Geometric Incidences and related problems

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Note: good & bad news

Good news:
Talk is simple & elementary

Bad news:
Audience is tired ...&
Speaker is jet-lagged
Reminder: Crossing -Lemma

- \( G = (V,E) \) arbitrary simple graph
- \( X = \# \text{ edge crossings} \)
- Thm: [Ajtai et al. '82, Leighton '83]
  \(|E| \geq 4|V| \Rightarrow X \geq \Omega(|E|^3/|V|^2)\)
Selection Lemmas: circles and points

- $|P| = n$ pts in $\mathbb{R}^2$
- $|C| = c > 4n$ arbitrary circles spanned by pairs of pts in $P$

**Thm:**
1. $\exists p \in \Omega \left( \frac{c^2}{n^2} \right)$
2. If circles spanned by triples,
   - $\exists p \in \Omega \left( \frac{c^{3/2}}{n^{3/2}} \right)$ circles
   - Bounds asymptotically tight!
Selection Lemmas: circles and points

\[ X = \# (pt, \text{circle}) \text{ pt inside circle} \]

\#empty circles (Delaunay circles) is \( O(n) \)

- **Bootstrapping Lemma:**
  \[ X > c - 3n \]

- \( X > \Omega (c^{3/2}/n^{1/2}) \) (using the sampling technique)

- \( \exists p \in \Omega (x/n) \) configurations

- \( \exists p \in \Omega (c^{3/2}/n^{3/2}) \) circles (asymptotically tight)

\(|P| = n \text{ pts in } \mathbb{R}^2\)

\(|C| = c > 4n \text{ triples circles}\)
Selection Lemmas: pseudo-circles and points

- Proof technique generalizes to **Pseudo-Circles**
- Replace circles by arbitrary simple closed Jordan curves:

\[ n \text{ pts and } c \text{ pseudo-circles} \]
Lemma: #empty pseudo-circles is $\leq 3n-6$

Proof:
Construct graph on pts as follows:
Selection Lemmas: pseudo-circles and points

Lemma: \#empty pseudo-circles is \( \leq 3n-6 \)

Drawing is not planar

**However:**
edges intersect even \#times

Combined with [Hanani, Tutte '70] implies planarity.
Unit Distances

Problem:
What is the maximum number of times, $f_d(n)$, that the same (say, the unit) distance can occur among $n$ points in $\mathbb{R}^d$.

Old Open and Hard:
Unit Distances (cont)

**Known bounds:**

\[ n^{1 + \frac{c}{\log \log n}} \leq f_2(n) \leq O(n^{\frac{3}{2}}) \quad \text{[Erdős '46]} \]

- \[ f_2(n) \leq O(n^{\frac{4}{3}}) \quad \text{[Spencer, Szemerédi, Trotter '84]} \]
  
  [Clarkson, Edelsbrunner, Guibas, Sharir, Welzl '90]

- \[ f_2(n) \leq O(n^{\frac{4}{3}}) \quad \text{Szekely '97 using the Crossing Lemma!} \]
Erdős's proof of the weaker upper bound $O(n^2)$:
The unit distance graph does not contain a subgraph $K_{2,3}$. Hence its size is at most $O(n^3)$. 
Unit Distances (cont)

- Simple proof of the $O(n^{4/3})$ upper bound.
  - Draw a unit circle around each pt
  - Connect every pair of consecutive pts with a circular arc along the circle.
Unit Distances (cont)

- We obtain a (almost simple) graph with \( n \) pts and \( e \) edges.
- \( I = 2 \) #unit distances
- \( X = \) #crossings so \( \Omega(e^3/n^2) = X = O(n^2) \) and \( e=O(n^{4/3}) \)
- Big open problem: Improve the bounds
More applications for **Crossing Lemma:** polygons spanned by pts

|P| = n pts in plane

|C| = k convex polygons with distinct edges

- Each polygon = convex hull of subset of P
- Bound the total # **polygon edges**
More applications for **Crossing Lemma:**

- $e =$ total #edges of all polygons
- We have: $\Omega(e^3/n^2) = X$
- $X = O(ek)$
- $e = O(nk^{1/2})$.
- Asymptotically tight!
The K-set problem in the plane

- **P**: \( n \) pts in \( \mathbb{R}^2 \) (assume \( n \) is even)
- In how many ways can we halve \( P \) with a line?
The K-set problem in the plane

|\(|P| = n\) pts in the plane

A halving-edge is a pair of points of \(P\) which spans a halving line (i.e., \((n-2)/2\) points on each side).

Enough to count halving-edges !!!

Bound the number \(F^2(n)\)
of halving-edges in the worst case

A halving edge
The K-set problem in the plane

A construction with “many” halving edges

\[ f(3n) > 3f(n) + \Omega(n) \]
\[ \Rightarrow f(n) = \Omega(n \log n) \]
The K-set problem in the plane

- Equivalent formulation
  - $|L| = n$ lines in the plane
  - Bound the number of median vertices
Example:

Algorithmic problem:

$P$ a set of $n$ points in the plane

Find a line $l$ (1-median line) that minimizes $\sum_{p \in P} d(p, l)$
If direction of \( l \) is fixed, and \( P \) is projected on a line orthogonal to \( l \),

Optimal placement of \( l \) is at the median of the projected points.

\[ \Rightarrow l \text{ is a halving line of } P \]
**Algorithm**: Vary the direction $\theta$ of $l$ from 0 to $\pi$

Keep track of the halving line $l(\theta)$

As long as the split of $P$ by $l(\theta)$ is unchanged,

Optimize $\sum_{p \in P} d(p, l(\theta))$ as a function of $\theta$ (easy task)

Output the best $\theta$ overall
How efficient is the algorithm?

Key question:

How many changes can occur in the splitting of $P$ into two equal halves by $l(\theta)$?

This is the famous $k$-set problem!
The K-set problem (Definition)

$S = n$ pts in $\mathbb{R}^d$

A $(d - 1)$-dimensional simplex $\sigma$ spanned by $d$ points $fo S$ is a **halving-facet** of $S$ if:

the hyperplane spanned by $\sigma$ contains exactly $(n-d)/2$ points of $S$ on each side

$F^d(n)$ = maximum # of halving-facets in a set of $n$ points in $d$-space in general position.

Goal: Obtain sharp bounds on $F^d(n)$

Still, after 40 years of research, very elusive
The K-set problem (History in $\mathbb{R}^2$)

[Lovász '71], [Erdős, Lovász, Simmons, Straus '73] posed +

Initial upper bound $F^2(n) = O(n^{3/2})$

Initial lower bound $F^2(n) = \Omega(n \log n)$

Slight improvement 20 years later

[Pach, Steiger, Szemerédi '92]

$F^2(n) = O(n^{3/2}/\log^*n)$

Record upper bound $F^2(n) = O(n^{4/3})$ [Dey, '98]

Record lower bound $F^2(n) = \Omega(n \cdot 2^{c \sqrt{\log n}})$ [Tóth, '00]

Closing the gap:

Major intriguing open problem in combinatorial geometry
History in $\mathbb{R}^3$

[Bárány, Füredi, Lovász '90]:
\[ F^3(n) = O(n^{3-1/343}) \]

[Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, Wenger '91]:
\[ F^3(n) = O(n^{8/3}\log^{5/3}n) \]

[Dey, Edelsbrunner '94]:
\[ F^3(n) = O(n^{8/3}) \]

[Sharir, Smorodinsky, Tardos '00]:
\[ F^3(n) = O(n^{5/2}) \] (current record)

Lower bound of [Tóth, '00] `lifted' from the plane:
\[ F^3(n) = \Omega \left( n^2 2^{c\sqrt{\log n}} \right) \]
History in $\mathbb{R}^d$ ($d \geq 4$)

[Alon, Bárány, Füredi, Kleitman '92]:

[Živaljević, Vrećica '92]:

$$F^d(n) = O(n^{d-\varepsilon(d)}) \text{ (algebraic topology)}$$

Where $\varepsilon(d)$ is exponentially small in $d$

[Matoušek, Sharir, Smorodinsky, Wagner '06]:

$$F^4(n) = O(n^4 - 2^{2/45}) \text{ (current record) elementary proof!}$$
The K-set problem in the plane reminder:

- $|P| = n$ pts in plane

- A halving-edge is a pair of points which spans a halving line (i.e., $(n-2)/2$ points on each side).

- Bound the number $F^2(n)$ of halving-edges in the worst case.
Upper Bound sketch of proofs

- Construct the halving-edge graph $G = (V,E)$

- Count the number of crossings:
  There are $\Omega(|E|^3/n^2)$ crossings.

- Lovász’ Lemma:
  A line can cross at most $n$ halving-edges!
The K-set problem in the plane

Claim: The halving-edge graph is antipodal
Lovásvz' Lemma

Any line intersects at most \( n \) edges

In particular any edge \( e \) crosses at most \( n \) other edges

So \(#\text{crossings} = O(|E|n)\)

Combined with lower bound of \( \Omega(|E|^3/n^2) \) yields

The \( O(n^{3/2}) \) bound of

[Lovász, 1971], [Erdős et al., 1973]
A simpler version of Dey's proof:
Claim: $G$ has only $O(n^2)$ crossings

Decompose the edges into concave chains
The K-set problem (cont)

- At most one chain ends at a given point
  \[ \Rightarrow \text{\#chains} \leq n \]
- Apply a symmetric decomposition into \( \leq n \) convex chains
The K-set problem (cont)

- Upper bounds on # of crossings
- Charge each crossing to the pair of concave and convex chains:
  - #such pairs is $O(n^2)$
  - Combined with lower bound yields $O(n^{4/3})$