Tutorial on the Calculus of Variations: Part I, Lagrangians

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Dot Notation and Other Simplifications

$$\dot{x}(t) \equiv \frac{d}{dt} x(t)$$

$$\mathcal{L}\big(x, \dot{x}\big) \equiv \mathcal{L}\big(x(t), \dot{x}(t), t\big)$$

Action Integrals and Lagrangians

Our goal is to find a differentiable curve of the form $x \equiv \{x(t) | 0 \le t \le T\}$ that is *extremal* for (maximizes or minimizes) the *action integral* as given below

$$\mathcal{A}(x, \dot{x}, T) = \int_0^T \mathcal{L}\big(x(t), \dot{x}(t), t\big)\,dt$$

The integrand function $\mathcal{L}$ is called the *Lagrangian* and is assumed to be continuously differentiable in all its arguments.

Application Interpretation for Action Integrals and Lagrangians: Operations Research

The curve $x \equiv \left\{ x(t) \middle| 0 \leq t \leq T \right\}$ is now the evolution of the *state of some system* and the Lagrangian $\mathcal{L}$ is the *profit (or cost) rate function*.

The action integral is the *total profit (cost)* over $[0, T]$.

We use calculus of variations to *maximize the profit (or minimize the cost)* over $[0, T]$.

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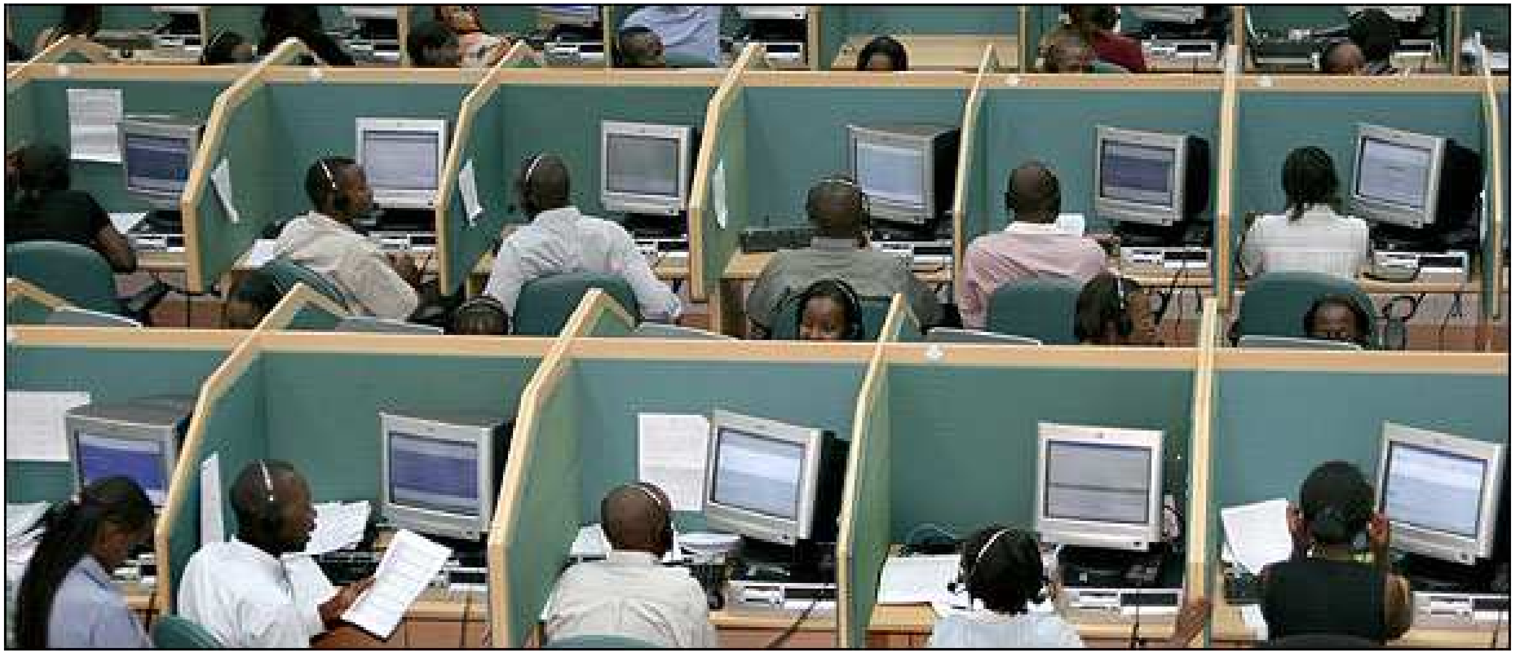
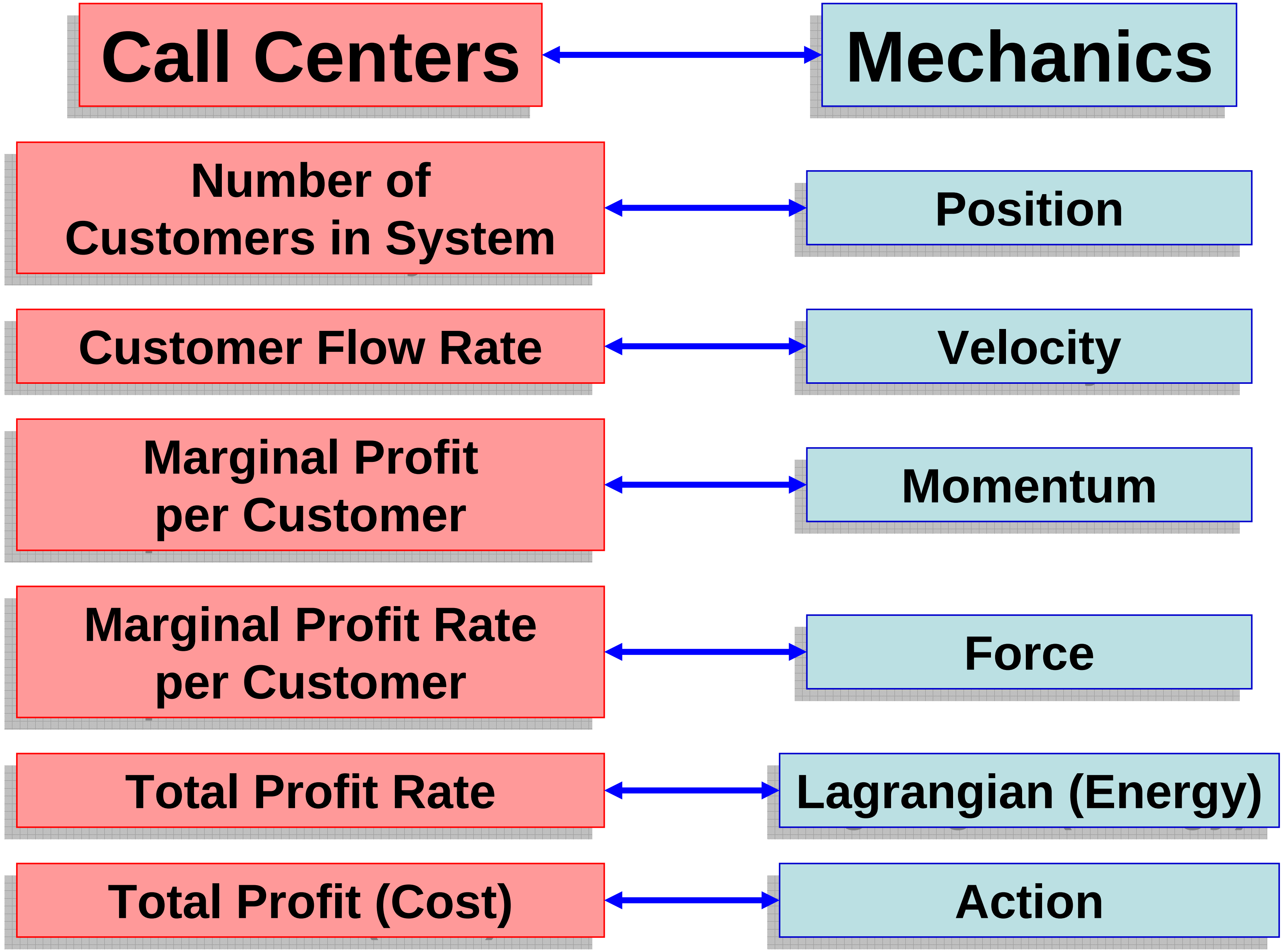



Call Centers
Mechanics
Number of Customers in System
Position
Customer Flow Rate
Velocity
Marginal Profit per Customer
Momentum
Marginal Profit Rate per Customer
Force
Total Profit Rate
Lagrangian (Energy)
Total Profit (Cost)
Action

Euler – Lagrange Equations

If $x \equiv \left\{ x(t) \middle| 0 \le t \le T \right\}$ is an extremal curve for the action integral, then it must solve the following ordinary differential equation on $[0, T]$:

$$\frac{\partial \mathcal{L}}{\partial x}\left(x(t), \dot{x}(t), t\right) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\left(x(t), \dot{x}(t), t\right).$$

Endpoint, Natural Boundary or Transversality Conditions

Moreover, if $x(T)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) = 0.$$

Finally, if $x(0)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(0), \dot{x}(0), 0\Big) = 0.$$

Let $\phi : [0,T] \rightarrow \mathbb{R}$ be any continuously differentiable function. Now define $I(\varepsilon)$ to be the perturbed action integral

$$I(\varepsilon) \equiv \mathcal{A}(x + \varepsilon \cdot \phi, \dot{x} + \varepsilon \cdot \dot{\phi}, T)$$

$$= \int_0^T \mathcal{L}\Big(x(t) + \varepsilon \cdot \phi(t), \dot{x}(t) + \varepsilon \cdot \dot{\phi}(t), t\Big)\, dt.$$

Differentiating with respect to ε gives us

$$I'(\varepsilon) = \int_0^T \phi(t) \frac{\partial \mathcal{L}}{\partial x}\Big(x(t) + \varepsilon \cdot \phi(t), \dot{x}(t) + \varepsilon \cdot \dot{\phi}(t), t\Big)\, dt$$

$$+ \int_0^T \dot{\phi}(t) \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(t) + \varepsilon \cdot \phi(t), \dot{x}(t) + \varepsilon \cdot \dot{\phi}(t), t\Big)\, dt.$$

$$I'(0) = \int_0^T \phi(t)\frac{\partial \mathcal{L}}{\partial x}\big(x(t),\dot{x}(t),t\big)\,dt + \int_0^T \dot{\phi}(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(t),\dot{x}(t),t\big)\,dt$$

$$= \int_0^T \phi(t)\frac{\partial \mathcal{L}}{\partial x}\big(x(t),\dot{x}(t),t\big)\,dt$$

$$+\phi(T)\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(T),\dot{x}(T),T\big) - \phi(0)\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(0),\dot{x}(0),0\big)$$

$$-\int_0^T \phi(t)\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(t),\dot{x}(t),t\big)\,dt$$

$$= \int_0^T \phi(t)\cdot\left[\frac{\partial \mathcal{L}}{\partial x}\big(x(t),\dot{x}(t),t\big) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(t),\dot{x}(t),t\big)\right]dt$$

$$+\phi(T)\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(T),\dot{x}(T),T\big) - \phi(0)\frac{\partial \mathcal{L}}{\partial \dot{x}}\big(x(0),\dot{x}(0),0\big)$$

Fundamental Lemma

If $f : [0,T] \to \mathbb{R}$ is a continuous function where

$$\int_0^T \phi(t) \cdot f(t)\, dt = 0$$

for all continuously differentiable functions $\phi : [0,T] \to \mathbb{R}$ with $\phi(0) = \phi(T) = 0,$ then f is the zero function.

Arc Length Example

$$\mathcal{L}\left(x(t), \dot{x}(t), t\right) = \sqrt{1 + \dot{x}(t)^2}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow 0 = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\sqrt{1 + \dot{x}(t)^2} = \frac{d}{dt}\frac{\dot{x}(t)}{\sqrt{1 + \dot{x}(t)^2}}$$

$$\Rightarrow \dot{x}(t) = \text{constant} = \frac{x(T) - x(0)}{T}$$

$x(0)$ & $x(T)$ fixed $\Rightarrow x(t) = \dfrac{x(T) - x(0)}{T} t + x(0), 0 \le t \le T$

only $x(0)$ is fixed $\Rightarrow \dot{x}(t) = 0 \Rightarrow x(t) = x(0), 0 \le t \le T.$

Sensitivity Analysis for Extremals

Let the Lagrangian be a function of a parameter a that is independent of t. If the endpoints of the extremal curve x are *only* a function of a when they do *not* have fixed values, then

$$\frac{d\mathcal{A}}{da}(x, \dot{x}, T) = \int_0^T \frac{\partial \mathcal{L}}{\partial a}\Big(x(t), \dot{x}(t), t\Big)\, dt$$

$$
\begin{aligned}
\frac{d\mathcal{A}}{da}(x,\dot{x},T) &= \int_0^T \frac{d\mathcal{L}}{da}(x,\dot{x})\,dt \\
&= \int_0^T \left[\frac{\partial \mathcal{L}}{\partial x}(x,\dot{x}) \cdot \frac{dx}{da} + \frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) \cdot \frac{d\dot{x}}{da} + \frac{\partial \mathcal{L}}{\partial a}(x,\dot{x}) \right] dt \\
&= \int_0^T \left[\frac{\partial \mathcal{L}}{\partial x}(x,\dot{x}) \cdot \frac{dx}{da} + \frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) \cdot \frac{d}{dt}\left(\frac{dx}{da}\right) \right] dt + \int_0^T \frac{\partial \mathcal{L}}{\partial a}(x,\dot{x})\,dt \\
&= \int_0^T \left[\frac{\partial \mathcal{L}}{\partial x}(x,\dot{x}) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) \right] \cdot \frac{dx}{da}\,dt + \frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) \cdot \frac{dx}{da}\bigg|_{t=0}^{t=T} \\
&\quad + \int_0^T \frac{\partial \mathcal{L}}{\partial a}(x,\dot{x})\,dt \\
&= \int_0^T \frac{\partial \mathcal{L}}{\partial a}(x,\dot{x})\,dt
\end{aligned}
$$

Optimizing Action Integrals with Endpoint Terms

Finding an extremal curve for

$$\mathcal{A}^*(x,\dot{x},T) = \ell(x(T),T) - \ell(x(0),0) + \int_0^T \mathcal{L}\big(x(t),\dot{x}(t),t\big)\,dt$$

is equivalent to finding an extremal for

$$\mathcal{A}^*(x,\dot{x},T) = \int_0^T \mathcal{L}^*\big(x,\dot{x}\big)\,dt$$

where

$$\mathcal{L}^*\big(x,\dot{x}\big) \equiv \mathcal{L}\big(x,\dot{x}\big) + \frac{d}{dt}\ell(x).$$

Automatic Solutions to the Euler-Lagrange Equations

Any curve $x \equiv \{x(t) \mid 0 \le t \le T\}$ satisfies the Euler-Lagrange equations for any Lagrangian of the form $\frac{d}{dt}\ell(x)$, where $\ell : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function in x and t, and

$$\frac{d}{dt}\ell(x(t),t) = \frac{\partial \ell}{\partial x}(x(t),t) \cdot \dot{x}(t) + \frac{\partial \ell}{\partial t}(x(t),t).$$

Equivalent Lagrangians

The two Lagrangians $\mathcal{L}$ and $\mathcal{L}^*$, where

$$\mathcal{L}(x,\dot{x}) \text{ and } \mathcal{L}^*(x,\dot{x}) = \mathcal{L}(x,\dot{x}) + \frac{d}{dt}\ell(x)$$

generate the same set of Euler-Lagrange equations.

Modified Endpoint Conditions

Moreover, if $x(T)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) + \frac{\partial \ell}{\partial x}\Big(x(T), T\Big) = 0.$$

Finally, if $x(0)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(0), \dot{x}(0), 0\Big) + \frac{\partial \ell}{\partial x}\Big(x(0), 0\Big) = 0.$$

Lagrange Multiplier for an Equality Constraint

Find an extremal for $\mathcal{A}(x, \dot{x}, T) = \int_0^T \mathcal{L}\big(x(t), \dot{x}(t), t\big)\, dt,$

subject to the constraint $\mathcal{M}\big(x(t), \dot{x}(t), t\big) = 0.$

This is equivalent to finding an extremal for

$$\mathcal{A}^* (x, \dot{x}, \lambda, T) = \int_0^T \mathcal{L}^* \big(x(t), \dot{x}(t), \lambda(t), t\big)\, dt,$$

where $\mathcal{L}^* \big(x, \dot{x}, \lambda\big) \equiv \mathcal{L}\big(x, \dot{x}\big) + \lambda \cdot \mathcal{M}\big(x, \dot{x}\big).$

Modified Euler – Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial x}(x,\dot{x}) + \lambda \frac{\partial \mathcal{M}}{\partial x}(x,\dot{x}) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) + \lambda \frac{\partial \mathcal{M}}{\partial \dot{x}}(x,\dot{x})\right).$$

$$\mathcal{M}(x,\dot{x}) = 0$$

Modified Endpoint Conditions

Moreover, if $x(T)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) + \lambda \frac{\partial \mathcal{M}}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) = 0.$$

Finally, if $x(0)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(0), \dot{x}(0), 0\Big) + \lambda \frac{\partial \mathcal{M}}{\partial \dot{x}}\Big(x(0), \dot{x}(0), 0\Big) = 0.$$

Lagrange Multiplier and Slack Variable for an Inequality Constraint

Find an extremal for $\mathcal{A}(x,\dot{x},T)=\int_0^T \mathcal{L}\big(x(t),\dot{x}(t),t\big)\,dt,$

subject to the constraint $\mathcal{N}\big(x(t),\dot{x}(t),t\big)\geq 0.$

This is equivalent to finding an extremal for

$$\mathcal{A}^*(x,\dot{x},\mu,\sigma,T)=\int_0^T \mathcal{L}^*\big(x(t),\dot{x}(t),\mu(t),\sigma(t),t\big)\,dt,$$

where $\mathcal{L}^*\big(x,\dot{x},\mu,\sigma\big)\equiv\mathcal{L}\big(x,\dot{x}\big)+\mu\cdot\big(\mathcal{N}\big(x,\dot{x}\big)-\sigma^2\big).$

Modified Euler – Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial x}(x,\dot{x}) + \mu \frac{\partial \mathcal{N}}{\partial x}(x,\dot{x}) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}(x,\dot{x}) + \mu \frac{\partial \mathcal{N}}{\partial \dot{x}}(x,\dot{x})\right).$$

$$\mathcal{N}(x,\dot{x}) = \sigma^2$$

$\sigma \cdot \mu = 0$ (*complementarity* condition)

Maximizing the action integral $\Rightarrow \mu \geq 0$

Minimizing the action integral $\Rightarrow \mu \leq 0$

Modified Endpoint Conditions

Moreover, if $x(T)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) + \mu \frac{\partial \mathcal{N}}{\partial \dot{x}}\Big(x(T), \dot{x}(T), T\Big) = 0.$$

Finally, if $x(0)$ has no fixed value, then

$$\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big(x(0), \dot{x}(0), 0\Big) + \mu \frac{\partial \mathcal{N}}{\partial \dot{x}}\Big(x(0), \dot{x}(0), 0\Big) = 0.$$

General Construction of a Lagrangian

$$\mathcal{L}^*\left(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma}\right)$$

$$\equiv \mathcal{L}\left(\mathbf{x}, \dot{\mathbf{x}}\right) + \sum_{i=1}^{m} \lambda_i \cdot \mathcal{M}_i\left(\mathbf{x}, \dot{\mathbf{x}}\right) + \sum_{j=1}^{n} \mu_j \cdot \left(\mathcal{N}_j\left(\mathbf{x}, \dot{\mathbf{x}}\right) - \sigma_j^2\right)$$

$\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_\ell \end{bmatrix}$ = the ℓ coordinate variables.

$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \dots & \lambda_m \end{bmatrix}$ = the m multiplier variables for the m equality constraints.

$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \dots & \mu_n \end{bmatrix}$ = the n multiplier variables for the n inequality constraints.

$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & \dots & \sigma_n \end{bmatrix}$ = the n corresponding slack variables.

Modified Euler – Lagrange Equations

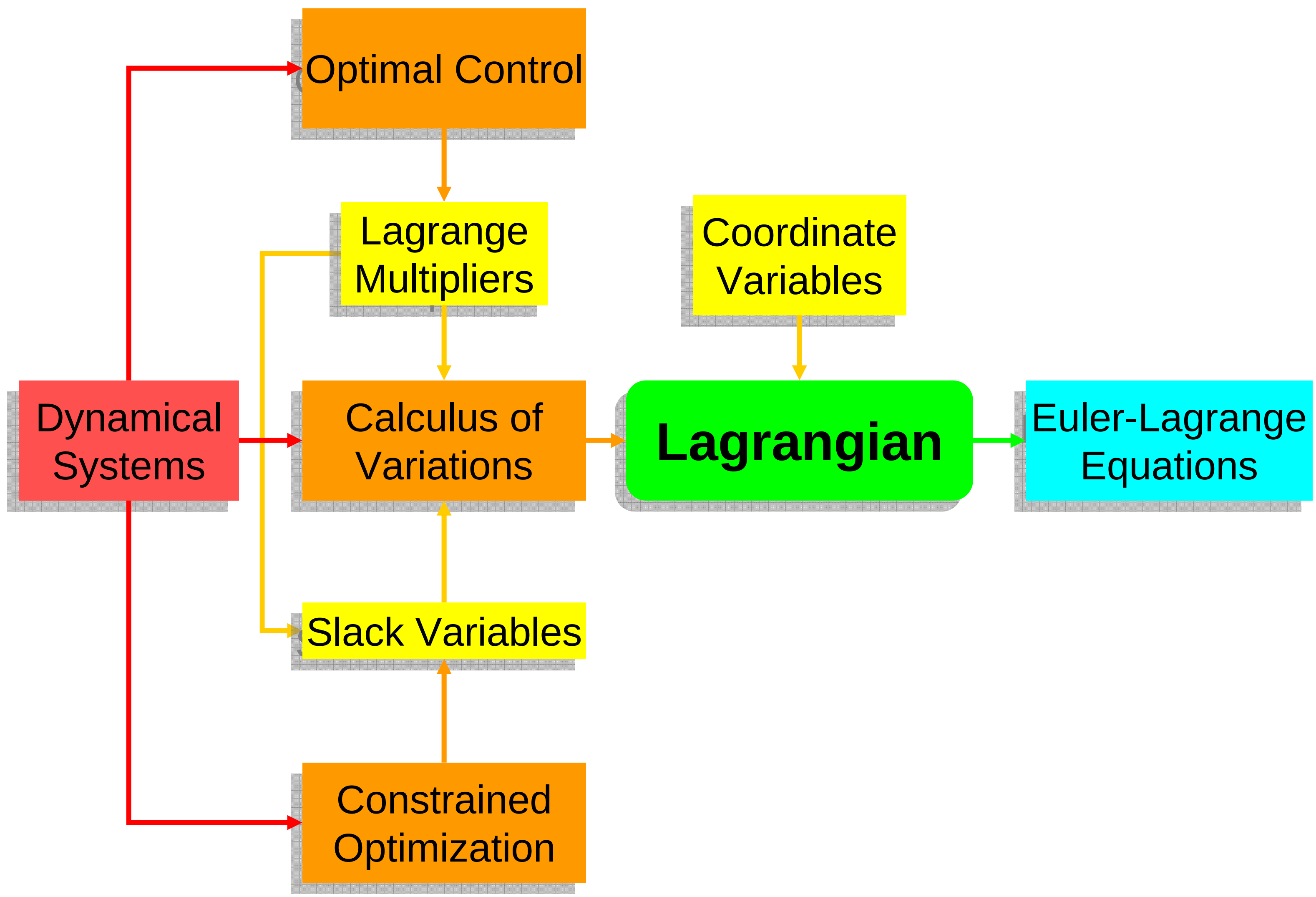
$$\frac{d}{dt}\nabla_{\dot{\mathbf{x}}}\left[\mathcal{L}(\mathbf{x},\dot{\mathbf{x}})+\sum_{i=1}^{m}\lambda_i\cdot\mathcal{M}_i(\mathbf{x},\dot{\mathbf{x}})+\sum_{j=1}^{n}\mu_j\cdot\mathcal{N}_j(\mathbf{x},\dot{\mathbf{x}})\right]$$

$$=\nabla_{\mathbf{x}}\left[\mathcal{L}(\mathbf{x},\dot{\mathbf{x}})+\sum_{i=1}^{m}\lambda_i\cdot\mathcal{M}_i(\mathbf{x},\dot{\mathbf{x}})+\sum_{j=1}^{n}\mu_j\cdot\mathcal{N}_j(\mathbf{x},\dot{\mathbf{x}})\right]$$

$$\mathcal{M}_1(\mathbf{x},\dot{\mathbf{x}})=\ldots=\mathcal{M}_m(\mathbf{x},\dot{\mathbf{x}})=0$$

$$\mathcal{N}_1(\mathbf{x},\dot{\mathbf{x}})-\sigma_1^2=\ldots=\mathcal{N}_n(\mathbf{x},\dot{\mathbf{x}})-\sigma_n^2=0$$

$$\mu_1\cdot\sigma_1=\ldots=\mu_n\cdot\sigma_n=0$$


Optimal Control
Lagrange Multipliers
Coordinate Variables
Dynamical Systems
Calculus of Variations
Lagrangian
Euler-Lagrange Equations
Slack Variables
Constrained Optimization