Geometric Incidences and related problems

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Note: good & bad news

Good news:

Last Talk

Bad news:

Pick your favorite one....

The K-set problem (Definition)

S = n pts in R^d

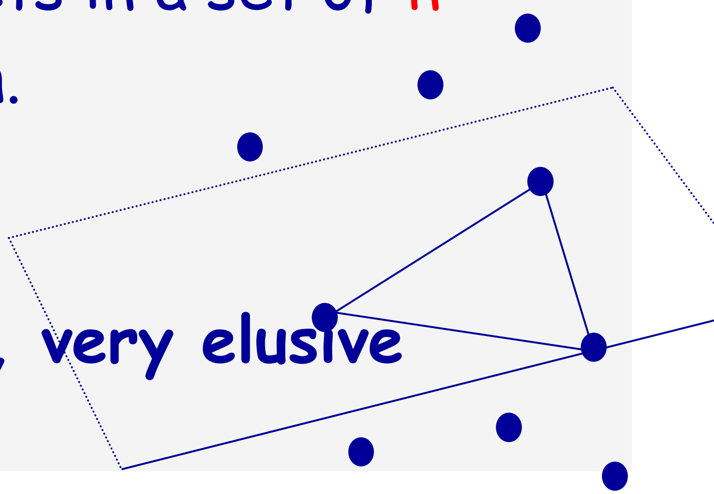
A $(d - 1)$-dimensional simplex σ spanned by d points fo S is a halving-facet of S if:

the hyperplane spanned by σ contains exactly $(n-d)/2$ points of S on each side

$F^d(n)$ = maximum # of halving-facets in a set of n points in d-space in general position.

Goal: Obtain sharp bounds on $F^d(n)$

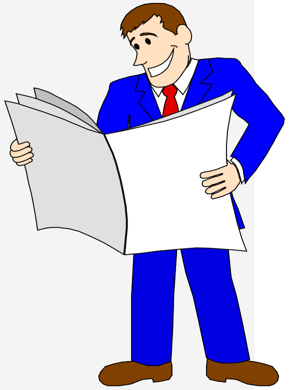
Still, after 40 years of research, very elusive



The K-set problem (History in R^2)

Record upper bound $F^2(n) = O(n^{4/3})$ [Dey, '98]

Record lower bound $F^2(n) = \Omega(n \cdot 2^{c\sqrt{\log n}})$ [Tóth, '00]



History in R^3

[Bárány, Füredi, Lovász '90]:

$$F^3(n) = O(n^{3-1/343})$$

[Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, Wenger '91

$$F^3(n) = O(n^{8/3}\log^{5/3} n)$$

[Dey, Edelsbrunner '94]:

$$F^3(n) = O(n^{8/3})$$

[Sharir, Smorodinsky, Tardos '00]:

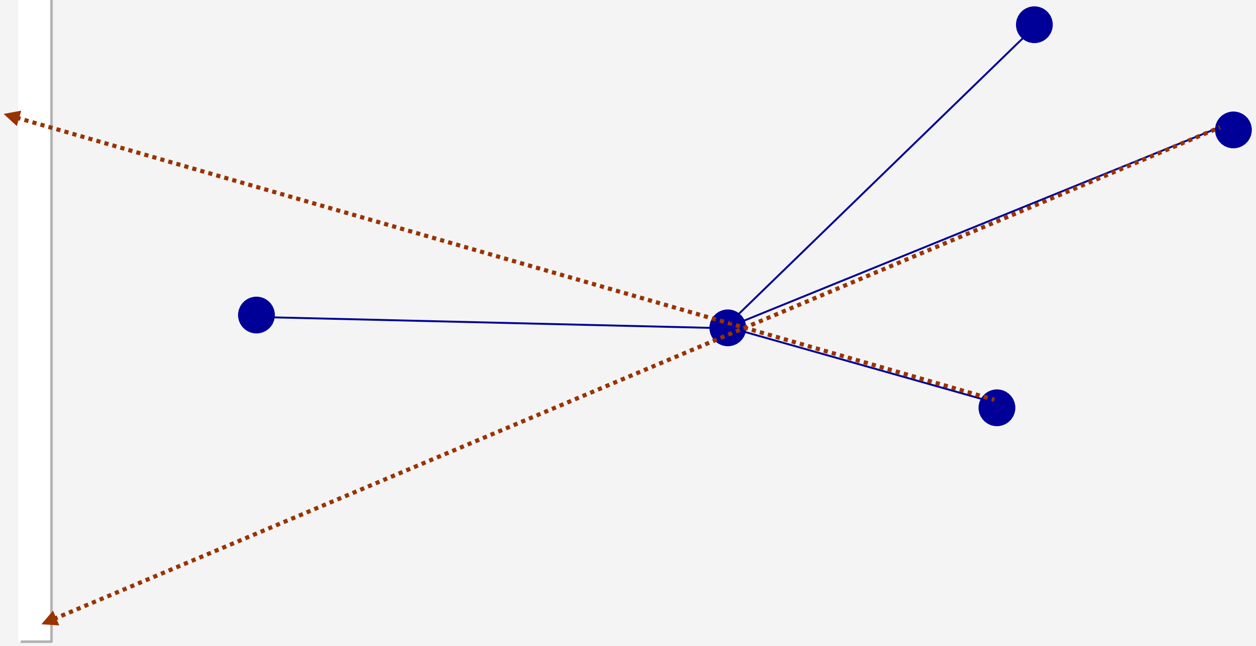
$F^3(n) = O(n^{5/2})$ **(current record)**

Lower bound of [Tóth, '00] `lifted' from the plane:

$$F^3(n) = \Omega\left(n^2 2^{c\sqrt{\log n}}\right)$$

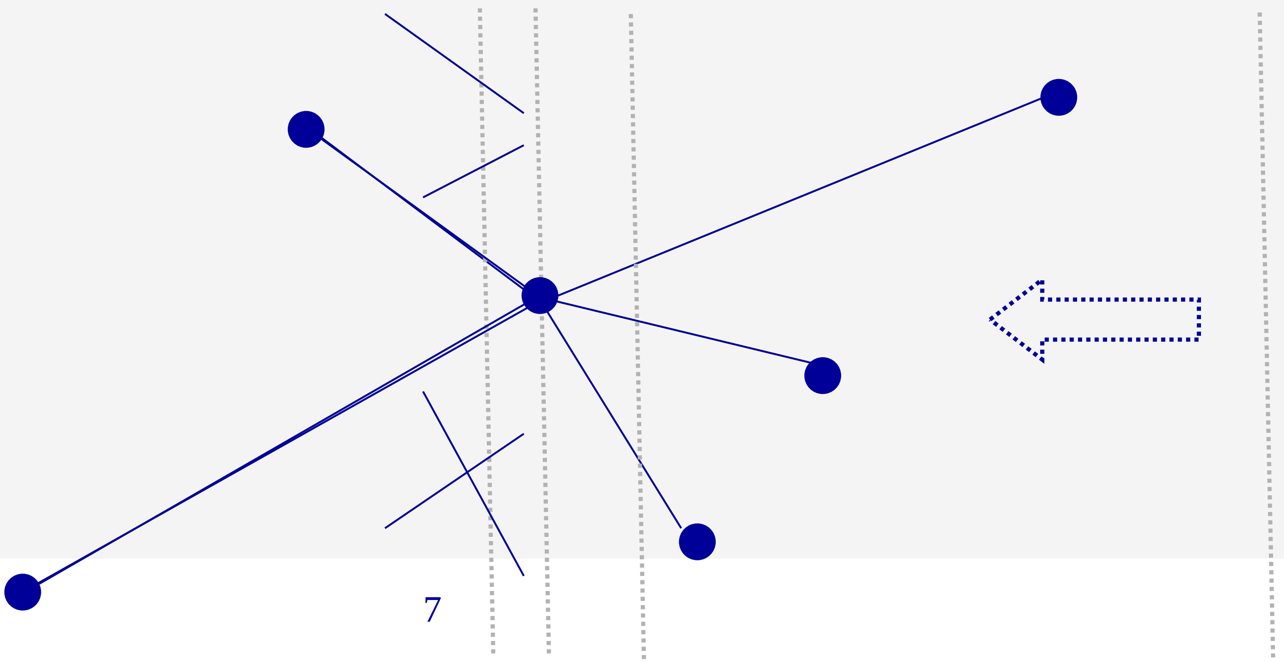
The K-set problem in the plane

Claim: The halving-edge graph is antipodal



Lovász' Lemma

Any line intersects at most $O(n^{d-1})$ halving simplices.

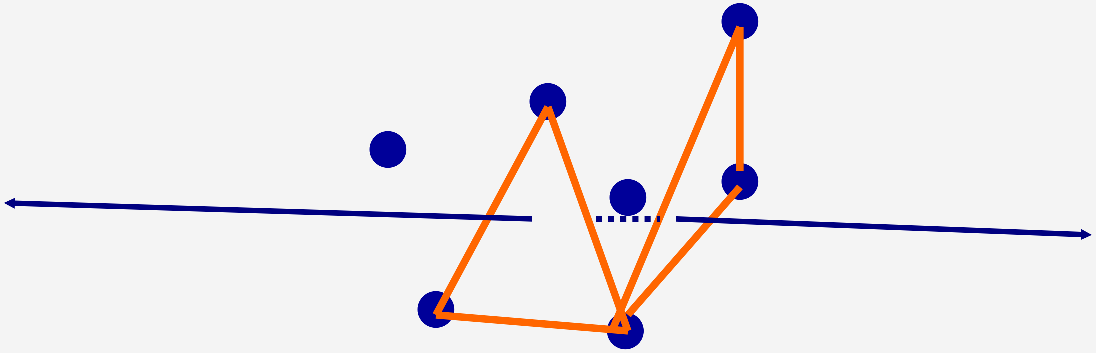


Points and triangles in 3D

- $|P| = n$ pts in R^3
- $|T| = t = \Omega(n^2)$ triangles spanned by P

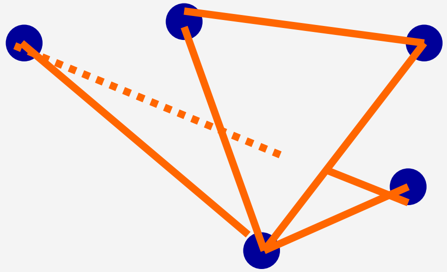
- THM: [Dey, Edelsbrunner '94]

Always ∃ line that stabs $\Omega(t^3/n^6)$ triangles



Points and triangles in 3D (cont)

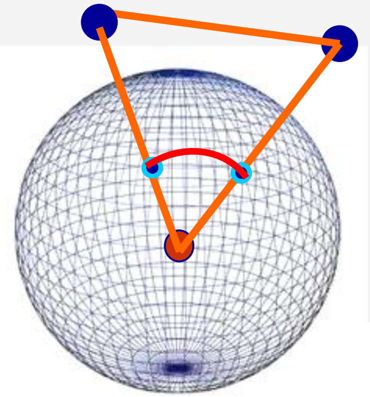
- ∃ line that stabs $\Omega(t^3/n^6)$ triangles
- Simple Proof:



- X = #crossing pairs with a common vertex

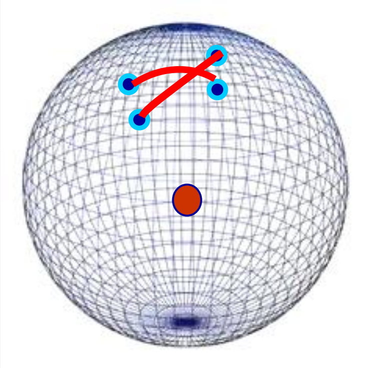
Points and triangles in 3D (cont)

- Consider T_p = triangles incident to p
- Intersect T_p with small sphere centered at p
- G_p = the induced graph
- points of G_p induced by segments
- Edges of G_p induced by triangles incident to p

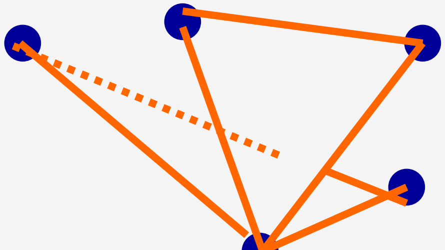


Points and triangles in 3D (cont)

- $\exists\ \Omega(|T_p|^3/n^2)$ crossing in G_p.



- A crossing corresponds to:

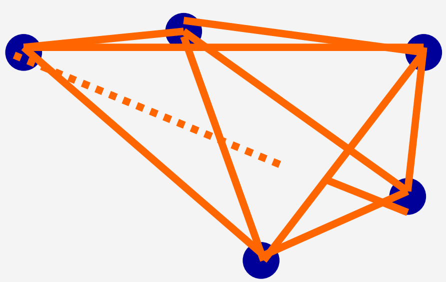


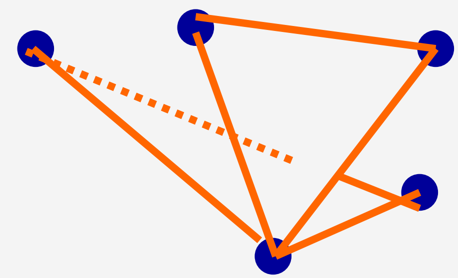
Points and triangles in 3D (cont)

- (By Hölder's inequality) we have:

$$\Sigma_{p\epsilon P}\, \Omega(|T_p|^3/n^2) \geq \Omega\,((\Sigma_{p\epsilon P}\, |T_p|)^3/n^4)$$

configurations of:





- $((\Sigma_{p\epsilon P}\, |T_p|)^3/n^4 = \Omega(t^3/n^4)$
- => $\exists$ edge in $\Omega(t^3/n^6)$ configurations

Points and triangles (cont)

- Remark:

  Best known upper bound construction $O(t^2/n^3)$

- Conjecture:

  Always ∃ line that intersects

  $\Omega(t^2/n^3)$ triangles ($\gg t^3/n^6$)

Applications (k-sets in 3D)

- $|P| = n$ pts in R^3
- $|T| = t$, the set of halving-triangles
  - Lovász' Lemma in 3D:
  - Any line stabs at most $O(n^2)$ halving-triangles
  - We have:

$$\Omega(t^3/n^6) \leq O(n^2)$$

$$t = O(n^{8/3})$$

Improved bounds on k-sets in 3D

- Thm: [Sharir, Smorodinsky, Tardos '00]

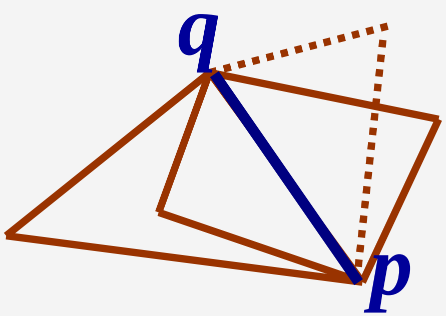
$$t = O(n^{5/2})$$

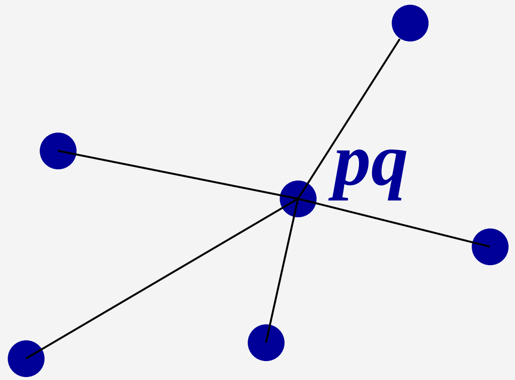
- Proof:

The halving-triangles are antipodal

Improved bounds on k-sets in 3D

- The halving-triangles are antipodal





- This property will imply:

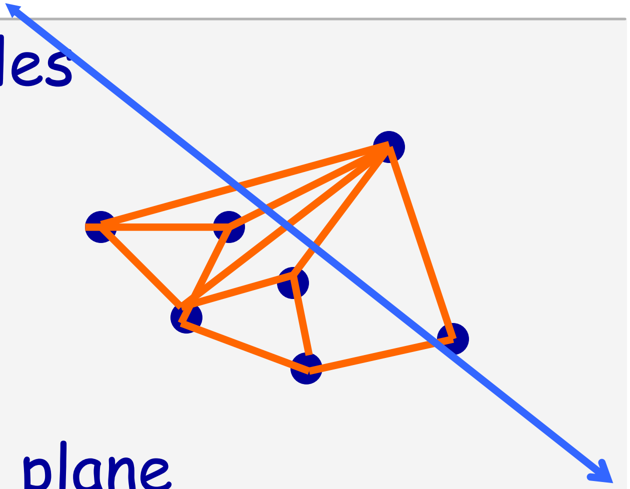
  ∃ line that stabs $\Omega(t^2/n^3)$ triangles

- Combined with $O(n^2)$ upper bound (Lovász)

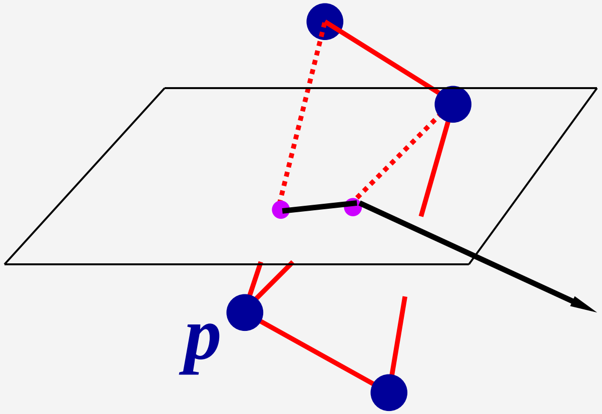
  We will get the $O(n^{5/2})$ upper bound.

Improved bounds on k-sets in 3D (cont)

- $\exists$ line that stabs $\Omega(t^2/n^3)$ triangles

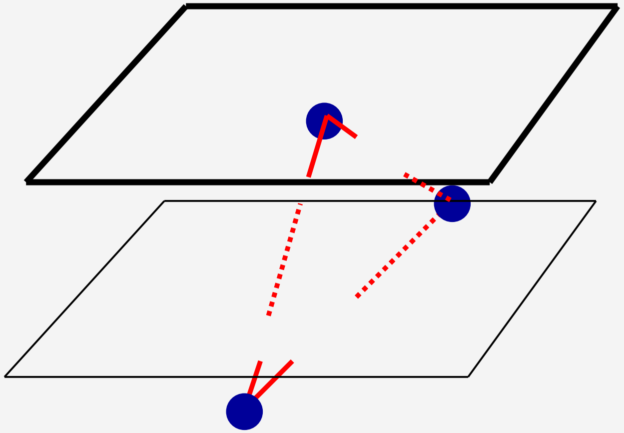


- Proof:
- For a point p, G_p is the stereographic projection of T_p on a plane above p

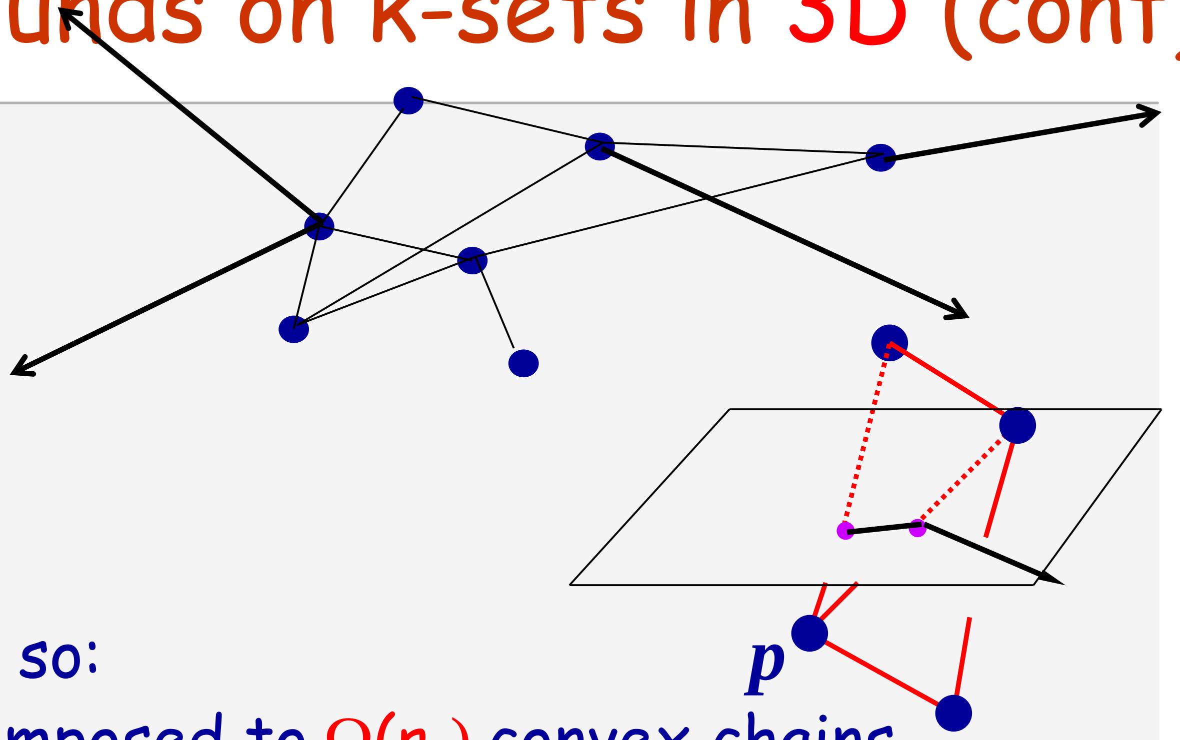


Improved bounds on k-sets in 3D (cont)

- $\sum_{p \in P} e_p = t$
- $\sum_{p \in P} r_p = t$



Improved bounds on k-sets in 3D (cont)



G_p consists of

- n vertices
- e_p edges
- r_p rays

- G_p is antipodal so:

  G_p is decomposed to $\Omega(r_p)$ convex chains

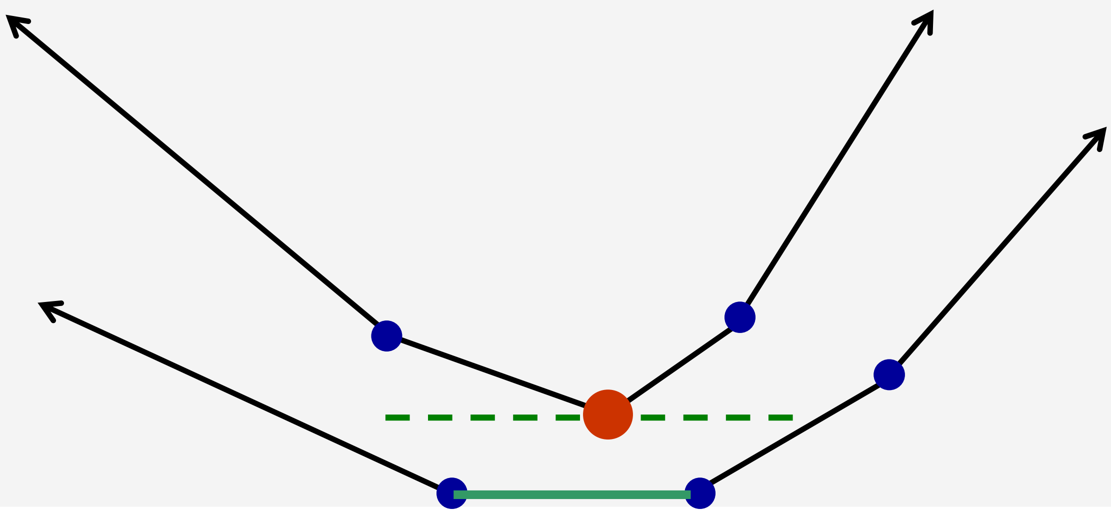
  Contains $\Omega(r_p^2 - nr_p)$ crossings

Improved bounds on k-sets in 3D (cont)

- G_p Contains $\Omega(r_p^2 - nr_p)$ crossings

Proof: bound the # pairs of convex chains which do not cross

- At most $O(nr_p)$ pairs out of the r_p^2



Improved bounds on k-sets in 3D (cont)

- G_p Contains $\Omega(r_p^2)$ crossings. Hence:
- X= #crossings $= \sum_{p\in P} r_p^2 = \Omega(\sum_{p\in P} r_p)^2/n$

  => $X = \Omega(t^2/n)$ and =>

  $\exists$ line that stabs $\Omega(t^2/n^3)$ triangles
- Combined with Lovász' Lemma we have:

  $t = O(n^{2.5})$

k-sets in 4D (Sketch)

- P := n pts in R^4
- S := the set of halving-simplices of P
- Thm: [Matoušek, Sharir, Smorodinsky, Wagner '05]

$$|S| = O(n^{4-2/45})$$

Proof uses two main lemmas:

Lemma 1:

There is a 2-plane that intersects $\Omega(|S|^3/n^8)$ simplices of S

Lemma 2:

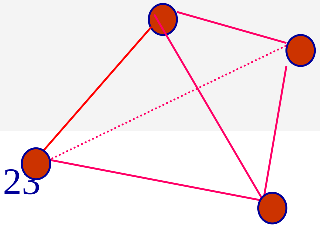
Any such 2-plane intersects $O(n^{4-2/15})$ simplices (main new ingredient)

k-sets in 4D (Sketch)

Lemma 1:
There is a 2-plane that intersects $\Omega(|S|^3/n^8)$ simplices of S

Proof (sketch): Project P and S orthogonally onto R^3

We have $|P'| = n$ pts in R^3 and $|S|$ tetrahedra spanned by P'



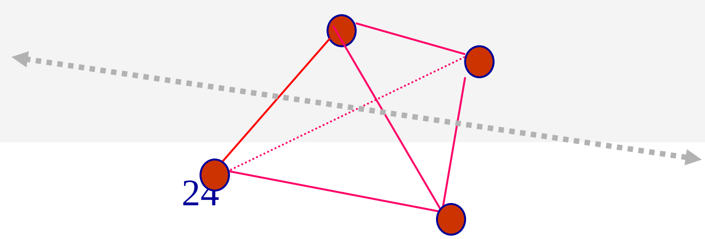
k-sets in 4D (Sketch)

Lemma 1:
There is a 2-plane that intersects $\Omega(|S|^3/n^8)$ simplices of S

We show that there is a line that intersects this many tetrahedra

:

`lifting' that line back to R^4 we get the desired plane



k-sets in 4D (Sketch)

Maybe Stop here???

Totally positive matrices

- A square matrix is called totally positive (TP) if:
  the determinant of any square sub matrix (minor) is positive

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 9 & 15 \\ 5 & 12 & 30 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 3 \\ 4 & 15 \end{pmatrix} = 3$$

$$\det\begin{pmatrix} 4 & 9 \\ 5 & 12 \end{pmatrix} = 3$$

$$\det(A) = 9$$

Example: Vandermonde matrix

$$V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^{\,2} & \cdots & \alpha_1^{\,m-1} \\ 1 & \alpha_2 & \alpha_2^{\,2} & \cdots & \alpha_2^{\,m-1} \\ 1 & \alpha_3 & \alpha_3^{\,2} & \cdots & \alpha_3^{\,m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^{\,2} & \cdots & \alpha_m^{\,m-1} \end{pmatrix}$$

$$\alpha_1 < \alpha_2 < \ldots\ldots < \alpha_m$$

$$\det(V) = \prod_{1\le i<j\le n} (\alpha_j - \alpha_i)$$

$$\left|\begin{pmatrix} \alpha_1 & \alpha_1^{\,2} \\ \alpha_2 & \alpha_2^{\,2} \end{pmatrix}\right| = \alpha_1\alpha_2 \left|\begin{pmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{pmatrix}\right|$$

Multiplicity of an entry

The multiplicity of an entry a in a matrix M is the # of occurrences of a in M

$$M = \begin{pmatrix} 30 & 2 & 3 \\ 4 & 9 & 15 \\ 5 & 12 & 30 \end{pmatrix}$$

The multiplicity of 30 is 2

Question (Farber et al.)

- What is the maximum # of equal entries in an N x N TP matrix?

$$N = 2, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{3 equal entries}$$

$$N = 3, \quad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \det(A) = \frac{3}{\sqrt{2}} - 2, \quad \text{6 equal entries}$$

Answer (Farber et al.)

- Theorem [Farber, Faulk, Johnson, Marzion '12]:
The max # of equal entries in N x N TP matrix is $\Theta(N^{4/3})$

PF of upper bound relies on a result of
J. Pach and G. Tardos '06,

PF of lower bound:
Construction using many pts-lines incidences…

Question (Farber et al.)

- Consider 2xN matrices with positive entries and positive 2x2 minors (TP_2)

- What is the max # of equal 2x2 minors in a 2xN TP_2 matrix?

- More generally: What is max # of equal 2x2 minors in NxN TP_2 matrix?

Equal 2x2 minors

Thm 1 [Farber-Ray-Smorodinsky '13]:

The max # of equal 2x2 minors in a 2xN TP_2 matrix is $\Theta(N^{4/3})$
(Reduction to pts-lines incidences)

- Thm 2 [Farber-Ray-Smorodinsky '13]:

The max # of equal dxd minors in a dxN TP_2 matrix is $\Theta(N^{d-(d/d+1)})$
(Reduction to pts-hyperplane incidences and a result of Apfelbaum-Sharir 2007)

A construction of TP_2

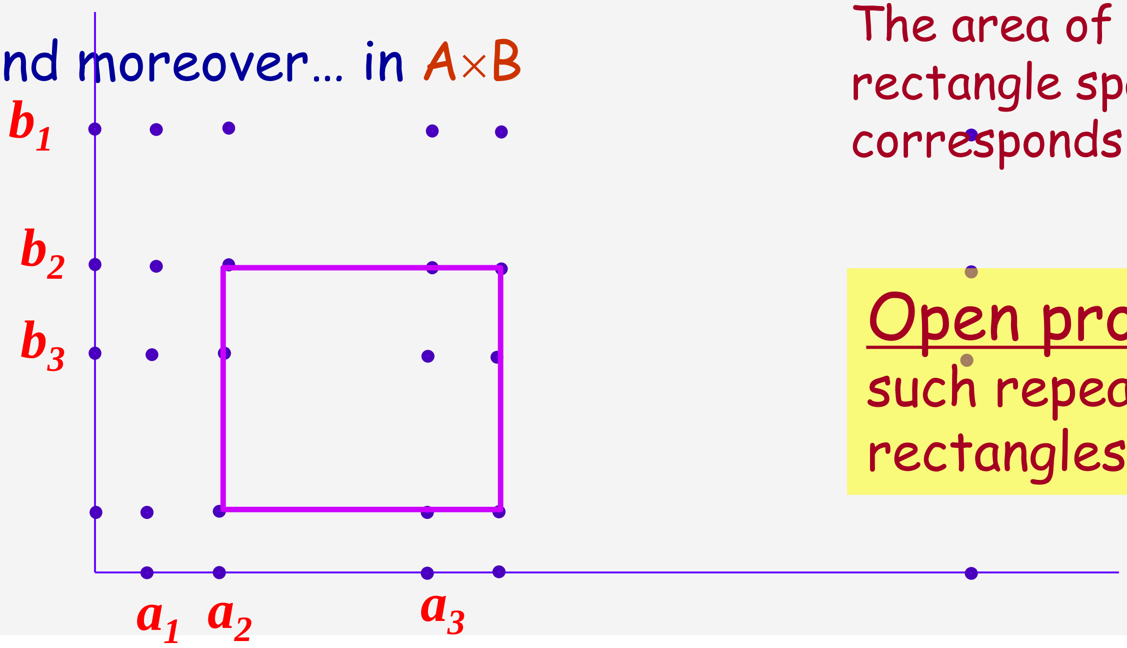
- Fix sets A,B of n reals;

$0<a_1<\ldots<a_n$ and $b_1>\ldots>b_n>0$

Define $A=(A_{i,j})$ where $A_{i,j}= b_i+a_j$

Claim: A is a TP_2 matrix (easy to check)

And moreover… in $A\times B$

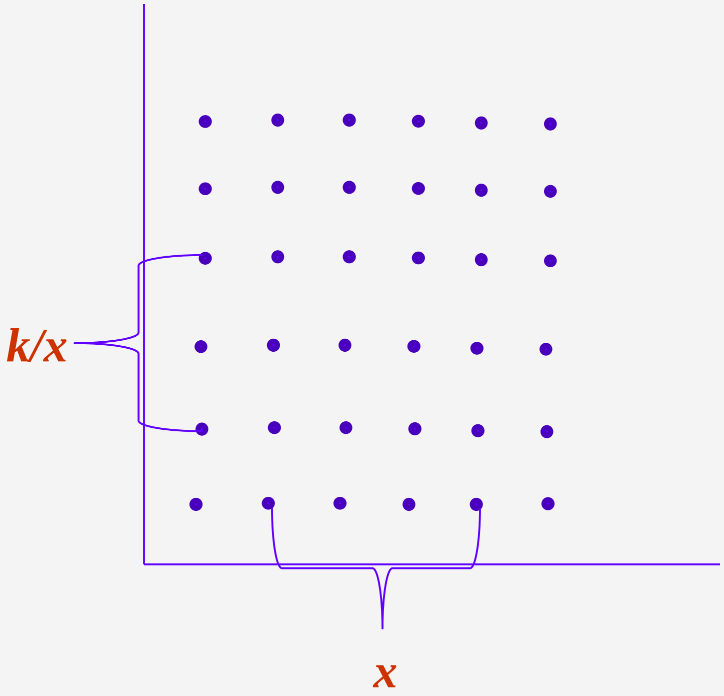


The area of an axis-parallel rectangle spanned by $A\times B$ corresponds to a 2×2 minor

<u>Open problem</u>: How many such repeated (unit)-area rectangles are possible?

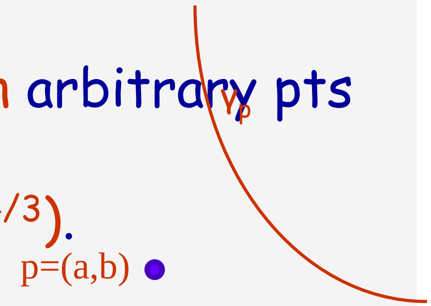
Lower bound of $\Omega(n^{2+1/\log\log n})$:

- Consider $A=B=\{1,2,\ldots,2n\}$: fix $1 \le k \le n$ s.t. $div(k) = \Omega(n^{1/\log\log n})$:
- For each divisor x of k we have $\Omega(n^2)$ rectangles with area k



Upper bound on repeated areas

- Note: $O(n^3)$ = trivial upper bound. Any point can be the lower left corner of at most n unit area rectangles!
- Thm 3 [Farber-Ray-Smorodinsky '13]: P = n arbitrary pts in the plane.
max #unit area axis-parallel rectangles in P is $O(n^{4/3})$.



Pf: For $p=(a,b) \in P$ define the hyperbola:

$$\gamma_p = \{(x,y) : (x-a)(y-b)=1\}$$

Observe: $(x,y) \in \gamma_p$ iff (a,b) and (x,y) opposite corners of unit area axis-parallel rectangle.

#rectangles ≤ # incidences between P and $\Gamma=\{\gamma_p | p \in P\}$
Luckily: Γ is a family of pseudo lines.

Upper bound on repeated areas

- **Corollary:** For $|P| = n$ points we have:

  $O(n^{4/3})$ repeated (unit) area rectangles.