Scalable Algorithms in the Age of Big Data and Network Sciences

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Asymptotic Complexity

\[ O( f(n) ) \]
\[ \Omega( g(n) ) \]
\[ \Theta( h(n) ) \]

for problems with massive input
Characterization of Efficient Algorithms

Polynomial Time

$O(n^c)$ for a constant $c$
Big Data and Massive Graphs

- Tera Web pages
- unbounded amount of Web logs
- billions of variables
- billions of transistors.
Big Data and Massive Graphs

- Tera Web pages
- unbounded amount of Web logs
- billions of variables
- billions of transistors.

Happy Asymptotic World for Theoreticians
Efficient Algorithms for Big Data

Quadratic time algorithms could be too slow!!!!
Modern Notion of Algorithmic Efficiency
Therefore, more than ever before, it is not just desirable, but essential, that efficient algorithms should be scalable. In other words, their complexity should be nearly linear or sub-linear with respect to the problem size.

Thus, scalability — not just polynomial-time computability — should be elevated as the central complexity notion for characterizing efficient computation.
Big Data and Scalable Algorithms

A Practical Match Made in the Digital Age
**Big Data and Scalable Algorithms**

\[
\text{scalability}(A, x) = \frac{T_A(x)}{\text{size}(x)}
\]
Big Data and Scalable Algorithms

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\text{scalability}(A, x) = \frac{T_A(x)}{\text{size}(x)}
\]

\[
\text{scalability}_A(n) = O(\log^c n)
\]
Big Data and Scalable Algorithms

- Nearly-Linear Time Algorithms
- Sub-Linear Time Algorithms

\[
\text{scalability}(A, x) = \frac{T_A(x)}{\text{size}(x)}
\]

\[
\text{scalability}_A(n) = O(\log^c n)
\]
Algorithmic Paradigms: Scorecard

- Greedy: often scalable (limited applications)
- Dynamic Programming: usually not scalable (even when applicable)
- Divide-and-Conquer: sometimes scalable
- Mathematical Programming: rarely scalable

- Branch-and-Bound: hardly scalable
- Multilevel Methods: mostly scalable (lack of proofs)
- Local Search and Simulated Annealing: can be scalable
Examples: Scalable Geometry Algorithms

Sorting
Nearest neighbors
Delaunay Triangulation/3D convex hull

Fixed Dimensional Linear Programming
ε–net in fixed-dimensional VC space

O(n log n)

O(n)
Examples: Scalable Graph Algorithms

Breadth-First Search
Depth-First Search
Shortest Path Tree
Minimum Spanning Tree
Planarity Testing
Bi-connected components
Topological sorting
Sparse matrix vector product

$O(|V| + |E|)$
Examples:

Scalable Numerical Algorithms

- N-Body simulation \( O(n) \)
- Sparse matrix vector product
- FFT/Multiplication \( O(n \log n) \)
- Multilevel algorithms
- Multigrid
We need more provably-good scalable algorithms for network analysis, data mining, and machine learning (in real-time applications)
**Scalable Methodology: Talk Outline**

- **Scalable Primitives and Reduction**
  - The Laplacian Paradigm
    - Electrical Flows & Maximum Flows; Spectral Approximation; Tutte’s embedding and Machine Learning

- **Scalable Technologies:**
  - Spectral Graph Sparsification
    - Sparse Newton’s Method and Sampling from Graphical Models
  - Computing Without the Whole Data: Local Exploration and Advanced Sampling
    - Significant PageRanks

- **Challenges:** Computation over Dense/High-Dimensional Models with Succinct/Sparse Representations
  - Social Influence; high order clustering;
Scalable Primitives and Reduction

Algorithm Design is like Building a Software Library

Scalable Reduction: Once scalable algorithms are developed, they can be used as primitives or subroutines for designing new scalable algorithms.
Laplacian Primitive

Solve $A \mathbf{x} = b$, where $A$ is a weighted Laplacian matrix
**Laplacian Primitive**

Solve $Ax = b$, where $A$ is a weighted Laplacian matrix

$A$ is Laplacian matrix: symmetric
- non-positive off diagonal
- row sums = 0

Isomorphic to weighed graphs

\[
A = \begin{pmatrix}
4.3 & -4 & 0 & -0.3 \\
-4 & 5.5 & -1.5 & 0 \\
0 & -1.5 & 4.6 & -3.1 \\
-0.3 & 0 & -3.1 & 3.4
\end{pmatrix}
\]
For symmetric diagonally dominant (SDD) $A$, any $b$

Compute $\left\| x - A^{-1}b \right\|_A < \epsilon \left\| x \right\|_A$ in time $m \log^{O(1)} n \log(1/\epsilon)$

Greatly improved by Koutis-Miller-Peng, Kelner-Orecchia-Sidford-Zhu, ..., Lee, Peng, and Spielman, to essentially $O(m \log (1/\epsilon))$
The Laplacian Paradigm

To apply the Laplacian Paradigm to solve a problem defined on massive networks or big matrices, we attempt to reduce the computational and optimization problem to one or more linear algebraic or spectral graph-theoretical problems.

Beyond scalable Laplacian solvers
Scalable Tutte’s Embedding

Learning from labeled data on directed graphs [Zhou-Huang-Schölkopf]
Approximate Fiedler Vector
For Laplacian $A$, is vector $v^T I = 0$ such that
\[
\frac{v^T A v}{v^T v} \leq (1 + \epsilon) \lambda_2(A)
\]

Can find $v$ using inverse power method, in time
\[
m \log^{O(1)} n \log(1/\epsilon)/\epsilon
\]
**Scalable Cheeger Cut**

**Theorem:** Constant degree graph $G$, Fiedler value $\lambda$: scalable computation of a cut of conductance $o(\sqrt{\lambda})$
Scalable Electrical Flows

Electrical potentials: \( L \varphi = \chi_{st} \)

in time \( \tilde{O}(m \log \varepsilon^{-1}) \)
Undirected Maximum Flow

Previously Best: $O(m^{3/2})$  
[Even-Tarjan 75]
Iterative Electrical Flows: $\varphi = L^{-1} \chi_{st} : \tilde{O}(m^{4/3} \varepsilon^{-3})$

Previously Best: $\tilde{O}(\min(m^{3/2}, m n^{2/3}))$ [Goldberg-Rao]
Path to Scalable Maximum Flow

Previously Best: $\tilde{O}(\min(m^{3/2}, mn^{2/3}))$ [Goldberg-Rao]

Iterative Electrical Flows: $\varphi = L^{-1} \chi_{st}: \tilde{O}(m^{4/3} \varepsilon^{-3})$

Scalable: Sherman; Kelner-Lee-Orecchia-Sidford, Peng
Applications of The Laplacian Paradigm

- Electrical flow computation
- Spectral approximation
- Tutte’s embedding
- Learning from labeled data on a directed graph [Zhou-Huang-Schölkopf]
- Cover time approximation [Ding-Lee-Peres]
- Maximum flows and minimum cuts [Christiano-Kelner-Madry-Spielman-Teng]
- Elliptic finite-element solver [Boman-Hendrickson-Vavasis]
- Rigidity solver [Shklarski-Toledo; Daitch-Spielman]
- Image processing [Koutis-Miller-Tolliver]
- Effective resistances of weighted graphs [Spielman-Srivastava]
- Generation of random spanning trees [Madry-Kelner]
- Generalized lossy flows [Daitch-Spielman]
- Geometric means [Miller-Pachocki]
- …
Scalable Techniques

• Algebraic Formulation of Network Problems

• Spectral Sparsification of Matrices and Networks

• Computing without the Whole Data: Local Exploration of Networks
For a graph $G$ (with Laplacian $L$), a sparsifier is a graph $\tilde{G}$ (with Laplacian $\tilde{L}$) with at most $n \log^{O(1)} n$ edges s.t.

$$\kappa_f(L, \tilde{L}) \leq (1 + \epsilon) \quad \forall x : x^T \tilde{L} x \leq x^T L x \leq (1 + \epsilon) x^T \tilde{L} x$$

Improved by Batson, Spielman, and Srivastava
Sampling From Graphical Models

Joint probability distribution of n-dimensional random variables $x$

Graphical model for local dependencies

Sampling according to the model
Gibbs’ Markov Chain Monte Carlo Process

Locally resample each variable, conditioned on the values of its graphical neighbors

- In limit, exact mean and covariance
  [Hammersley-Clifford]
- Easy to implement
- Many iterations
- Sequential
A Holy Grail Sampling Question

Characterization of graphical models that have scalable parallel sampling algorithms with poly-logarithmic depth?
Gaussian Markov Random Fields

Pr \[x|\Lambda, h] \propto \exp(-\frac{1}{2}x^T \Lambda x + h^T x)

- Precision matrix – symmetric positive definite
- Potential vector
- Goal: Sampling from Gaussian distribution \(\mathcal{N}(\Lambda^{-1} h, \Lambda^{-1})\)
GMRF with $H$-Precision Matrices

Johnson-Saunderson-Willsky (*NIPS* 2013)

**DAD** is SDD

If the precision matrix $\Lambda$ is (generalized) diagonally dominant, then Hogwild Gibbs distributed sampling process converges
Scalable Parallel Gaussian Sampling?

- Time complexity: $O(\text{nnz}(\Lambda))$
- Parallel complexity: $O(\log n)$
- Randomness complexity: $n$

It remains open even if the precision matrix is symmetric diagonally dominant (SDD).
A Numerical Program for Gaussian Sampling

1. Find the mean:
   \[ \mu = \Lambda^{-1} h \]

2. Compute an inverse square-root factor:
   \[ CC^T = \Lambda^{-1} \]

3. Sampling:
   
   generate standard Gaussian variables \( z \)

   \[ x = C z + \mu \]
Canonical Inverse Square-Root

$$\Lambda^{-1} = C C^T$$
Focus on normalized Laplacian:

\[ L = D - W = D^{1/2}(I - D^{-1/2}WD^{-1/2})D^{1/2} \]

\[ \Lambda^{-1} = C C^T \]

\[ I - X \]
Newton’s Method

\[(I - X)^{-1} = \left( I + \frac{1}{2}X \right) \left( I - \frac{3}{4}X^2 - \frac{1}{4}X^3 \right)^{-1} \left( I + \frac{1}{2}X \right) \]
Newton’s Method

\[(I - X)^{-1} = \left( I + \frac{1}{2}X \right) \left( I - \frac{3}{4}X^2 - \frac{1}{4}X^3 \right)^{-1} \left( I + \frac{1}{2}X \right) \]

Newton’s method uses dense matrix multiplications, even when the original matrix is sparse. This is particularly the case in network analysis, where input graphs are usually sparse. Although Newton’s method may converge rapidly, which provides a numerical framework for designing not only sequential but also parallel algorithms, its intermediate computation could be prohibitively expensive for handling big data.
Sparse Newton’s Method

$$(I - X)^{-1} = \left( I + \frac{1}{2}X \right) \left( I - \frac{3}{4}X^2 - \frac{1}{4}X^3 \right)^{-1} \left( I + \frac{1}{2}X \right)$$

Spectral Sparsification
Sparse Newton Chain

\[(I - X)^{-1} = \left( I + \frac{1}{2}X \right) \left( I - \frac{3}{4}X^2 - \frac{1}{4}X^3 \right)^{-1} \left( I + \frac{1}{2}X \right) \]

\[ [X_0, X_1, \ldots, X_{d-1}] \]

\(X_t\) is a spectral sparsifier of \(\left( \frac{3}{4}X_{t-1}^2 + \frac{1}{4}X_{t-1}^3 \right)\)

\[ C = \prod_{i=0}^{d-1} \left( I + \frac{X_i}{2} \right) \]
Random-Walk Polynomials and Sparsification

\[ D - \sum_{r=1}^{t} \alpha_r D \cdot (D^{-1} W)^r \]
Path Sampling

Scalable Sparsification of Random-Walk Polynomials

\[ O(t^2 \cdot m \cdot \log^2 n \cdot \frac{1}{\epsilon^2}) \]
Scalable Parallel Gaussian Sampling for H-Precision Matrices


• Time complexity: \( O(\text{nnz}(\Lambda)) \)

• Parallel complexity: \( O(\text{polylog } n) \)

• Randomness complexity: \( 2 \, n \)
Scalable Sparse Newton’s Method

Matrix $p^{th}$-Power Factorization:
Given an $n \times n$ Laplacian matrix $M$ and a constant $-1 \leq p \leq 1$, compute an $n \times n$ linear operator $C$ such that $M^p = CC^\top$.

$$(I - X)^{-\frac{1}{q}} = \left(I + \frac{1}{2q}X\right)\left[(I + \frac{1}{2q}X)^{2q}(I - X)\right]^{-\frac{1}{q}}\left(I + \frac{1}{2q}X\right)$$
Nick Higham at Brain Davies’ 65 Birthday: An email from a power company regarding the usage of electricity networks

“I have an Excel spreadsheet containing the transition matrix of how a company’s [Standard & Poor’s] credit rating charges from one year to the next. I’d like to be working in eighths of a year, so the aim is to find the eighth root of the matrix.”
Family of Scalable Techniques

• Algebraic Formulation of Network Problems

• Spectral Sparsification of Matrices and Networks

• Computing without the Whole Data: Local Exploration of Networks
Local Network Algorithms
Local Network Algorithms
Local Network Algorithms
Local Network Algorithms
Local Network Algorithms
PageRank

- **PageRank:** Stationary Distribution of the Markov Process:
  - Probability $1 - \alpha$: random walk on the network
  - Probability $\alpha$: random restarting

- Stationary Distribution:

$$\text{PR}_{W, \alpha} = \alpha \cdot 1 + (1 - \alpha) \cdot \text{PR}_{W, \alpha} \cdot (D_{W}^{out})^{-1} W$$
Significant PageRank without Explore the Entire Network?

Input: \( G, 1 \leq \Delta \leq n, \text{ and } c > 1 \)

Output: Identify a subset \( S \subseteq V \) containing:

- all nodes of PageRank at least \( \Delta \)
- no nodes with PageRank less than \( \Delta / c \)

\( O(n/\Delta) \) time algorithm?
**Personalized PageRank Matrix**

**Personalized PageRank**

\[
p_u = \alpha \cdot 1_u + (1 - \alpha) \cdot p_u \cdot (D_W^{out})^{-1} \cdot W
\]

\[
p_u = (p_{u\rightarrow 1}, \cdots, p_{u\rightarrow n})
\]

\[
\text{PPR}_{W, \alpha} = [p_1, \ldots, p_n]^T = \\
\begin{bmatrix}
p_{1\rightarrow 1} & \cdots & p_{1\rightarrow n} \\
\vdots & \ddots & \vdots \\
p_{n\rightarrow 1} & \cdots & p_{n\rightarrow n}
\end{bmatrix}
\]
Scalable Local Personalized PageRank

\[ p_u = \alpha \cdot 1_u + (1 - \alpha) \cdot p_u \cdot (D_{W}^{\text{out}})^{-1} \cdot W \]

Jed-Widom
Andersen-Chung-Lang
\[ O(d_{\text{max}}/\varepsilon) \]

Fogaras-Racz-Csalogany-Sarlos
Borgs-Brautbar-Chayes-Teng
\[ O(\log n/(\varepsilon \rho^2)) \]
**An Abstract Problem: Vector Sum**

**Input:** $\nu$ (an unknown vector from $[0,1]^n$)

$1 \leq \Delta \leq n$ (a threshold value)

**Query Model:** $?(\nu, i, \varepsilon)$

**Cost:** $\frac{1}{\varepsilon}$

**Output:** Is $\text{sum}(\nu)$ more than $\Delta$ or less than $\Delta/2$?

**Question:** $O(n/\Delta)$ cost algorithm?
Riemann Estimator
Borgs-Brautbar-Chayes-Teng

\[ S_Q = \frac{n}{T} \sum_{t=1}^{T} I[q_t \geq \epsilon_t], \text{ where } \forall t \in [T], \epsilon_t = \frac{t}{T} \]

\[ E[q] = \int_0^1 \Pr[q \geq x] \, dx. \]

\[ \tilde{O}\left(\frac{n}{\Delta}\right) \]
Scalable Methodology: Talk Outline

• Scalable Primitives and Reduction
  – The Laplacian Paradigm
    • Electrical Flows & Maximum Flows; Spectral Approximation; Tutte’s embedding and Machine Learning

• Scalable Technologies:
  – Spectral Graph Sparsification
    • Sparse Newton’s Method and Sampling from Graphical Models
  – Computing Without the Whole Data: Local Exploration and Advanced Sampling
    • Significant PageRanks
Challenges

• Computation over dense models with succinct/sparse representations
  • High order clustering;

• Computation over high-dimensional models with succinct/sparse representations
  • Social Influence;

• Computation over incomplete data
  • ML
Clustering Based on Personalized Page-Rank Matrix

\[
\begin{bmatrix}
p_1 \rightarrow 1 & \cdots & p_1 \rightarrow n \\
\vdots & \ddots & \vdots \\
p_n \rightarrow 1 & \cdots & p_n \rightarrow n
\end{bmatrix}
\]

PPR-Density_{\mathbf{w}, \alpha}(S) = \frac{1}{|S|} \cdot \sum_{u,v \in S} \text{PPR}_{\mathbf{w}, \alpha}[u,v]

suspiciousness measure (Hooi-Song-Beutel-Shah-Shin-Faloutsos)

**Open Question:** scalable 2-approximation?
Reversed Diffusion Structure and Process

\[
\begin{bmatrix}
p_{1\rightarrow 1} & \cdots & p_{1\rightarrow n} \\
\vdots & \ddots & \vdots \\
p_{n\rightarrow 1} & \cdots & p_{n\rightarrow n}
\end{bmatrix}
\]
Influence Through Social Networks
Reversed Diffusion Process

- Scalable Influence Maximization
  - Borgs, Brautbar, Chayes, Lucier
  - Tang, Shi, Xiao
- Scalable Shapley Centrality of Social Influence
  - Chen and Teng
Given a vertex $v$ of interest in a massive network

find a provably-good cluster near $v$

in time $O$\text{(cluster size)}

**Open Question:** What other clustering problems can it be extended to?

High order clustering?
Scalable Algorithms for Data and Network Analysis

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Subjects
Algorithmic game theory, Computational aspects of combinatorics and graph theory, Computational complexity, Computational geometry, Computational Models and Complexity, Data structures, Design and analysis of algorithms, Operations Research, Parallel algorithms, Randomness in Computation, Data Mining, Economics of information and the Web, Scalability, Social Networking, Spectral methods, Robustness, Optimization, Markov chain Monte Carlo, Graphical models, Game theoretic learning, Dimensionality reduction, Data mining, Clustering, Web search, Natural language processing for IR

Abstract
In the age of Big Data, efficient algorithms are now in higher demand more than ever before. While Big Data takes us into the asymptotic world envisioned by our pioneers, it also challenges the classical notion of efficient algorithms: Algorithms that used to be considered efficient, according to polynomial-time characterization, may no longer be adequate for solving today's problems. It is not just desirable, but essential, that efficient algorithms should be scalable. In other words, their complexity should be nearly linear or sub-linear with respect to the problem size. Thus, scalability, not just polynomial-time computability, should be elevated as the central complexity notion for characterizing efficient computation. In this tutorial, I will survey a family of algorithmic techniques for the design of provably-good scalable algorithms. These techniques include local network exploration, advanced sampling, sparsification, and geometric partitioning. They also include spectral graph-theoretical methods, such as those used for computing electrical flows and sampling from Gaussian Markov random fields. These methods exemplify the fusion of combinatorial, numerical, and statistical thinking in network analysis. I will illustrate the use of these techniques by a few basic problems that are fundamental in network analysis, particularly for the identification of significant nodes and coherent clusters/communities in social and information networks. I also take this opportunity to discuss some frameworks beyond graph-theoretical models for studying conceptual questions to understand multifaceted network data that arise in social influence, network dynamics, and Internet economics.

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Scalable Algorithms for Data and Network Analysis

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Abstract

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In this article, I will survey a family of algorithmic techniques
Big Data and Network Sciences:
Going Beyond Graph Theory

- Set Functions
- Distributions
- Dynamics
- Multilayer Networks
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Scalable Algorithms for Big Data and Network Sciences: Characterization, Primitives, and Techniques

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