Geometric Incidences and related problems

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Note: good & bad news

Good news:

Talk is simple & elementary

(so be happy)

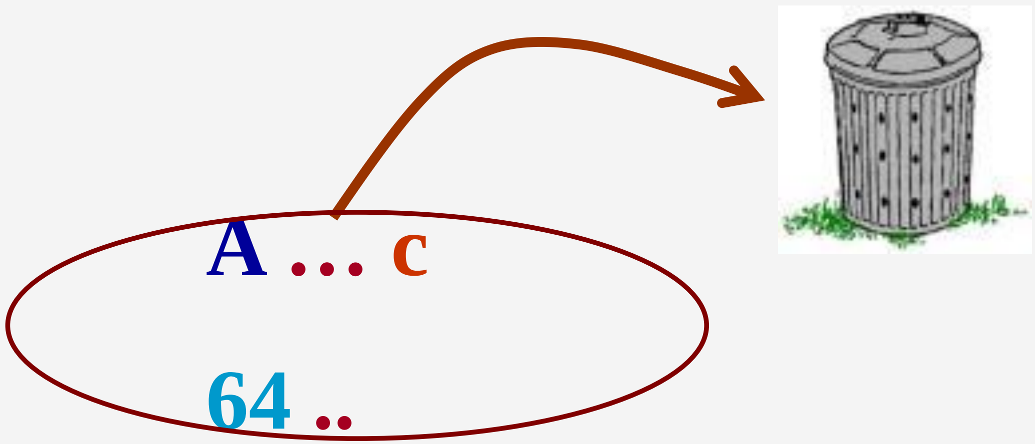
Bad news:

Speaker is jet-lagged

Note:

We don't care about details (such as constants)

asymptotic notations- $O(f), \Omega(f)$, etc



Consider the following problems:

Given $|P| = n$ pts in the plane:

- At most how many pairs in P can be at distance 1 ?
- At least how many distances are determined by P ?
- At most how many ways to ``halve" P by a line ?

For $|P| = m$ pts and $|L| = n$ lines in the plane:

- At most how many pairs of point incident with a line ?

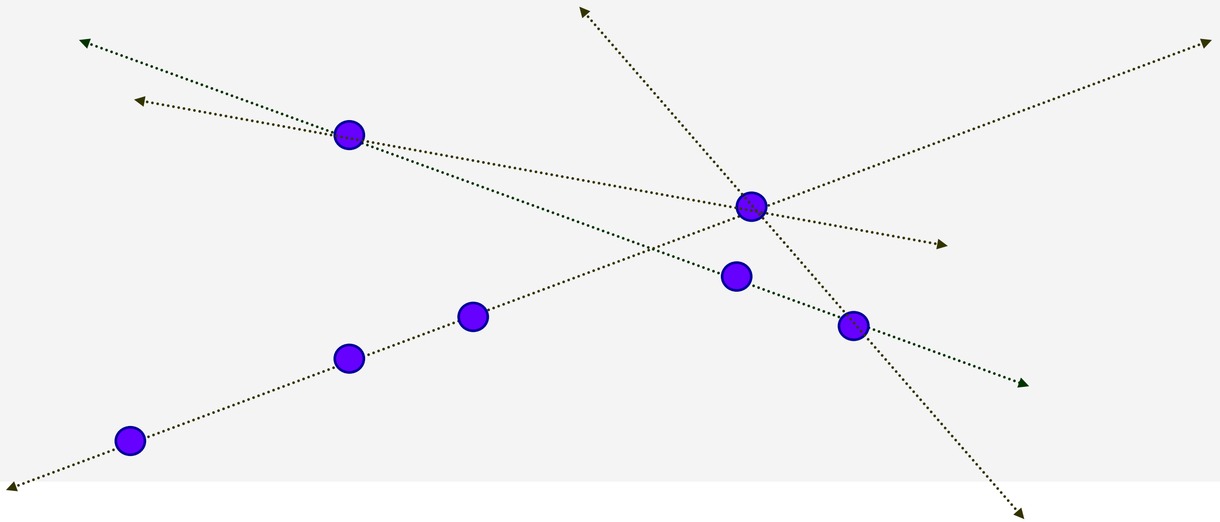
Erdős

Easy (to describe)

Hard (to solve)

Incidences between points and lines.

- P a finite set of pts in R^2.
- L a finite set of lines in R^2
- $I(P,L)$ = # pairs of $p \in P$, $l \in L$ s.t. p is incident to l.



Incidences between points and lines.

- $I(m,n) = \max_{|P|=m,\,|L|=n} I(P,L)$
- Theorem [Szemerédi Trotter '83] :

  $I(m,n) = O(m^{2/3}n^{2/3} + m + n)$

  (for m=n $I(n,n) = O(n^{\frac{4}{3}})$)
- Asymptotically tight !
- Original proof: COMPLICATED!!!
- Székely '98: An ingenious simple proof

  using crossings

bounds on Incidences
Why is it interesting?

- Erdős **repeated** and **distinct** distances problems
- Elekes reduction from a sum-product problem to pts lines incidences
- Gutz-Katz solution....

Example: Sum-Product

- Let $A \subset R$ with $|A| = n$
- Put:

$$A + A = \{a + b |\ a,b \in A\}$$

$$A \cdot A = \{a \cdot b | a, b \in A\}$$

- Problem [Erdős, Szemerédi]:

  is $\max\{|A + A|, |A \cdot A|\}$ always big?

- Note: if *A* is an A.P. then $|A + A| \leq 2n$
- If *A* is geometric then $|A \cdot A| \leq 2n$

Example: Sum-Product

- **The Sum-Product Thm** [Erdős, Szemerédi '83] :

  $\exists \epsilon > 0 \ s.t.$ if A is a finite set of reals:

  $$\max\{|A+A|, |A \cdot A|\} \geq |A|^{1+\epsilon}$$

- Best known bound [Solymosi '08]:

  $$\max\{|A+A|, |A \cdot A|\} \geq \Omega(n^{\frac{4}{3}} \log^{-\frac{1}{3}} n)$$

- Conjecture: $\max\{|A+A|, |A \cdot A|\} \geq \Omega(n^{2-o(1)})$

  This conjecture is widely open!!!

Example: Sum-Product

- **Elekes: Sum-Product from point-line incidences:**

Given $A = \{a_1, a_2, \dots, a_n\}$

- Put: $n = |A|$ $s = |A + A|$, $t = |A \cdot A|$
- We wish to lower bound $s \cdot t$

- Consider the planar set $\mathrm{P} = (A + A) \times (A \cdot A)$ with $s \cdot t$ points.
- Define a set L of n^2 lines of the form: $y = a_i(x - a_j)$

  Consider the number of incidences: $\mathrm{I(P, L)}$

Example: Sum-Product

- $n = |A|$ $s = |A + A|$, $t = |A \cdot \mathrm{A}|$
- $\mathrm{P} = (A + A) \times (A \cdot A)$
- $|P| = s \cdot \mathrm{t}$ $|L| = n^2$
- Each line $y = a_i(x - a_j)$ contains n points of P

  Note: for $x \in \{a_j + a_l\}_{l=1,\dots,n} \subset A + A$

  We have: $y \in \{a_i \cdot a_l\}_{l=1,\dots,n} \subset A \cdot A$

So $\mathrm{n}^3 \leq \mathrm{I(P, L)}$

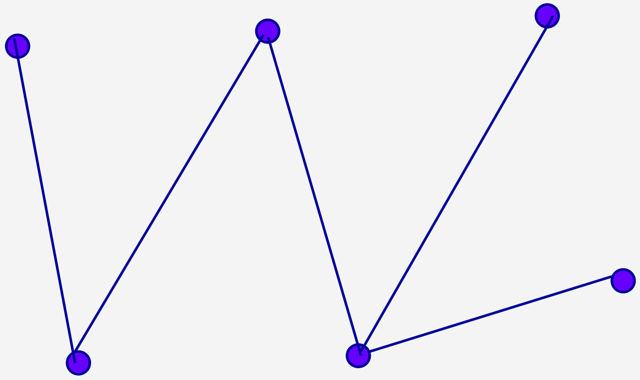
But from Szemerédi-Trotter:

$$\mathrm{I(P, L)} = O((s \cdot t)^{\frac{2}{3}} (n^2)^{2/3}) \text{ or } n^{\frac{5}{3}} = O((\mathrm{s} \cdot t)^{\frac{2}{3}})$$

$$n^{5/2} = O(s \cdot t)$$

The Crossing Lemma:

- $G = (V,E)$ simple graph is planar if can be "drawn " with no crossings

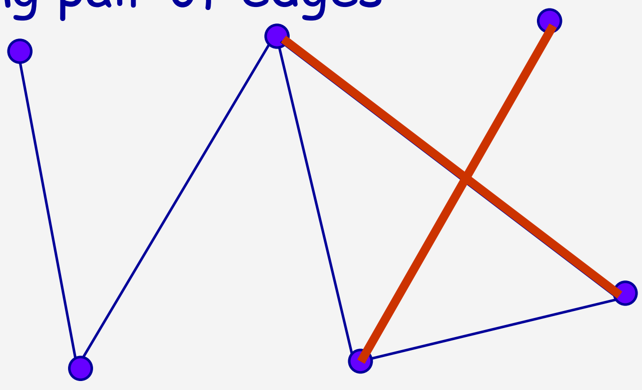


- Euler's Formula imply:
- $|E| \leq 3|V|-6$

Crossing in arbitrary graphs

- $G = (V,E)$ arbitrary simple graph

$|E| > 3|V|$ implies at least one crossing pair of edges



- X = # edge crossings

- Thm: [Ajtai et al. '82, Leighton '83]

$$(|E| \geq 4|V|) \Rightarrow X \geq \Omega(|E|^3/|V|^2)$$

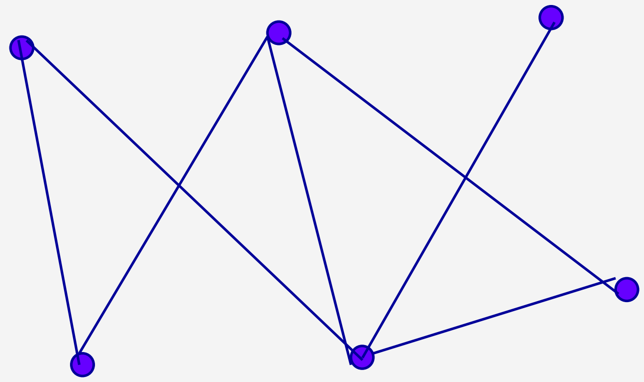
Crossing in graphs (cont)

$X = \Omega(|E|^3/|V|^2)$

Pf (probabilistic):

Bootstrapping Lemma:

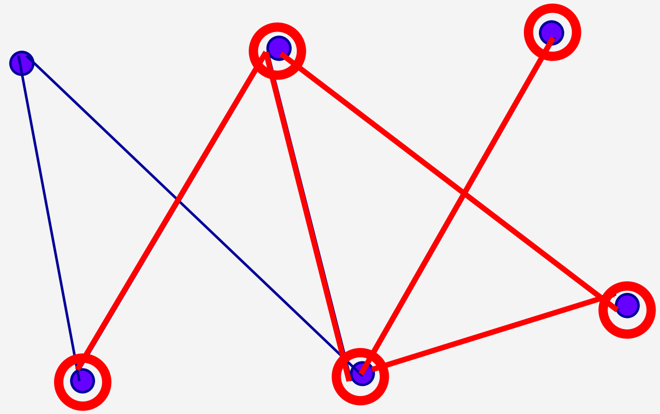
$X \geq |E| - 3|V|$



As long as $|E| > 3|V|$ $\exists$ crossings

Crossing in graphs (cont)

- Sample each vertex of G with prob. p

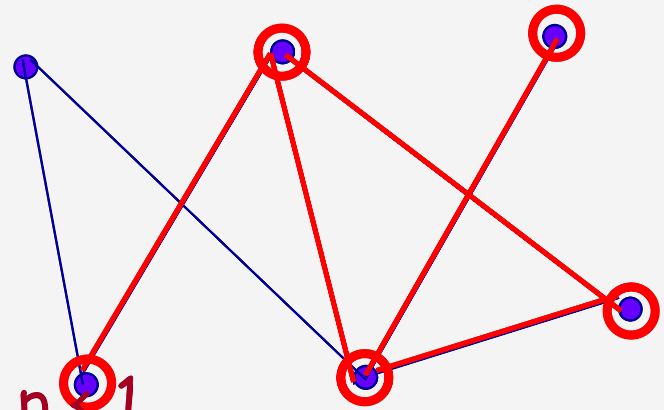


- $|V'|$, $|E'|$, $|X'|$ random variables
- $|X'| \geq |E'| - 3|V'|$
- $\mathrm{Exp}[|X'|] \geq \mathrm{Exp}[|E'|] - 3\mathrm{Exp}[|V'|]$

Crossing in graphs (cont)

- $Exp[|X'|] \geq Exp[|E'|\ ] - 3Exp[|V'|]$

- $Exp[|V'|] = p|V|$
- $Exp[|E'|] = p^2|E|$
- $Exp[|X'|] = p^4|X|$
- $p^4|X| \geq p^2|E| - 3\ p|V|$
- $X \geq |E|/p^2 - 3|V|/p3$ $\quad \forall\ 0 < p < 1$
- Choose $p = 4|V|/|E|$
- We get $X \geq \Omega(|E|^3/|V|^2)$



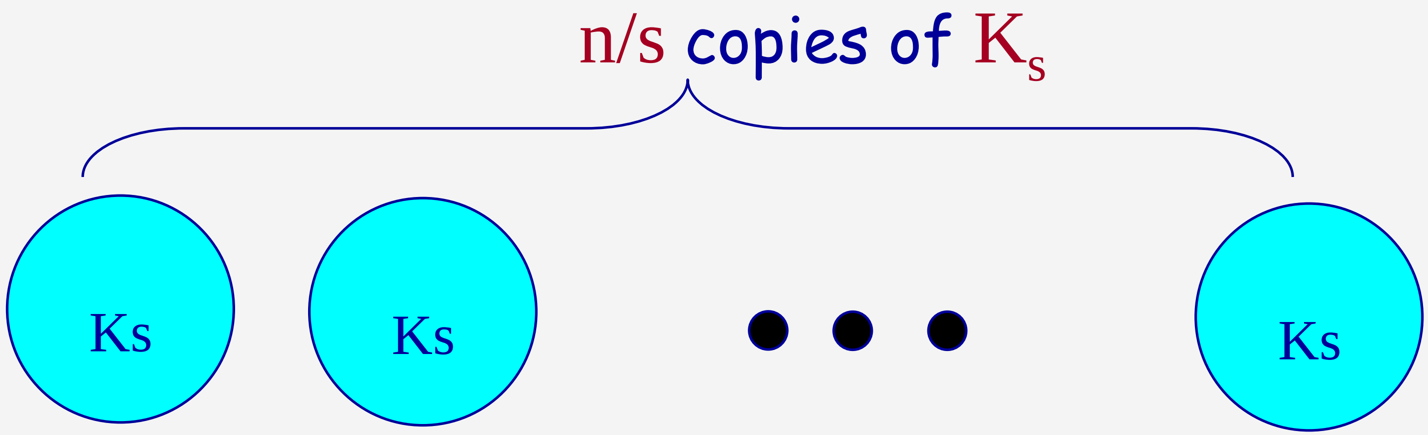
Crossing in graphs (cont)

- Remarks: Asymptotically tight:
- For e and n (e > 4n) ∃ G with n vertices e edges.

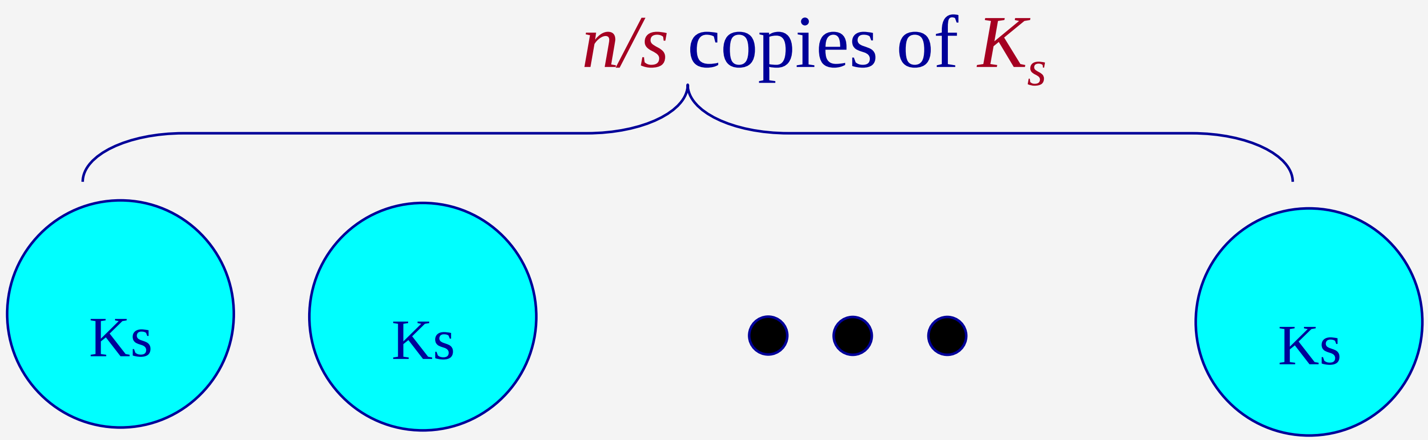
$cr(G) \leq O(e^3/n^2)$

$S=e/n$

- take n/s copies of K_s



Crossing in graphs (cont)



#edges = $s^2 n/s = ns = e$

#crossings $\leq s^4 n/s = s^3 n = e^3/n^2$.

Lower bounds on crossings
Why is it interesting?

- Simple proofs for bounds on some problems.

Example 1: Incidences between points and lines.

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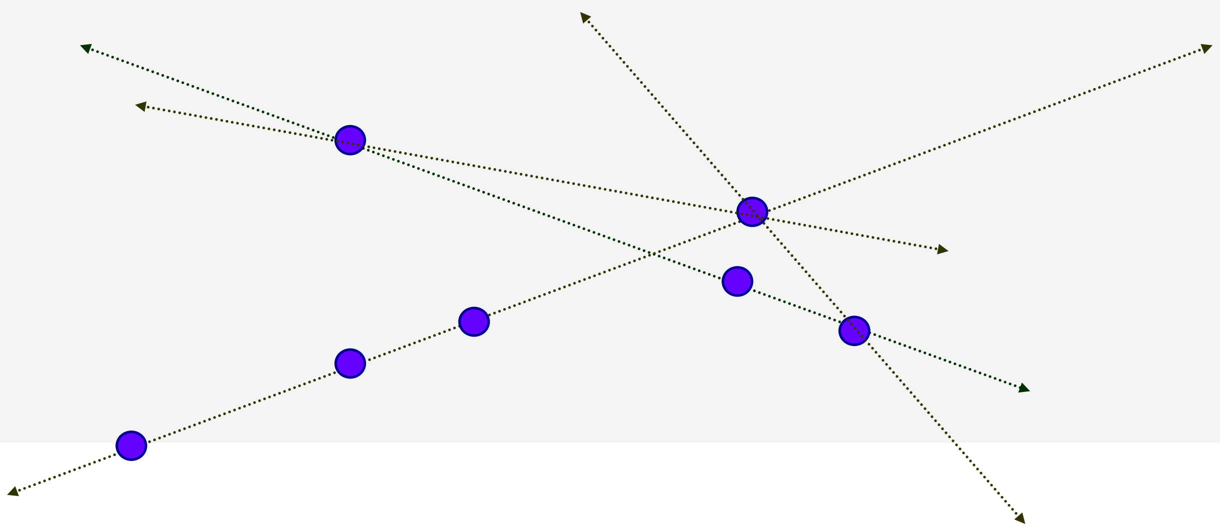
- $I(P,L)$ = # pairs of $p \in P$, $l \in L$ s.t. p is incident to l.



Example 1: Incidences between points and lines. (cont)

- $I(m,n) = \max_{|P|=m, |L|=n|} I(P,L)$
- Theorem:

  $I(m,n) = O(m^{2/3}n^{2/3} + m + n)$ (for m=n $O(n^{\frac{4}{3}})$)
- Proof: [Szemerédi Trotter '83] COMPLICATED!!!

- [Székely '98] SIMPLE!!! (using crossings)
- Remark: $O(n^{3/2})$ easy using extremal graph theory

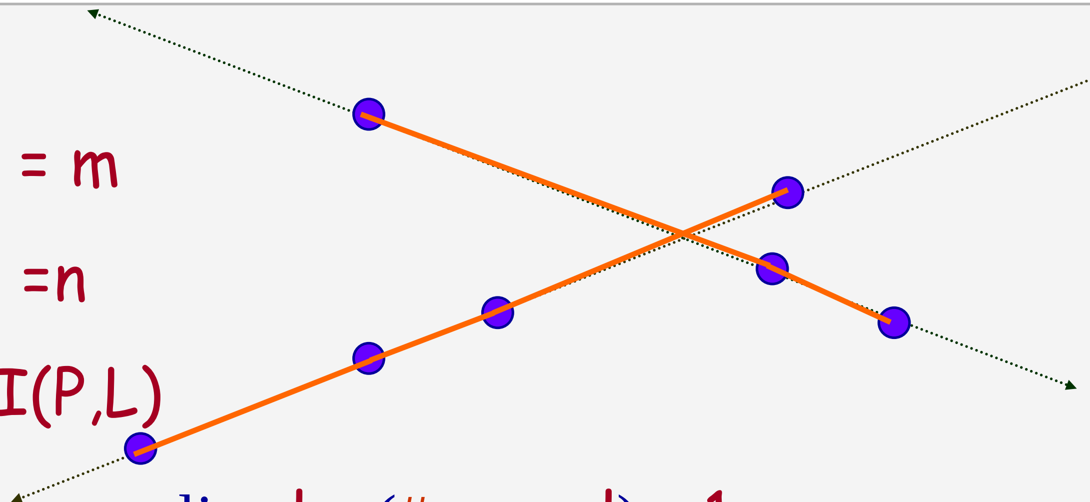
  The incidence graph does not contain a $K_{2,2}$

Example 1: Incidences between points and lines. (cont)

$|P| = m$

$|L| = n$

$I = I(P, L)$



$G = (P, E)$

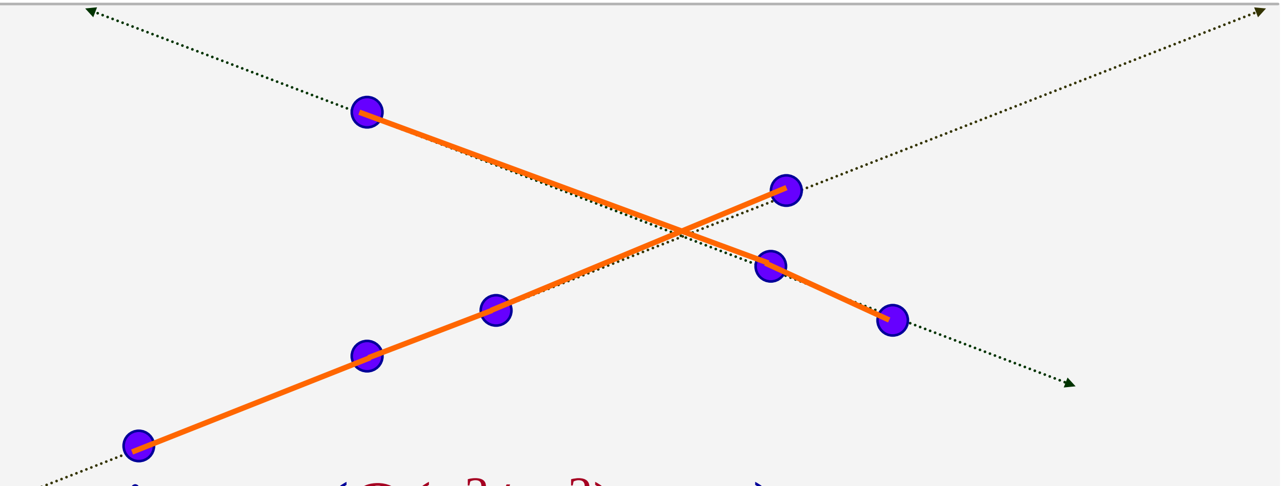
#edges on a line l = (#pts on l) – 1

$e = I - n$

We want to bound e

Example 1: Incidences between points and lines. (cont)

If $e \geq 4m$



X = #edge crossings $\quad (\Omega(e^3/m^2) \leq x)$

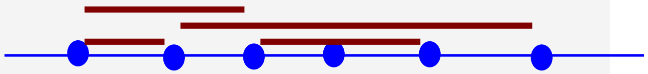
$X \leq O(n^2)$

$e^3/m^2 < n^2$

$e = I\text{-}n = O(m^{2/3}n^{2/3}) \Rightarrow I = O(m^{2/3}n^{2/3} + m + n)$

Using the proof technique
Some warm-up exercises

- Example: n pts and I intervals



- Selection Lemma [Aronov et al. '91]:

$\exists P \in \Omega(I^2/n^2)$ intervals

- We give a **simple** Proof: X = # (pt,interval)
  point inside interval.

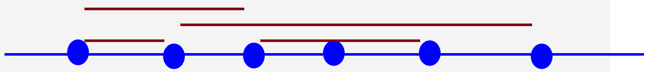
The Bootstrapping Lemma:

- $X > I - n$
- $p^3x > p^2I - pn$
- $X > I/p-n/p^2 \qquad (\forall\ 0< p < 1)$

Selection Lemmas: points and intervals

- $X > I/p - n/p^2$ $(\forall\ 0 < p < 1)$
- Choose $p = 2n/I$
- $X > I^2/n$
- $\exists$ $P \in X/n$ configurations.

Hence $\exists$ $p \in \Omega(I^2/n^2)$ intervals.



Selection Lemmas: circles and points

- $|P|=n$ pts in R^2
- $|C| = c > 4n$ circles spanned by pts in P
- Thm[Chazelle et al. '94]:

If circles spanned by **pairs** and are **diametrical**

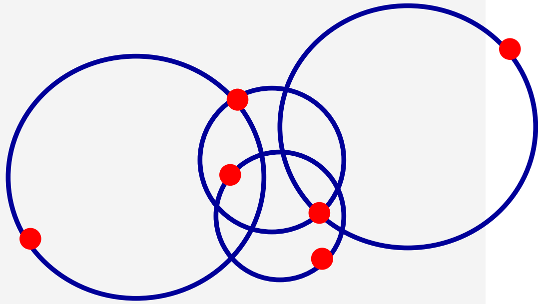
then:

- $\exists\ p \in \Omega\ (c^2/n^2)$ circles.

For non diametrical circles

- $\exists\ p \in \Omega\ (c^2/(n^2 \log^2 n))$

Proof: complicated!

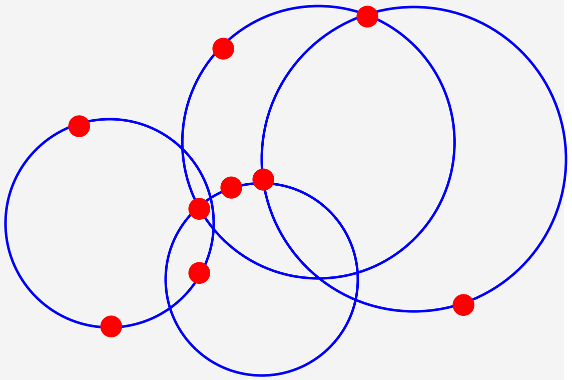


Selection Lemmas: circles and points

- $|P|=n$ pts in R^2
- $|C| = c > 4n$ **arbitrary** circles spanned by pairs of pts in P
- ***Thm:***

1. $\exists$ p $\in \Omega(c^2/n^2)$
2. If circles spanned by **triples**,

- $\exists$ p $\in \Omega(c^{3/2}/n^{3/2})$ circles
- Bounds asymptotically tight!



Selection Lemmas: circles and points

X = # (pt,circle) pt inside circle

$|P|=n$ pts in R^2

$|C| = c > 4n$ **triples** circles

#empty circles (Delaunay circles) is O(n)

- Bootstrapping Lemma:

$X > c - 3n$

- $X > \Omega(c^{3/2}/n^{1/2})$ (using the same sampling technique)
- $\exists p \in \Omega(x/n)$ configurations
- $\exists p \in \Omega(c^{3/2}/n^{3/2})$ circles (asymptotically tight)