- First, new proofs of old results (simpler, different)
  [Kaplan, Matoušek, Sharir, 2011]

Then new results:

- Unit distances in three dimensions
  [Zahl], [Kaplan, Matoušek, Safernová, Sharir]

- Point-circle incidences in three dimensions
  [Sharir, Sheffer, Zahl 2013]

- Complex Szemerédi-Trotter incidence bound and related bounds
  [Solymosi, Tao], [Zahl]

- Range searching with semi-algebraic ranges
  An algorithmic application;
  [Agarwal, Matoušek, Sharir, 2013]

- Sums vs. products
  [Iosevich, Roche-Newton, Rudnev]

And even newer results:

- Incidences between points and lines in four dimensions
  [Sharir, Solomon, 2014]

- Distinct distances between two lines
  [Sharir, Sheffer, Solymosi, 2013]

- Distinct distances: Other special configurations
  [Sharir, Solymosi, 2013], [Pach, de Zeeuw, 2013],
  [Charalambides, 2013]

And still newer results:

- The Elekes–Rónyai problem revisited
  [Raz, Sharir, Solymosi, 2014]

- Triple intersections of three families of unit circles
  [Raz, Sharir, Solymosi, 2014]

- Unit-area triangles in the plane
  [Raz, Sharir, 2014]

Old-new Machinery from Algebraic Geometry

- Low-degree polynomial vanishing on a given set of points
- Polynomial ham sandwich cuts
- Polynomial partitioning
- Miscellany (Thom-Milnor, Bézout, Harnack, Warren, and co.)
- Miscellany of newer results on the algebra of polynomials
- And just plain good old stuff from the time when algebraic geometry was algebraic geometry
(Cayley–Salmon, Severi; circa end of 19th century)

Erdős's distinct distances problem

It is all about incidences between points and lines (or curves, or surfaces) in three or higher dimensions

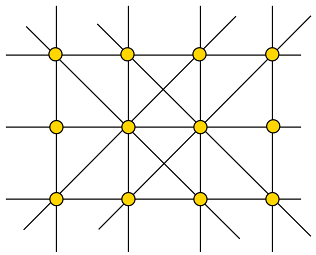
- General theme in most of the recent results:

Beat (or join) the Szemerédi-Trotter incidence bound in the plane

Szemerédi-Trotter (1983): Incidences between points and lines in the plane

P: Set of m distinct points in the plane
L: Set of n distinct lines

$I(P, L) =$ Number of incidences between P and L
$= |\{(p, \ell) \in P \times L \mid p \in \ell\}|$



$I(m, n) = \max \{I(P, L) \mid |P| = m,\ |L| = n\}$

$I(m, n) = \Theta(m^{2/3} n^{2/3} + m + n)$

Three decades of incidences

- Many proofs, some beautiful
([Székely, 1997], new polynomial method [Kaplan et al.])

- Many extensions (other curves, higher dimensions)

- Many connections (distances, repeated configurations, algorithmic problems, Kakeya)

- Many open problems

Beating Szemerédi-Trotter in higher dimensions

"Leading term": $O(m^{2/3}n^{2/3})$

- Elekes / Guth–Katz:

Incidences between m points and n lines in 3D
Assuming a "truly 3-D" scenario:
Not too many lines in a common plane (or regulus)
"Leading term": $O(m^{1/2}n^{3/4})$

- [Sharir, Solomon, 2014]:

Incidences between m points and n lines in 4D
Similar assumptions ("truly 4-D")
"Leading term": $O(m^{2/5}n^{4/5})$

Beating Szemerédi-Trotter in higher dimensions

- Conjecture:

"Leading term" for incidences between m points and n lines in d dimensions:
$O(m^{2/(d+1)}n^{d/(d+1)})$

- And what about

"Leading term" for incidences between m points and n k-flats in d dimensions?

Joining Szemerédi-Trotter in higher dimensions

- Incidences between points and lines in the complex plane
Same bound $O(m^{2/3}n^{2/3} + m + n)$ [Tóth, 2003]

- Special instance of incidences between points and 2-flats in $\mathbb{R}^4$

- (Almost) established as a special case in [Solymosi, Tao, 2012]
(Incidences between points and "pseudo d-flats"
in $2d$ dimensions; (almost) the same bound)

- Established in [Zahl, 2013] (special 2-flats in $\mathbb{R}^4$)

Joining Szemerédi-Trotter in higher dimensions

- And some of the new developments (to be unfolded):

Incidences between points and pseudo-line-like curves in the plane

From distinct distances to incidences in 3D
Elekes's transformation

P: Ground set of n points in the plane

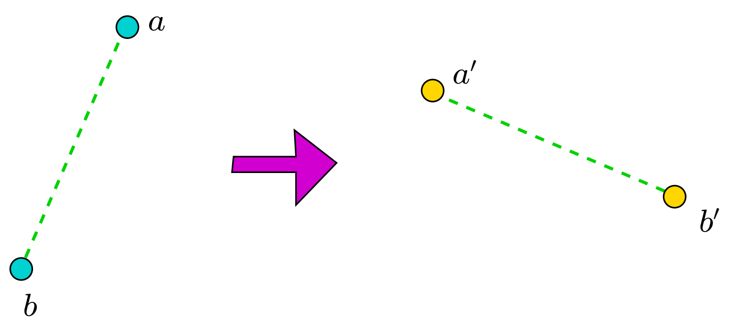
$x = d(P)$: Number of distinct distances in P

$\delta_1, \ldots, \delta_x$: The distinct distances

$E_i = \{(a, b) \mid \|ab\| = \delta_i\}$

From Distinct Distances to Incidences in 3D

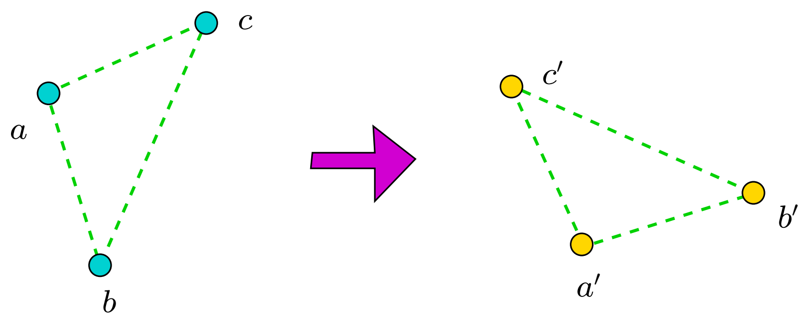
There is a rotation (rigid motion) mapping a to a' and b to b'
$\Leftrightarrow \|ab\| = \|a'b'\|$



Every quadruple (a, b, a', b') in $E_i \times E_i$ generates such a rotation

From Distinct Distances to Incidences in 3D

Multiplicity of a rotation τ:
$|\tau(P) \cap P|$ = Number of points of P mapped by τ to other points of P



From Distinct Distances to Incidences in 3D

N_k: Number of rotations of multiplicity k

$N_{\geq k}$: Number of rotations of multiplicity at least k

$$\sum_{k\geq 2} \binom{k}{2} N_k = \sum_{i=1}^{x} |E_i|(|E_i|-1)$$

Both sides count the 5-tuples (τ, a, b, c, d),
with $\tau(a) = c$ and $\tau(b) = d$
$\equiv$ Quadruples (a, b, c, d) (τ implicitly defined)

From Distinct Distances to Incidences in 3D

$$\sum_{k\geq 2}\binom{k}{2}N_k = \sum_{i=1}^{x}|E_i|(|E_i|-1)$$

Cauchy-Schwarz for RHS:

$$\sum_{i=1}^{x}|E_i|(|E_i|-1) = \Omega\left(\frac{1}{x}\left(\sum_i |E_i|\right)^2\right) = \Omega\left(\frac{n^4}{x}\right)$$

And rearranging LHS:

$$\sum_{k\geq 2}\binom{k}{2}N_k = N_{\geq 2} + \sum_{k\geq 3}(k-1)N_{\geq k}$$

so	$N_{\geq 2} + \sum_{k\geq 3}(k-1)N_{\geq k} = \Omega\left(\frac{n^4}{x}\right)$

From Distinct Distances to Incidences in 3D

$$N_{\geq 2} + \sum_{k\geq 3}(k-1)N_{\geq k} = \Omega\left(\frac{n^4}{x}\right)$$

Challenge: Upper bound LHS by $\approx n^3$

Main Conjecture (Elekes): $N_{\geq k} = O(n^3/k^2)$

If true $\Rightarrow \dfrac{n^4}{x} = O\left(\sum_k \dfrac{n^3}{k}\right) = O\left(n^3 \log n\right)$

Or $x = \Omega(n/\log n)$

Second part of Elekes's transformation

- A rotation (rigid motion) has three degrees of freedom $\Rightarrow$
A point in 3D

- Locus $h_{a,b}$ of rotations that map a to b:
A line in 3D (n^2 such lines)

- $N_{\geq k}$: Number of points incident to at least k lines

- Show that this is $O(n^3/k^2) = O(\#\text{lines}^{3/2}/k^2)$

- And this is what Guth and Katz have done

Point-line Incidences in $\mathbb{R}^3$

Theorem: ([Guth-Katz 10])
For a set P of m points
And a set L of n lines in $\mathbb{R}^3$, such that
No plane contains more than $O(n^{1/2})$ lines of L
(Holds in the Elekes setup)

$$\max I(P,L) = \Theta(m^{1/2}n^{3/4} + m + n)$$

Many (heroic) details missing,
Most notably: the Polynomial partitioning technique
Major new tool introduced by Guth and Katz

(mentioned in other talks; e.g. [Sheffer])

But a major insight to take home:

Elekes's main idea: Double count **quadruples**:

(a, b, a', b') such that $\|ab\| = \|a'b'\|$

Surprisingly, this is the key ingredient in much of the newer stuff

Counting quadruples

- The general idea is not new
- Goes back to [Elekes, Rónyai], [Elekes, Szabó], ???
- And recently by [Tao] (over finite fields)

The Elekes–Rónyai problem
And its many variants

(Polynomial) interaction between three sets of real numbers

A, B, C: Three sets, each of n real numbers

$F(x, y, z)$: A real trivariate polynomial (constant degree)

How many zeroes can F have on $A \times B \times C$?

The Elekes–Rónyai problem

$Z(F) = \{(a, b, c) \in A \times B \times C \mid F(a, b, c) = 0\}$

$|Z(F)| = O(n^2)$

And the bound is worst-case tight:

$A = B = C = \{1, \ldots, n\}$ and $F(x, y, z) = x + y - z$

The Elekes–Rónyai problem

$Z(F) = \{(a,b,c) \in A \times B \times C \mid F(a,b,c) = 0\}$

- Characterize the situations where $|Z(F)| = \Theta(n^2)$

- Establish a gap:

If $|Z(F)| = o(n^2)$ then $|Z(F)| = O(n^{2-\beta})$ for some $\beta > 0$

- Develop a general theory ([Elekes, Szabó, 2012]), and / or

Bypass it in specific (interesting) instances, to improve the upper bound

Background

- For $F(x, y, z) = z - f(x, y)$:

If $|Z(f)| = \Omega(n^2)$ then
$f(x, y) = p(q(x) + r(y))$ or $f(x, y) = p(q(x) \cdot r(y))$
for suitable polynomials p, q, r
[Elekes, Rónyai, 2000]

- Similar (but more complicated) structural results for the general case $F(x, y, z)$
[Elekes, Szabó, 2012]

New approach in a nutshell

- Double count quadruples

- Lower bound: Use Cauchy-Schwarz

- Upper bound:
Reduce to incidences between points and curves in the plane
[Raz, Sharir, Solymosi, 2014]

- If the curves are (essentially) distinct
Szemerédi-Trotter-like incidence upper bound $\Rightarrow$
Subquadratic bound for $|Z(f)|$

- If the curves overlap a lot
f must have the special forms

An example (a starting point)

[Sharir, Sheffer, Solymosi, 2013]

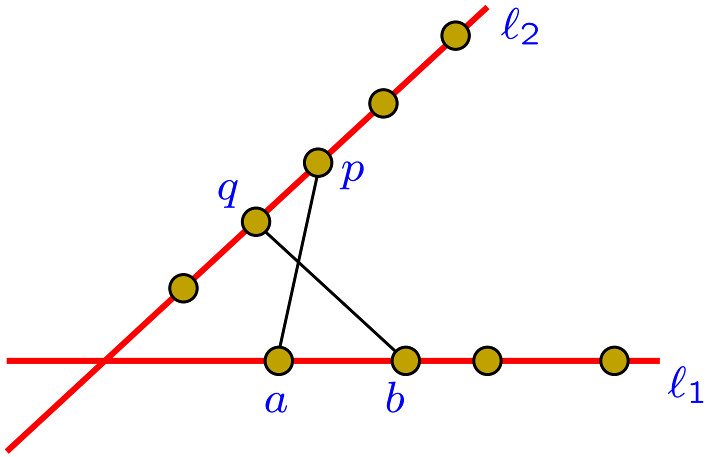
ℓ_1, ℓ_2: Two lines in $\mathbb{R}^2$, non-parallel, non-orthogonal

P_1, P_2: Two finite point sets, $P_1 \subset \ell_1$, $P_2 \subset \ell_2$

$D(P_1, P_2)$: Number of distinct distances between P_1 and P_2

Theorem: $D(P_1, P_2) = \Omega\left(\min\{|P_1|^{2/3}|P_2|^{2/3},\ |P_1|^2,\ |P_2|^2\}\right)$

In particular, $D(P_1, P_2) = \Omega(n^{4/3})$ for $|P_1| = |P_2| = n$



- A superlinear bound conjectured by [Purdy]
- And proved by [Elekes, Rónyai, 2000]
- And improved to $\Omega(n^{5/4})$ (equal-size sets) by [Elekes, 1999]

What's the connection?

$Z(F) = \{(a, b, c) \in A \times B \times C \mid F(a, b, c) = 0\}$

$A = P_1$, $B = P_2$
$C =$ Set of (squared) distinct distances between P_1 and P_2

$F(a, b, c) = c - \|a - b\|^2$

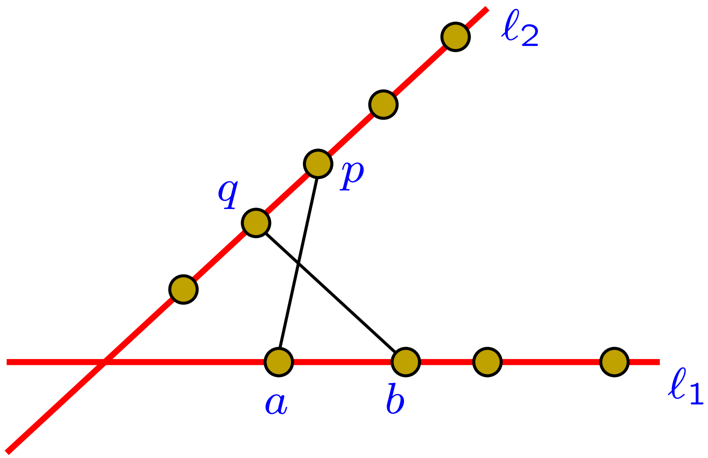
(Here A, B, C have different sizes!)

Intuition: "Subquadratic" upper bound on $|Z(F)| \Rightarrow$
Sharp lower bound on $|C| = D(P_1, P_2)$

How does it work?
Quadruples!

Double count size of

$Q = \{(a, b, p, q) \mid a, b \in P_1,\ p, q \in P_2,\ \|a - p\| = \|b - q\|\}$



How does it work?

Double count size of
$Q = \{(a,b,p,q) \mid a,b \in P_1,\ p,q \in P_2,\ \|a-p\| = \|b-q\|\}$

Put $x = D(P_1, P_2)$ (number of (squared) distinct distances)
$\{\delta_1, \ldots, \delta_x\}$: The distances themselves
Assume $|P_1| = |P_2| = n$

$E_i = \{(a,p) \in P_1 \times P_2 \mid \|a-p\|^2 = \delta_i\}$ for each i

Then good old Cauchy-Schwarz says
(Exactly as in [Elekes / Guth–Katz])

$$|Q| = \sum_{i=1}^{x} |E_i|^2 \geq \frac{1}{x}\left(\sum_{i=1}^{x} |E_i|\right)^2 = \frac{n^4}{x}$$

Now for an upper bound on $|Q|$

Turn into an incidence problem between points and curves in the plane

(Think of ℓ_1 and ℓ_2 as two copies of the real line)

P_1^2: Set of n^2 points in $\mathbb{R}^2$
(Pair of points on $\ell_1 \Rightarrow x$ and y-coordinates)

For each $(p, q) \in P_2^2$ define a curve

$$\gamma_{p,q} = \{(x, y) \in \mathbb{R}^2 \mid \|x - p\| = \|y - q\|\}$$

Points, curves, and incidences

$(a, b, p, q) \in Q$ iff

point (a, b) incident to curve $\gamma_{p,q}$

(Recall: $Q = \{(a, b, p, q) \mid a, b \in P_1,\ p, q \in P_2,\ \|a - p\| = \|b - q\|\}$
and

$\gamma_{p,q} = \{(x, y) \in \mathbb{R}^2 \mid \|x - p\| = \|y - q\|\}$)

Points, curves, and incidences

$\gamma_{p,q}$: $\quad x^2 + p^2 - 2px \cos\alpha = y^2 + q^2 - 2qy \cos\alpha$

n^2 hyperbolas, all **distinct** (when $\alpha \neq 0, \pi/2$)

Every pair of hyperbolas intersect at most twice

Every pair of points incident to at most two common hyperbolas

$\Rightarrow$ Essentially, system of "pseudo-lines"

Points, curves, and incidences

Standard incidence bounds
[Szemerédi–Trotter, 1983], via [Székely, 1997]:

# of incidences between M points and N such curves is

$$O(M^{2/3}N^{2/3} + M + N)$$

We have $M = N = n^2 \Rightarrow O(n^{8/3})$ incidences $\Rightarrow |Q| = O(n^{8/3})$

Together with $|Q| = \Omega(n^4/x) \Rightarrow x = \Omega(n^{4/3})$

Works in general, in principle

$Z(F) = \{(a, b, c) \in A \times B \times C \mid F(a, b, c) = 0\}$

Consider only the case $z = f(x, y)$ "for simplicity"

$Z(f) = \{(a, b) \in A \times B \mid f(a, b) \in C\}$

$Q = \{(a, b, p, q) \mid a, b \in A,\ p, q \in B,\ f(a, p) = f(b, q)\}$

Works in general, in principle

$Z(f) = \{(a,b) \in A \times B \mid f(a,b) \in C\}$

$Q = \{(a,b,p,q) \mid a,b \in A,\ p,q \in B,\ f(a,p) = f(b,q)\}$

$M = |Z(f)|$
and $M_c = |\{(a,p) \in A \times B \mid f(a,p) = c\}|$ for $c \in C$

Again, Cauchy–Schwarz:

$$|Q| = \sum_c M_c^2 \geq \frac{1}{|C|}\left(\sum_c M_c\right)^2 = \frac{M^2}{|C|}$$

Upper bound on $|Q|$

For each $(p, q) \in B^2$ define a curve

$\gamma_{p,q} = \{(x, y) \mid f(x, p) = f(y, q)\}$

$(a, b, p, q) \in Q$ iff $(a, b) \in \gamma_{p,q}$

So $|Q|$ = number of incidences between the $|A|^2$ points of A^2 and the $|B^2|$ curves of B^2

Point-curve incidences

Suppose that all the curves are **distinct**

- Every pair of curves intersect in $O(1)$ points ([Bézout])
- Every pair of points incident to $O(1)$ curves
  Duality = symmetry between points and curves

So again they are "pseudo-lines with bounded multiplicity"

And we get the same Szemerédi–Trotter-like bound

$$|Q| = O((|A|^2)^{2/3}(|B|^2)^{2/3} + |A|^2 + |B|^2) = O((|A||B|)^{4/3})$$

$$|Q| = O((|A|^2)^{2/3}(|B|^2)^{2/3} + |A|^2 + |B|^2) = O((|A||B|)^{4/3})$$

and $|Q| \geq \frac{M^2}{|C|}$, so

$$\boxed{M = O(|A|^{2/3}|B|^{2/3}|C|^{1/2})}$$

And when $|A| = |B| = |C| = n$, we have $M = O(n^{11/6})$

In words: $z = f(x, y)$ vanishes at only $O(n^{11/6})$ points of $A \times B \times C$

Some reflections

- Fairly simple derivation of a concrete and improved bound
(Not concrete and only implicit in [Elekes and co.])

- **The catch:** Who said the curves
$\gamma_{p,q} = \{(x, y) \mid f(x, p) = f(y, q)\}$
need be distinct (or nonoverlapping)?!

(The incidence infrastructure collapses when curves coincide)

- Need an argument:

If there is significant coincidence / overlap of curves
Then f must have a special form
$f(x,y) = p(q(x) + r(y))$ or $f(x,y) = p(q(x) \cdot r(y))$
And $\Theta(n^2)$ zeros of f on $A \times B \times C$ are then possible

- And if not, we get the improved bound $O(n^{11/6})$

- A different approach to the Elekes–Rónyai–Szabó theory

- Meanwhile, sound general theory for $z = f(x,y)$
- Only ad-hoc arguments, problem by problem, for $F(x,y,z) = 0$

Recall: Distinct distances between two lines

$\gamma_{p,q} = \{(x, y) \in P_1^2 \mid \|x - p\| = \|y - q\|\}$

$\gamma_{p,q}$: $\quad x^2 + p^2 - 2px\cos\alpha = y^2 + q^2 - 2qy\cos\alpha$

n^2 hyperbolas, all **distinct** (when $\alpha \neq 0, \pi/2$)

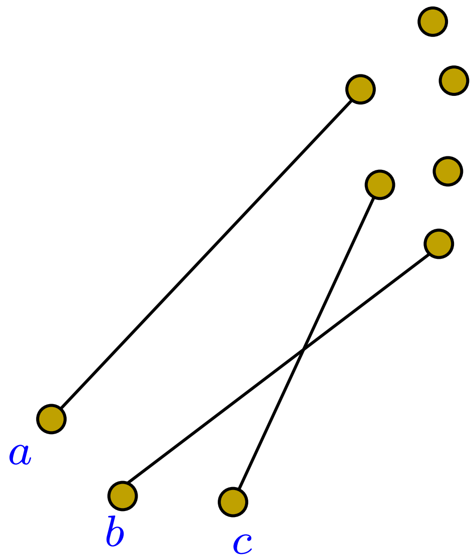
But when $\alpha = 0$: $\quad \gamma_{p,q}$: $\quad |x - p| = |y - q|$

And when $\alpha = \pi/2$: $\quad \gamma_{p,q}$: $\quad x^2 - y^2 = q^2 - p^2$

Many potential coincidences / overlaps

Other problems:
Distinct distances from three points

Distinct distances between three points a, b, c and a set P of n other points in the plane



Other problems:
Distinct distances from three points

Distinct distances between three points a, b, c
and a set P of n other points in the plane

$\Omega(n^{6/11})$ if a, b, c not collinear
[Sharir, Solymosi, 2013]

Improving $\Omega(n^{0.502})$
[Elekes, Szabó, 2012]

(Can be $\Theta(n^{1/2})$ if a, b, c collinear)
[Elekes, 1995]

Other problems:
Triple points on unit circles through 3 points

Three points a, b, c,

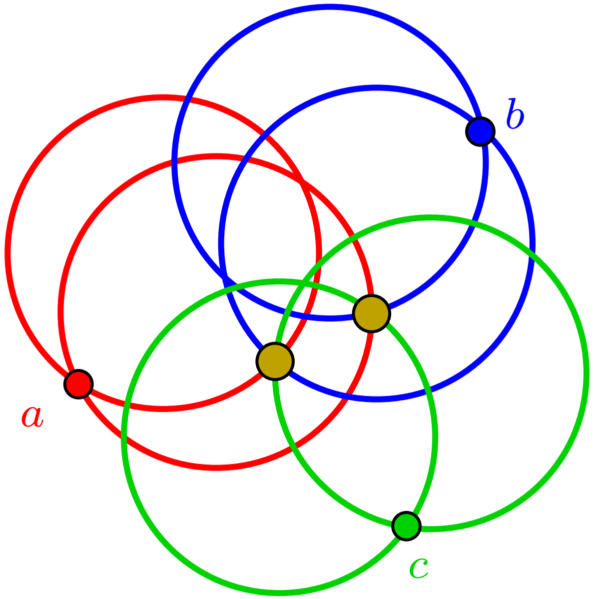
n red unit circles pass through a

n blue unit circles pass through b

n green unit circles pass through c

How many triple points, incident to a red, a blue, and a green circle?

Triple points



Previous bound: $O(n^{2-\eta})$, for some unspecified $\eta > 0$
[Elekes, Simonovits, Szabó, 2012]
Settling a conjecture of [Székely]

We have: $O(n^{11/6})$
[Raz, Sharir, Solymosi, 2013]

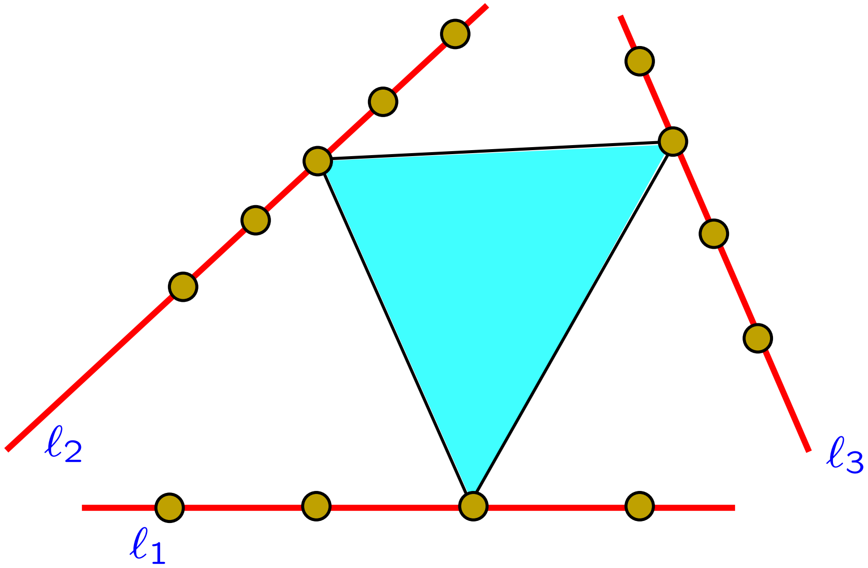
Other problems
Unit-area triangles on three lines

ℓ_1, ℓ_2, ℓ_3: Three non-parallel lines in the plane

$P_i \subset \ell_i$, $i = 1, 2, 3$: Three sets of n points each

How many unit-area triangles spanned by triples in $P_1 \times P_2 \times P_3$?

Unit-area triangles on three lines



Suprise: Can be $\Theta(n^2)$
(The curves can overlap!) [Raz, Sharir 2014]

Thank You