Geometric Incidences and related problems

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Note: *good* & *bad* news

Good news: Last Talk

Bad news: Pick your favorite one....
The K-set problem (Definition)

$S = n$ pts in $\mathbb{R}^d$

A $(d - 1)$-dimensional simplex $\sigma$ spanned by $d$ points for $S$ is a halving-facet of $S$ if:

- the hyperplane spanned by $\sigma$ contains exactly $(n-d)/2$ points of $S$ on each side

$F^d(n) =$ maximum $\#$ of halving-facets in a set of $n$ points in $d$-space in general position.

Goal: Obtain sharp bounds on $F^d(n)$

Still, after 40 years of research, very elusive
The K-set problem (History in $\mathbb{R}^2$)

Record upper bound $F^2(n) = \mathcal{O}(n^{4/3})$ [Dey, '98]

Record lower bound $F^2(n) = \Omega(n \cdot 2^{c \sqrt{\log n}})$ [Tóth, '00]
History in $\mathbb{R}^3$

[Bárány, Füredi, Lovász '90]:
$$F^3(n) = O(n^{3-1/343})$$

[Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, Wenger '91]:
$$F^3(n) = O(n^{8/3}\log^{5/3} n)$$

[Dey, Edelsbrunner '94]:
$$F^3(n) = O(n^{8/3})$$

[Sharir, Smorodinsky, Tardos '00]:
$$F^3(n) = O(n^{5/2}) \text{ (current record)}$$

Lower bound of [Tóth, '00] `lifted' from the plane:
$$F^3(n) = \Omega \left( n^2 2^c \sqrt[3]{\log n} \right)$$
The K-set problem in the plane

Claim: The halving-edge graph is antipodal
Lovász' Lemma

Any line intersects at most $O(n^{d-1})$ halving simplices.
Points and triangles in 3D

- $|P| = n$ pts in $\mathbb{R}^3$
- $|T| = t = \Omega(n^2)$ triangles spanned by $P$

**THM:** [Dey, Edelsbrunner ’94]
Always $\exists$ line that stabs $\Omega(t^3/n^6)$ triangles
Points and triangles in 3D (cont)

- There exists a line that stabs $\Omega(t^3/n^6)$ triangles

Simple Proof:

- $X = \#$crossing pairs with a common vertex
Points and triangles in 3D (cont)

- Consider $T_p =$ triangles incident to $p$
- Intersect $T_p$ with small sphere centered at $p$
- $G_p =$ the induced graph
- Points of $G_p$ induced by segments
- Edges of $G_p$ induced by triangles incident to $p$
Points and triangles in 3D (cont)

- $\exists \Omega(|T_p|^3/n^2)$ crossing in $G_p$.

- A crossing corresponds to:
Points and triangles in 3D (cont)

- (By Hölder’s inequality) we have:
  \[ \sum_{p \in \mathcal{P}} \Omega(|T_p|^3/n^2) \geq \Omega \left( (\sum_{p \in \mathcal{P}} |T_p|)^3/n^4 \right) \]

configurations of:

\[ ((\sum_{p \in \mathcal{P}} |T_p|)^3/n^4 = \Omega(t^3/n^4) \]

=> \( \exists \) edge in \( \Omega(t^3/n^6) \) configurations
Points and triangles (cont)

- **Remark:**
  Best known upper bound construction $O(t^2/n^3)$

- **Conjecture:**
  Always $\exists$ line that intersects $\Omega(t^2/n^3)$ triangles ($\gg t^3/n^6$)
Applications (k-sets in 3D)

- $|P| = n$ pts in $\mathbb{R}^3$
- $|T| = t$, the set of halving-triangles

Lovász’ Lemma in 3D:
- Any line stabs at most $O(n^2)$ halving-triangles

We have:
\[ \Omega(t^3/n^6) \leq O(n^2) \]
\[ t = O(n^{8/3}) \]
Improved bounds on $k$-sets in 3D

- **Thm:** [Sharir, Smorodinsky, Tardos '00]
  \[ t = O(n^{5/2}) \]

- **Proof:**

  The halving-triangles are antipodal
Improved bounds on $k$-sets in 3D

The halving-triangles are antipodal

This property will imply:

- \exists \text{ line that stabs } \Omega(t^2/n^3) \text{ triangles}
- \text{Combined with } O(n^2) \text{ upper bound (Lovász)}
  
  We will get the $O(n^{5/2})$ upper bound.
Improved bounds on k-sets in 3D (cont)

- ∃ line that stabs $\Omega(t^2/n^3)$ triangles

Proof:
- For a point $p$, $G_p$ is the stereographic projection of $T_p$ on a plane above $p$
Improved bounds on $k$-sets in 3D (cont)

- $\sum_{p \in P} e_p = t$
- $\sum_{p \in P} r_p = t$
Improved bounds on k-sets in 3D (cont)

$G_p$ consists of
- $n$ vertices
- $e_p$ edges
- $r_p$ rays

$G_p$ is antipodal so:
- $G_p$ is decomposed to $\Omega(r_p)$ convex chains
- Contains $\Omega(r_p^2 - nr_p)$ crossings
Improved bounds on k-sets in 3D (cont)

- $G_p$ Contains $\Omega(r_p^2 - nr_p)$ crossings
  Proof: bound the # pairs of convex chains which do not cross
- At most $O(nr_p)$ pairs out of the $r_p^2$
Improved bounds on k-sets in 3D (cont)

- \( G_p \) contains \( \Omega(r_p^2) \) crossings. Hence:

- \( X = \# \text{crossings} = \sum_{p \in P} r_p^2 = \Omega(\sum_{p \in P} r_p)^2 / n \)

  \[ \Rightarrow X = \Omega(t^2 / n) \text{ and } \Rightarrow \]

  \( \exists \) line that stabs \( \Omega(t^2 / n^3) \) triangles

- Combined with Lovász' Lemma we have:
  \( t = O(n^{2.5}) \)
**k-sets in 4D (Sketch)**

- $P := n$ pts in $\mathbb{R}^4$
- $S :=$ the set of halving-simplices of $P$
- **Thm:** [Matoušek, Sharir, Smorodinsky, Wagner '05]
  \[ |S| = O(n^{4-2/45}) \]

Proof uses two main lemmas:

**Lemma 1:**
There is a $2$-plane that intersects $\Omega(|S|^3/n^8)$ simplices of $S$

**Lemma 2:**
Any such $2$-plane intersects $O(n^{4-2/15})$ simplices (main new ingredient)
Lemma 1:
There is a 2-plane that intersects $\Omega(|S|^3/n^8)$ simplices of $S$

Proof (sketch): Project $P$ and $S$ orthogonally onto $R^3$

We have $|P'|=n$ pts in $R^3$ and $|S|$ tetrahedra spanned by $P'$
Lemma 1:
There is a 2-plane that intersects $\Omega(|S|^3/n^8)$ simplices of $S$

We show that there is a line that intersects this many tetrahedra

`lifting' that line back to $\mathbb{R}^4$ we get the desired plane
k-sets in 4D (Sketch)

Maybe Stop here???
A square matrix is called totally positive (TP) if:
the determinant of any square sub matrix (minor) is positive

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
4 & 9 & 15 \\
5 & 12 & 30
\end{pmatrix}
\]

\[
det \begin{pmatrix}
1 & 3 \\
4 & 15
\end{pmatrix} = 3
\]

\[
det \begin{pmatrix}
4 & 9 \\
5 & 12
\end{pmatrix} = 3
\]

\[
det(A) = 9
\]
Example: Vandermonde matrix

\[ V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{m-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{m-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{m-1} \end{pmatrix} \]

\[ \alpha_1 < \alpha_2 < \cdots < \alpha_m \]

\[ \det(V) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i) \]

\[ \begin{vmatrix} \alpha_1 & \alpha_1^2 \\ \alpha_2 & \alpha_2^2 \end{vmatrix} = \alpha_1 \alpha_2 \begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix} \]
The multiplicity of an entry \( a \) in a matrix \( M \) is the number of occurrences of \( a \) in \( M \).

\[
M = \begin{pmatrix}
30 & 2 & 3 \\
4 & 9 & 15 \\
5 & 12 & 30 \\
\end{pmatrix}
\]

The multiplicity of 30 is 2.
Question (Farber et al.)

What is the maximum number of equal entries in an $N \times N$ TP matrix?

- $N = 2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, 3 equal entries
- $N = 3$, $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\text{det}(A) = \frac{3}{\sqrt{2}} - 2$, 6 equal entries
Answer (Farber et al.)

Theorem [Farber, Faulk, Johnson, Marzion '12]:
The max # of equal entries in $N \times N$ TP matrix is $\Theta(N^{4/3})$

PF of upper bound relies on a result of J. Pach and G. Tardos '06,

PF of lower bound:
Construction using many pts-lines incidences...
Question (Farber et al.)

Consider $2 \times N$ matrices with positive entries and positive $2 \times 2$ minors ($TP_2$)

What is the max # of equal $2 \times 2$ minors in a $2 \times N$ $TP_2$ matrix?

More generally: What is max # of equal $2 \times 2$ minors in $N \times N$ $TP_2$ matrix?
Equal $2 \times 2$ minors

**Thm 1** [Farber-Ray-Smorodinsky '13]:

The max # of equal $2 \times 2$ minors in a $2 \times N$ TP$_2$ matrix is $\Theta(N^{4/3})$

(Reduction to pts-lines incidences)

**Thm 2** [Farber-Ray-Smorodinsky '13]:

The max # of equal $d \times d$ minors in a $d \times N$ TP$_2$ matrix is $\Theta(N^{d-(d/d+1)})$

(Reduction to pts-hyperplane incidences and a result of Apfelbaum-Sharir 2007)
A construction of TP$_2$

Fix sets $A, B$ of $n$ reals;

$0 < a_1 < ... < a_n$ and $b_1 > ... > b_n > 0$

Define $A = (A_{i,j})$ where $A_{i,j} = b_i + a_j$

Claim: $A$ is a TP$_2$ matrix (easy to check)

And moreover... in $A \times B$

The area of an axis-parallel rectangle spanned by $A \times B$ corresponds to a $2 \times 2$ minor

Open problem: How many such repeated (unit)-area rectangles are possible?
Lower bound of $\Omega(n^{2+1/\log\log n})$:

- Consider $A=B=\{1,2,\ldots,2n\}$: fix $1 \leq k \leq n$ s.t. $\text{div}(k) = \Omega(n^{1/\log\log n})$.
- For each divisor $x$ of $k$ we have $\Omega(n^2)$ rectangles with area $k$. 

![Diagram showing the relationship between $k$, $x$, and $k/x$.]
Upper bound on repeated areas

Note: $O(n^3) =$ trivial upper bound. Any point can be the lower left corner of at most $n$ unit area rectangles!

Thm 3 [Farber-Ray-Smorodinsky '13]: $P = n$ arbitrary pts in the plane.
max #unit area axis-parallel rectangles in $P$ is $O(n^{4/3})$.

Pf: For $p=(a,b) \in P$ define the hyperbola:

$$\gamma_p = \{(x,y) : (x-a)(y-b)=1\}$$

Observe: $(x,y) \in \gamma_p$ iff $(a,b)$ and $(x,y)$ opposite corners of unit area axis-parallel rectangle.

#rectangles $\leq$ # incidences between $P$ and $\Gamma=\{\gamma_p | p \in P\}$

Luckily: $\Gamma$ is a family of pseudo lines.
Corollary: For $|P| = n$ points we have:

$O(n^{4/3})$ repeated (unit) area rectangles.