Geometric Incidences and related problems

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Note: good & bad news

Good news:

Talk is simple & elementary

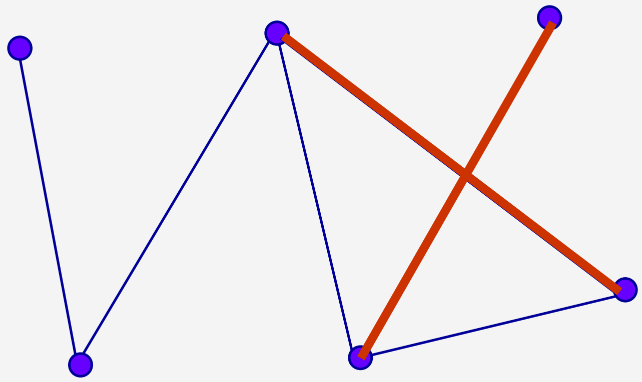
Bad news:

Audience is tired ...&

Speaker is jet-lagged

Reminder: Crossing -Lemma

- ***G* = (*V*,*E*)** arbitrary simple graph
- ***X*** = # edge crossings



- Thm: [Ajtai et al. '82, Leighton '83]

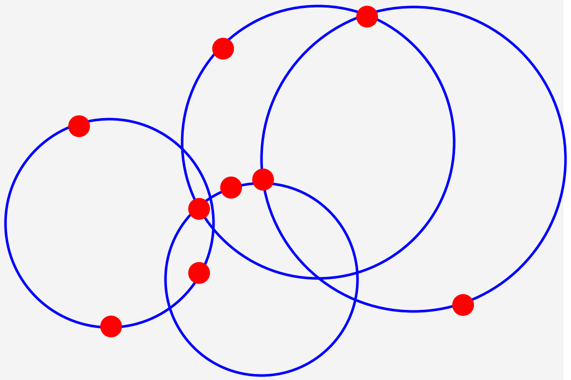
$$(|E| \geq 4|V|) \Rightarrow X \geq \Omega(|E|^3/|V|^2)$$

Selection Lemmas: circles and points

- $|P|=n$ pts in R^2
- $|C| = c > 4n$ **arbitrary** circles spanned by pairs of pts in P
- ***Thm:***

1. $\exists\, p \in \Omega\,(c^2/n^2)$
2. If circles spanned by **triples**,

- $\exists\, p \in \Omega\,(c^{3/2}/n^{3/2})$ circles
- Bounds asymptotically tight!



Selection Lemmas: circles and points

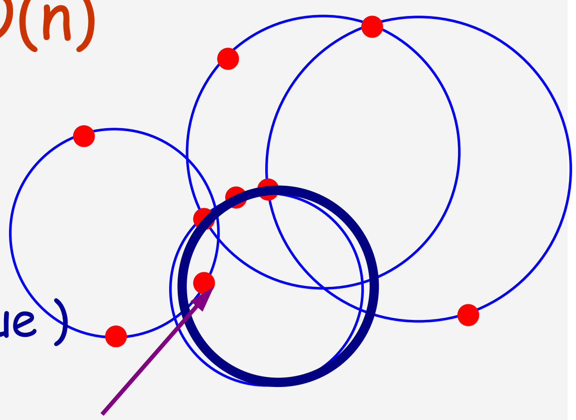
X = # (pt,circle) pt inside circle

$|P|=n$ pts in R^2

$|C| = c > 4n$ **triples** circles

#empty circles (Delaunay circles) is $O(n)$

- Bootstrapping Lemma:

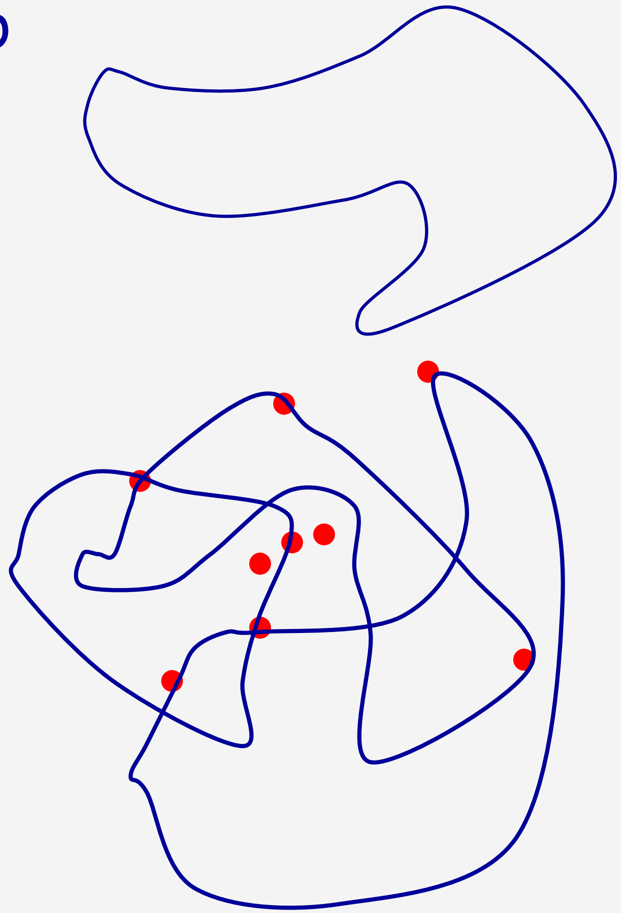


$X > c - 3n$

- $X > \Omega(c^{3/2}/n^{1/2})$ (using the sampling technique)
- $\exists p \in \Omega(x/n)$ configurations
- $\exists p \in \Omega(c^{3/2}/n^{3/2})$ circles (asymptotically tight)

Selection Lemmas: pseudo-circles and points

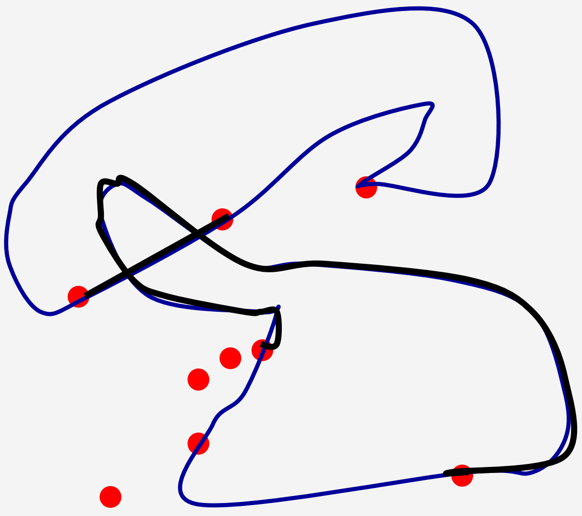
- Proof technique generalizes to **Pseudo-Circles**
- Replace circles by arbitrary simple closed Jordan curves:



n pts and c pseudo-circles

Selection Lemmas: pseudo-circles and points

Lemma: #empty pseudo-circles is $\leq 3n-6$



Proof:

Construct graph on pts as follows:

Selection Lemmas: pseudo-circles and points

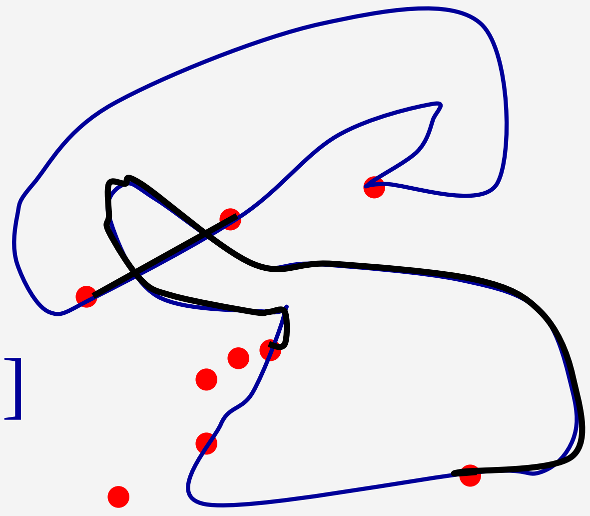
Lemma: #empty pseudo-circles is $\leq 3n-6$

Drawing is not planar

<u>However:</u>

edges intersect even #times

Combined with [Hanani, Tutte '70] implies planarity.



Unit Distances

- Problem:

What is the maximum number of times, $f_d(n)$, that the same (say, the **unit**) distance can occur among n points in R^d

- **Old Open and Hard:**

Unit Distances (cont)

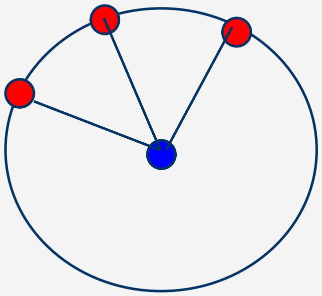
Known bounds:

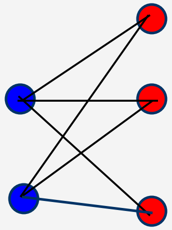
$n^{1+\frac{c}{\log\log n}} \leq f_2(n) \leq O(n^{\frac{3}{2}})$ [Erdős '46]

- $f_2(n) \leq O(n^{\frac{4}{3}})$ [Spencer, Szemerédi, Trotter '84] [Clarkson, Edelsbrunner, Guibas, Sharir, Welzl '90]
- $f_2(n) \leq O(n^{\frac{4}{3}})$ Székely '97 using the Crossing Lemma!

Unit Distances (cont)

- Erdős's proof of the weaker upper bound $O(n^{\frac{3}{2}})$:
  The unit distance graph does not contain a subgraph $K_{2,3}$. Hence its size is at most $O(n^{\frac{3}{2}})$



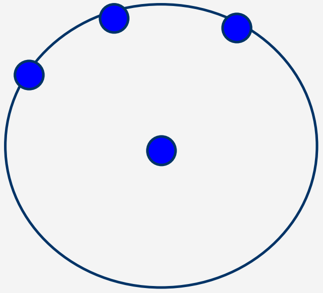


Unit Distances (cont)

- Simple proof of the $O(n^{4/3})$ upper bound.

- Draw a unit circle around each pt

- Connect every pair of consecutive pts with a circular arc along the circle.

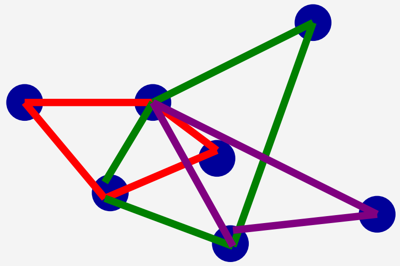


Unit Distances (cont)

- We obtain a (almost simple) graph with ***n*** pts and ***e*** edges.
- I = 2 #unit distances
- X = #crossings so $\Omega(e^3/n^2) = X = O(n^2)$ and $e=O(n^{4/3})$
- Big open problem: Improve the bounds

More applications for Crossing Lemma: polygons spanned by pts

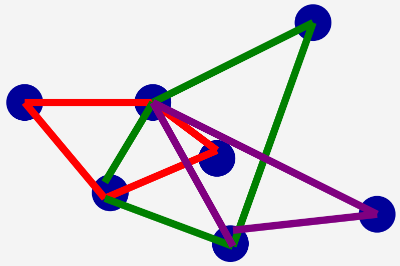
$|P|=n$ pts in plane



- $|C| = k$ convex polygons with distinct edges
  - Each polygon = convex hull of subset of P
  - Bound the total # **polygon edges**

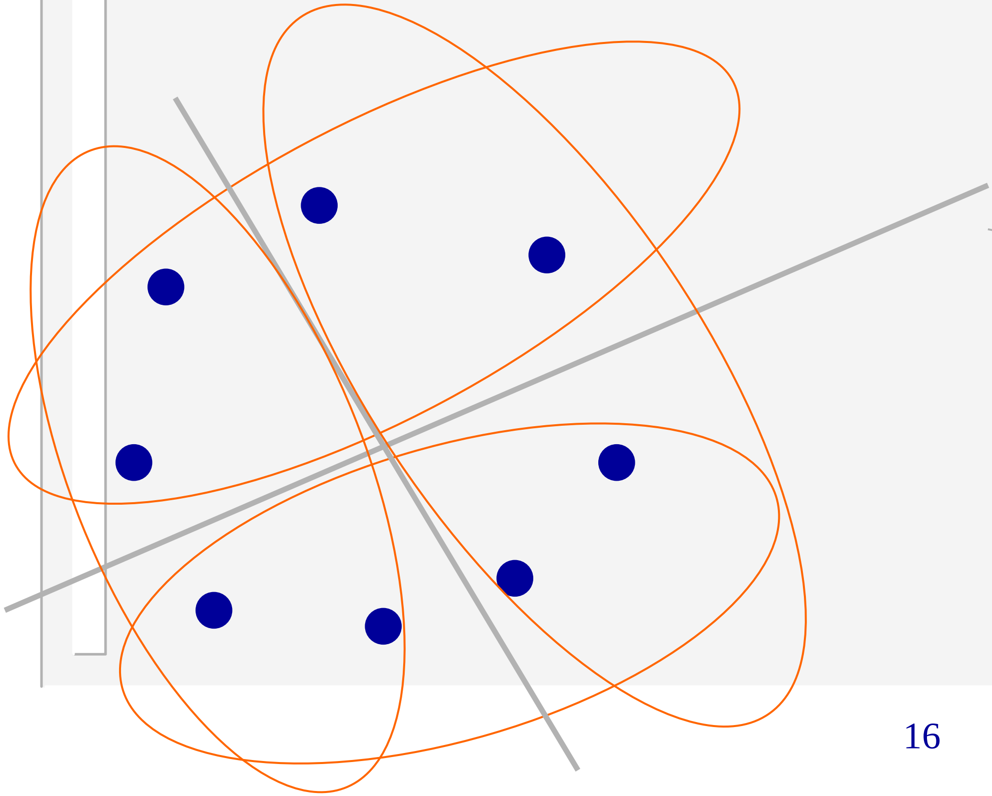
More applications for Crossing Lemma: polygons spanned by pts

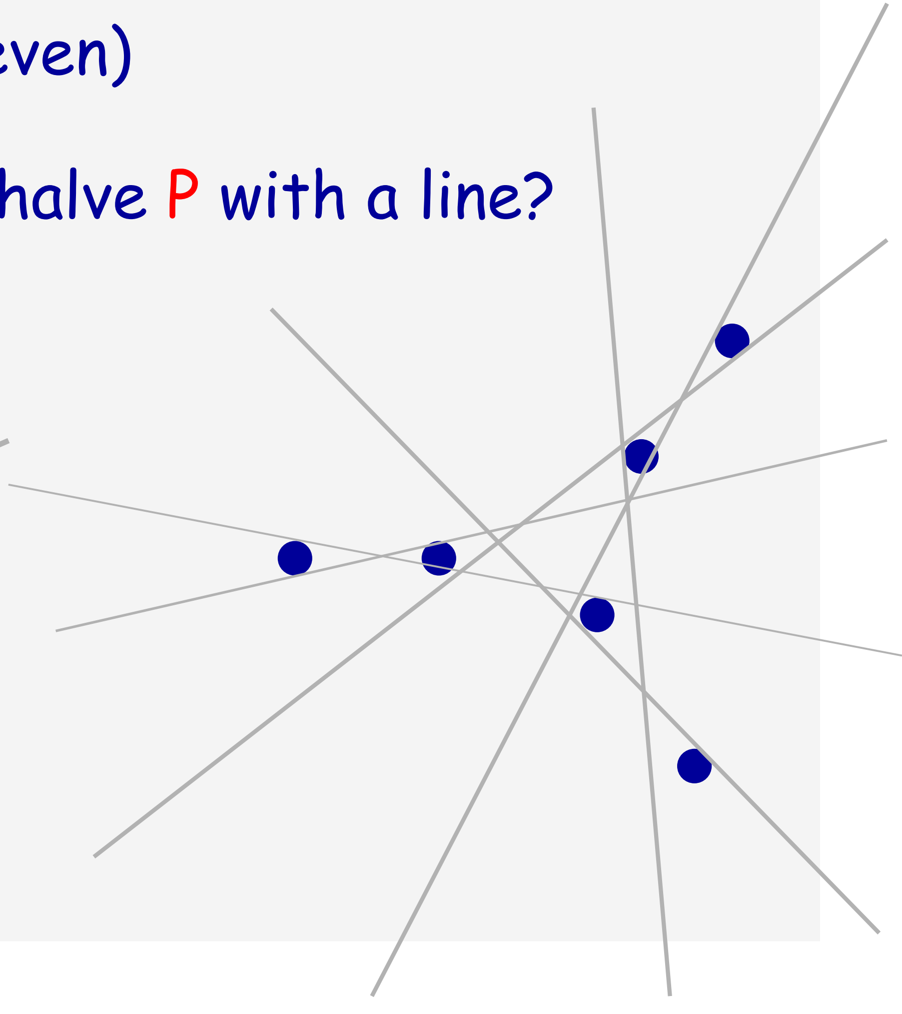
- e = total #edges of all polygons
- We have: $\Omega(e^3/n^2) = X$
- $X = O(ek)$
- $e = O(nk^{1/2})$.
- Asymptotically tight!



The K-set problem in the plane

- P: n pts in R^2 (assume n is even)
- In how many ways can we halve P with a line?

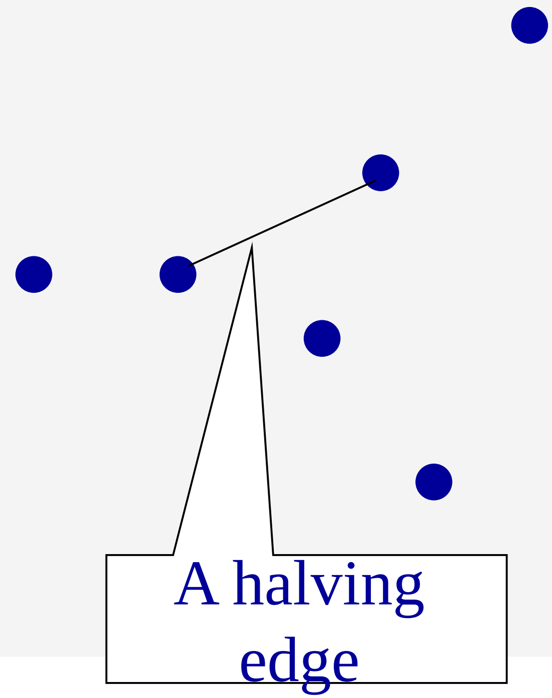




The K-set problem in the plane

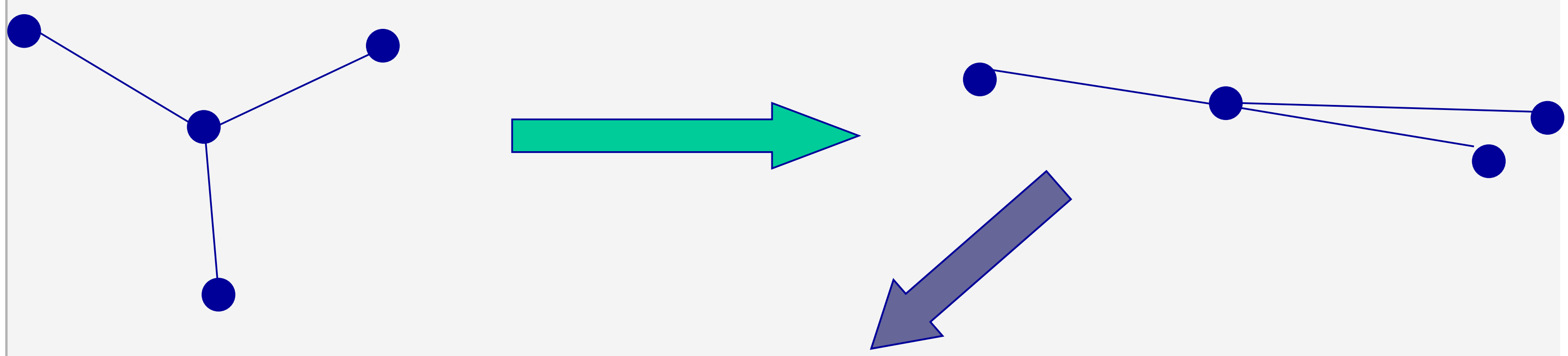
- $|P| = n$ pts in the plane
- A halving-edge is a pair of points of P which spans a halving line (i.e., $(n-2)/2$ points on each side).
- Enough to count halving-edges !!!
- Bound the number $F^2(n)$

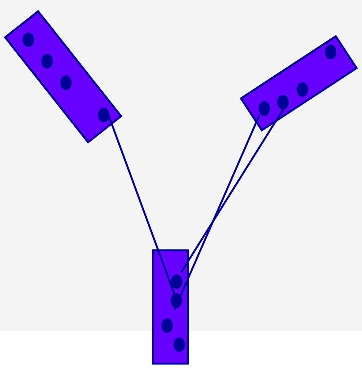
of halving-edges in the worst case



The K-set problem in the plane

- A construction with “many” halving edges

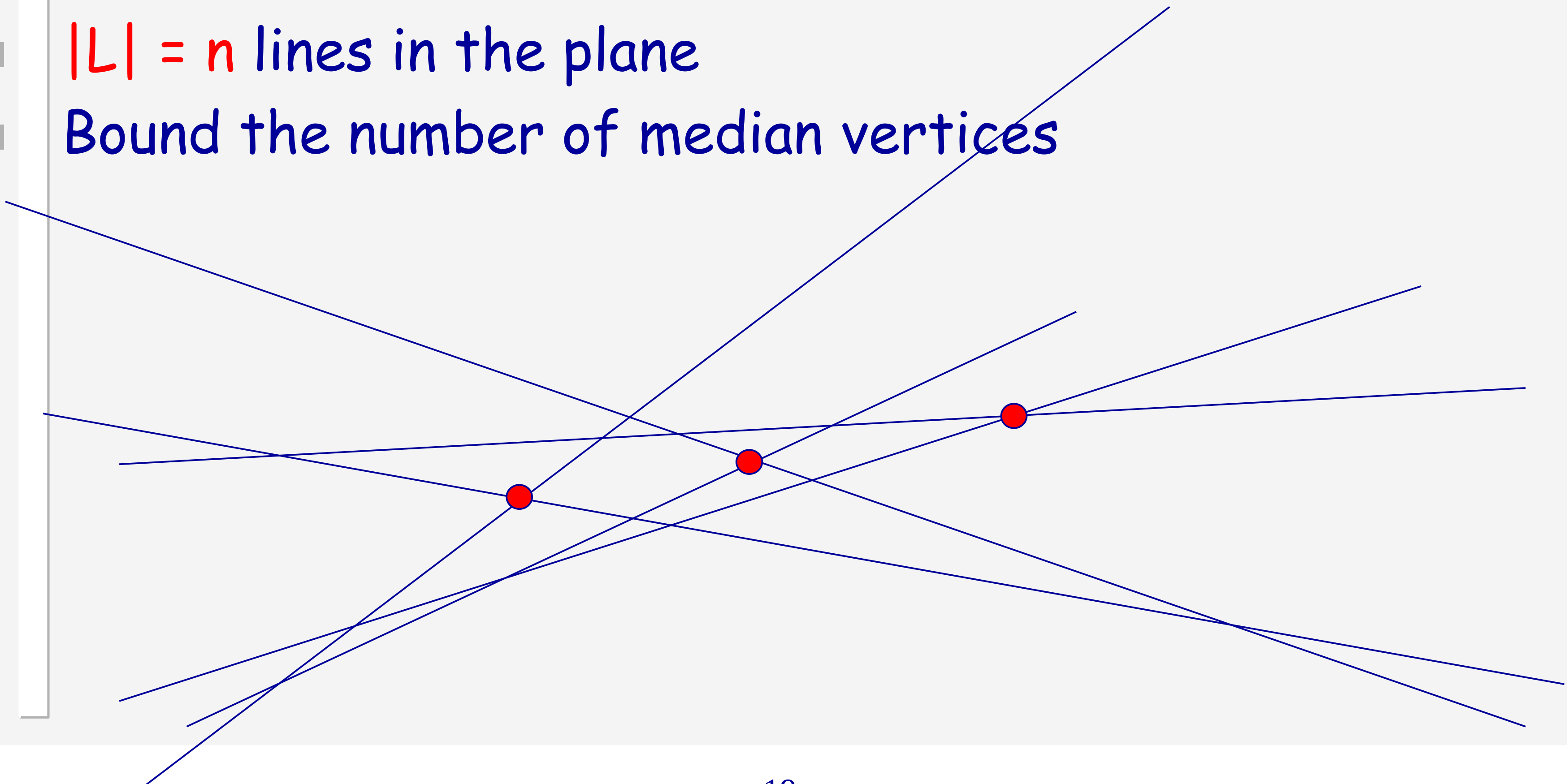




$$f(3n) > 3f(n) + \Omega(n)$$
$$\Rightarrow f(n) = \Omega(n \log n)$$

The K-set problem in the plane

- Equivalent formulation
- |L| = n lines in the plane
- Bound the number of median vertices

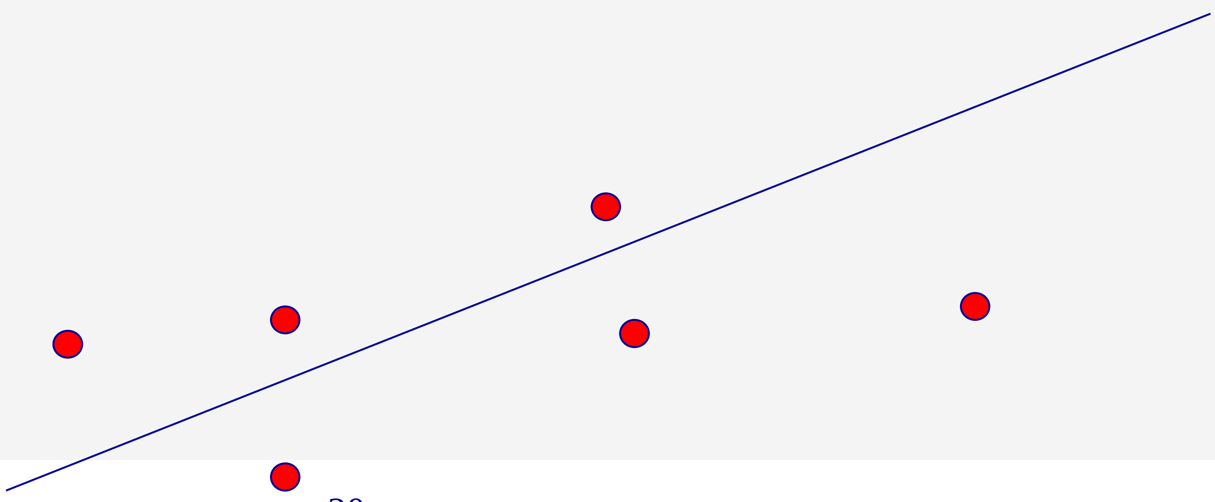


Example:

Algorithmic problem:

P a set of n points in the plane

Find a line l (1-median line) that minimizes $\sum_{p \in P} d(p, l)$

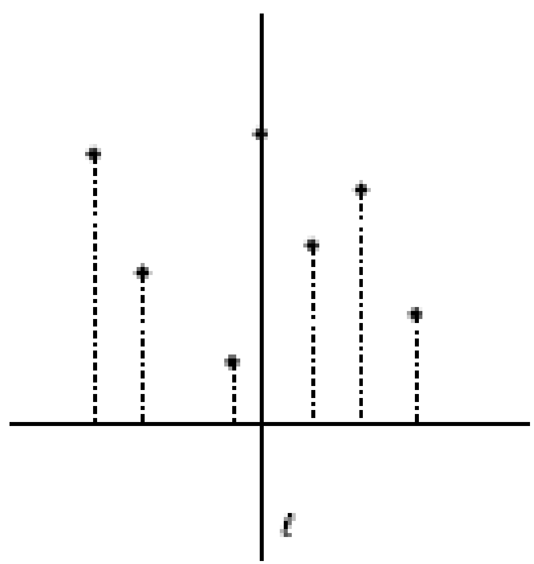


If direction of l is fixed,

and P is projected on a line orthogonal to l

Optimal placement of l is at the median of the projected points

$\Rightarrow$ ***l*** *is a halving line of P*



Algorithm: Vary the direction θ of l from 0 to π

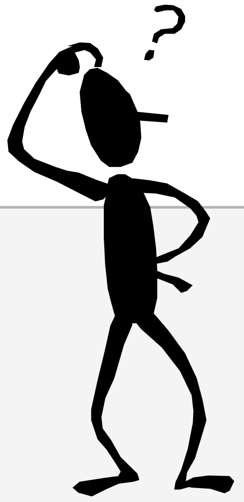
Keep track of the halving line $l(\theta)$

As long as the split of P by $l(\theta)$ is unchanged,

Optimize $\sum_{p \in P} d(p, l(\theta))$ as a function of θ (easy task)

Output the best θ overall

How efficient is the algorithm?



key question:

How many changes can occur in the splitting of P into two equal halves by $l(\theta)$?

This is the famous k-set problem !

The K-set problem (Definition)

S = n pts in R^d

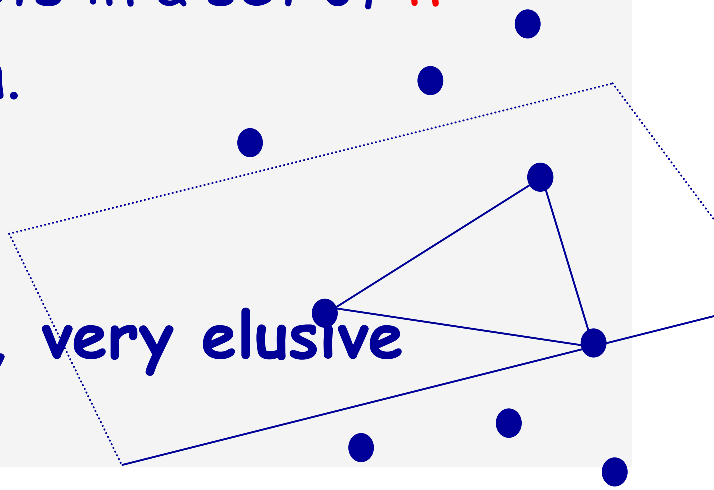
A $(d - 1)$-dimensional simplex σ spanned by d points fo S is a halving-facet of S if:

the hyperplane spanned by σ contains exactly $(n-d)/2$ points of S on each side

$F^d(n)$ = maximum # of halving-facets in a set of n points in d-space in general position.

Goal: Obtain sharp bounds on $F^d(n)$

Still, after 40 years of research, very elusive

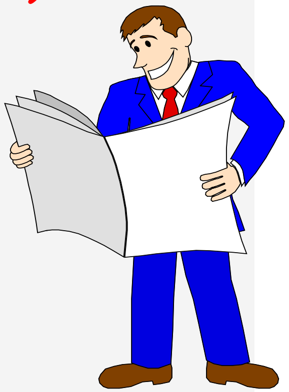


The K-set problem (History in R^2)

[Lovász '71], [Erdős, Lovász, Simmons, Straus '73] posed +

Initial upper bound $F^2(n) = O(n^{3/2})$

Initial lower bound $F^2(n) = \Omega(n \log n)$



Slight improvement 20 years later

[Pach, Steiger, Szemerédi '92]

$$F^2(n) = O(n^{3/2}/\log^* n)$$

Record upper bound $F^2(n) = O(n^{4/3})$ [Dey, '98]

Record lower bound $F^2(n) = \Omega(n \cdot 2^{c\sqrt{\log n}})$ [Tóth, '00]

Closing the gap:

Major intriguing open problem in combinatorial geometry

History in R^3

[Bárány, Füredi, Lovász '90]:

$$F^3(n) = O(n^{3-1/343})$$

[Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, Wenger '91]

$$F^3(n) = O(n^{8/3}\log^{5/3} n)$$

[Dey, Edelsbrunner '94]:

$$F^3(n) = O(n^{8/3})$$

[Sharir, Smorodinsky, Tardos '00]:

$F^3(n) = O(n^{5/2})$ **(current record)**

Lower bound of [Tóth, '00] `lifted' from the plane:

$$F^3(n) = \Omega\left(n^2 2^{c\sqrt{\log n}}\right)$$

History in R^d ($d \geq 4$)

[Alon, Bárány, Füredi, Kleitman '92]:

[Živaljević, Vrećica '92]:

$F^d(n) = O(n^{d-\varepsilon(d)})$ (algebraic topology)

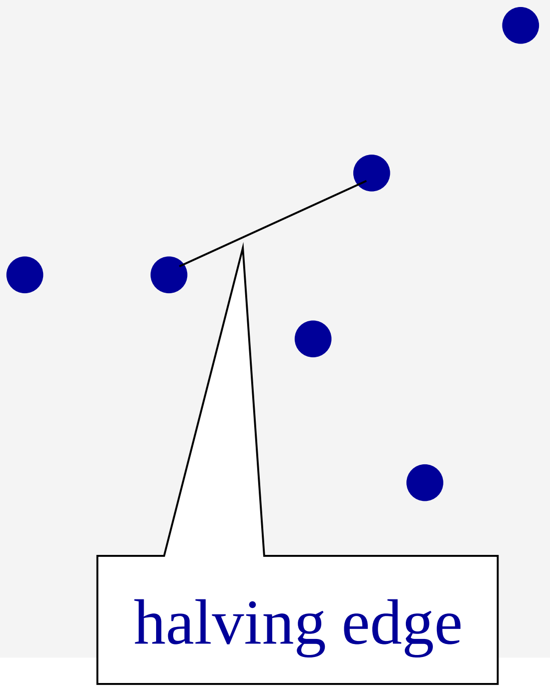
Where $\varepsilon(d)$ is exponentially small in d

[Matoušek, Sharir, Smorodinsky, Wagner '06]:

$F^4(n)=O(n^{4 - 2/45})$ **(current record)** elementary proof!

The K-set problem in the plane reminder:

- $|P| = n$ pts in plane
- A halving-edge is a pair of points which spans a halving line (i.e., $(n-2)/2$ points on each side).
- Bound the number $F^2(n)$ of halving-edges in the worst case

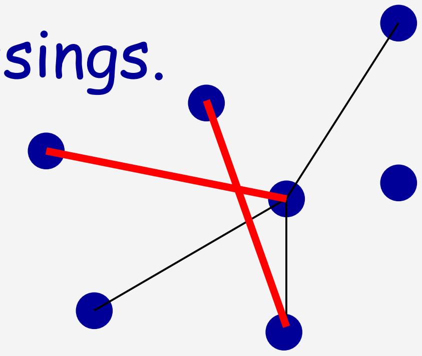


The K-set problem in the plane

Upper Bound sketch of proofs

- Construct the halving-edge graph G = (V,E)
- Count the number of crossings:

  There are $\Omega(|E|^3/n^2)$ crossings.

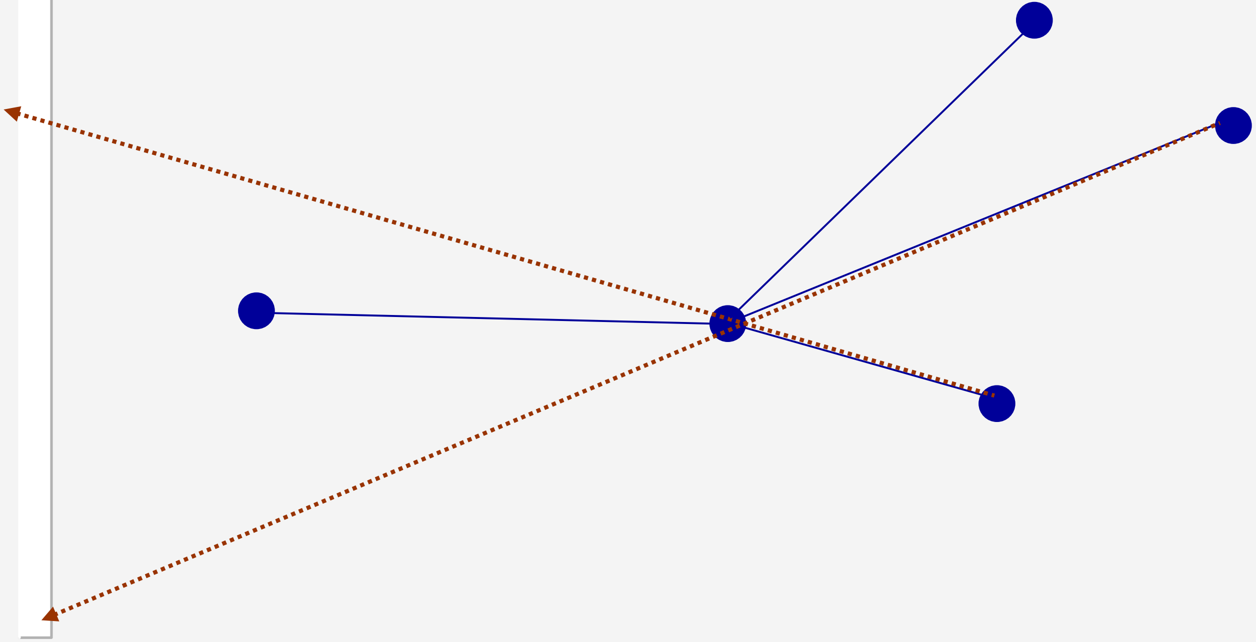


- Lovász' Lemma:

  A line can cross at most n halving-edges!

The K-set problem in the plane

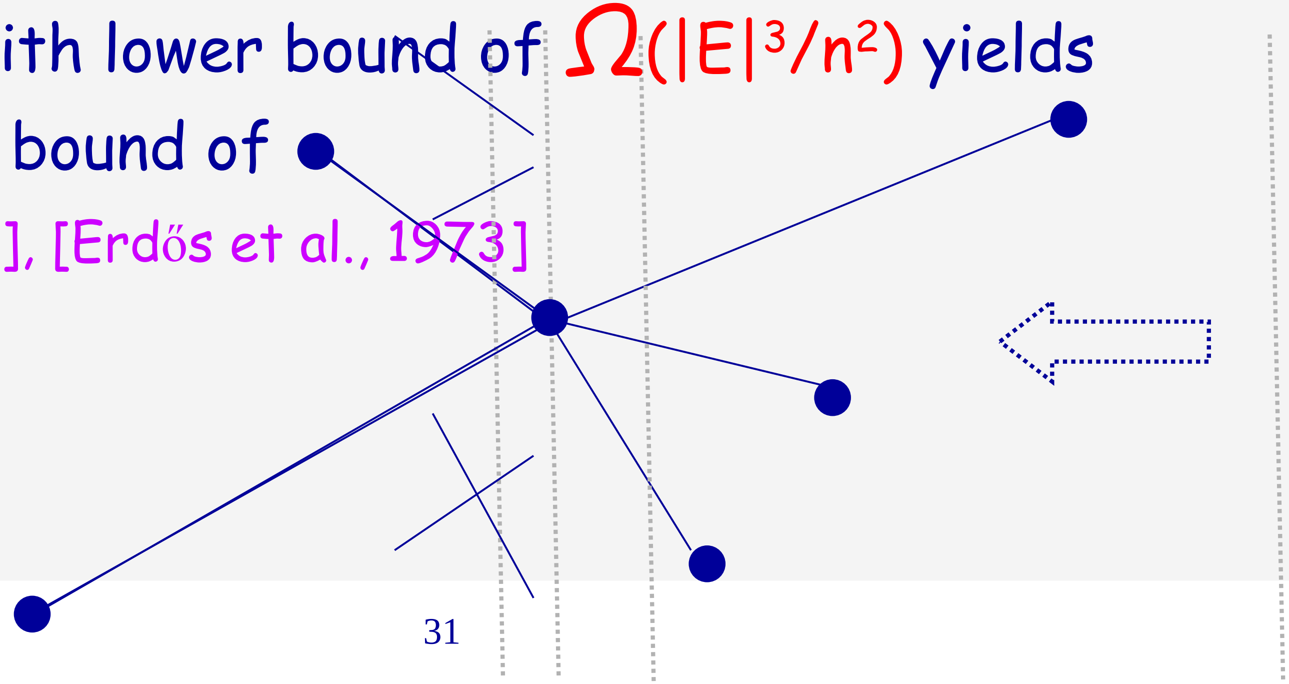
Claim: The halving-edge graph is antipodal



Lovász' Lemma

- Any line intersects at most n edges
- In particular any edge e crosses at most n other edges
- So #crossings = $O(|E|n)$
- Combined with lower bound of $\Omega(|E|^3/n^2)$ yields
- The $O(n^{3/2})$ bound of

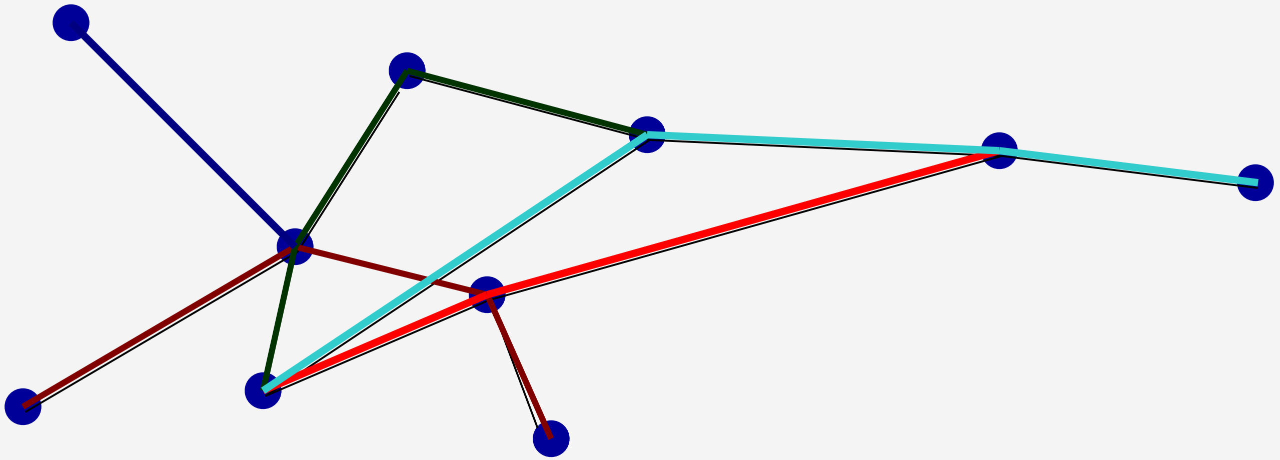
[Lovász, 1971], [Erdős et al., 1973]



The K-set problem (cont)

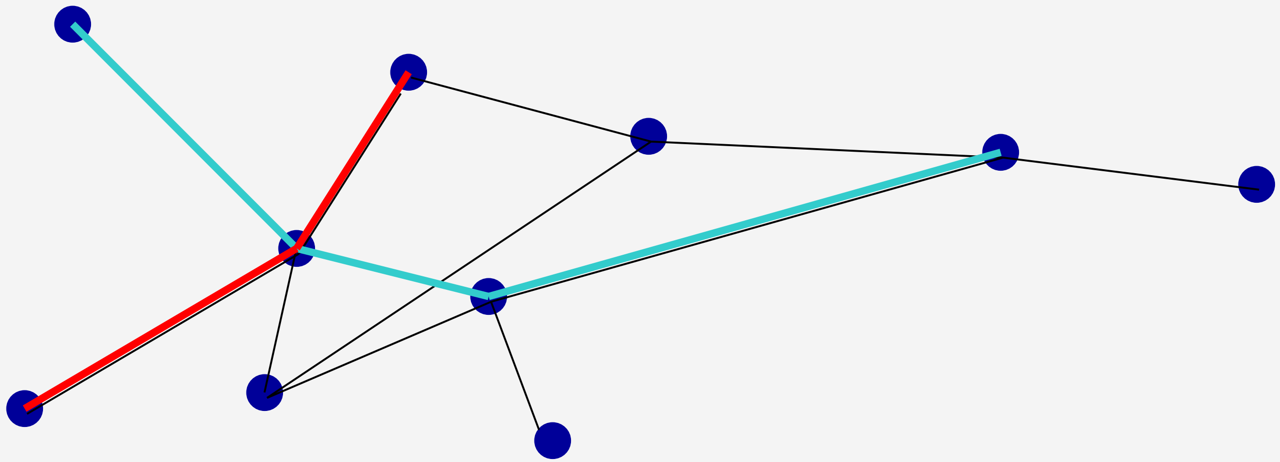
A simpler version of Dey's proof:
Claim: G has only $O(n^2)$ crossings

Decompose the edges into **concave** chains



The K-set problem (cont)

- At most one chain ends at a given point
  => #chains ≤ **n**
- Apply a symmetric decomposition into ≤ **n** convex chains



The K-set problem (cont)

- Upper bounds on # of crossings
- Charge each crossing to the pair of concave and convex chains:
- #such pairs is $O(n^2)$
- Combined with lower bound yields $O(n^{4/3})$