Perturbations of quasi-periodic orbits: From Theory to computations

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Work in collaboration with
R. Calleja, A. Celletti, A. Haro, A. González Enríquez, D. Rana. Building on work of many other people.

Main theme:
The theory of persisentence of quasi-periodic orbits was developed in the 60's to study perturbations of integrable systems.
Henon (letter to V. A. Arnol'd) [Paraphased] "The [numerical values required are so small] that the theory, in spite of its considerable theoretical interest, has no relevance for practical problems".

The small constants was a problem in the 60's, but not true anymore. Recent proofs lead to:

- Methods to validate numerical results.
- Very efficient algorithms.
- They have been implemented and in test cases there are rigorous upper and lower bounds agreeing in 10^{-4} (KAM tori, Figueras-Haro-Luque) or 10^{-6} with numerics (Whiskered tori, Figueras-Haro)

It seems that there is a clear path to getting results in more realistic systems.
Much — non-rutine – work remains to be done to get to real world pplications.
Numerical explorations have suggested new mathematical phenomena that deserve exploration.

We recall that a quasiperiodic orbit (for a map f) is a sequence $\{x_j = f^j(x_0)\}_{j\in\mathbb{Z}}$ such that

$$
\begin{aligned}
x_j &= \sum_{k\in\mathbb{Z}^d} \hat{x}_k e^{2\pi i k\cdot\omega j} \\
&= K(j\omega)
\end{aligned}
$$

where $K : \mathbb{T}^d \to \mathbb{R}^n$ is given by

$$
K(\theta) = \sum_{k\in\mathbb{Z}^d} e^{2\pi i k\cdot\theta} \hat{x}_k
$$

Such K satisfies the invariance

$$
f \circ K = K \circ T_\omega \tag{1}
$$

Where $T_\omega(\theta) = \theta + \omega$.

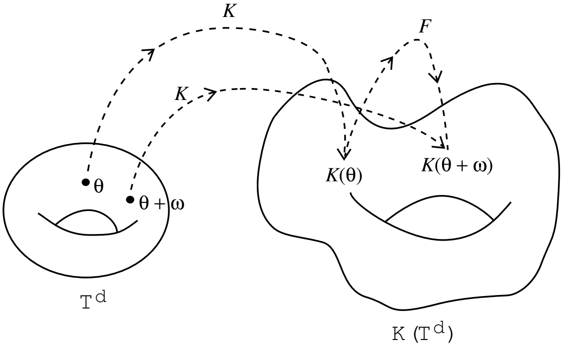


Figure: Geometric interpretation of the invariance equation (1)

A-posteriori theorems

We can often reduce the existence of invariant objects to finding solutions of a functional equation.

$$\mathcal{F}(x) = 0$$

(x can be a parameterization of the invariant object and $\mathcal{F}$ be a formulation of the fact that the object is invariant.)

Following standard notation in Numerical Analysis.
We want theorems that tell us that if for some suitable $\mathcal{F}$ we have:

- An approximate solution

$$||\mathcal{F}(x_0)||_1 = \varepsilon$$

- The approximate solution has some good condition numbers

$$\mathcal{M}_1(x_0) \leq m_1, \quad \dots \quad , \mathcal{M}_n(x_0) \leq m_L$$

- The error in the invariance equation is smaller than an explicit function of the condition numbers.

$$\varepsilon \leq A(m_1, \dots, m_L)$$

Then,

- There is a true solution:

$$\exists x^*, \mathcal{F}(x^*) = 0$$

- The solution is close to the approxiate solution.

$$||x^* - x_0||_2 \leq C\varepsilon$$

  Also that the solution is locally unique in a ball of radius $\tilde{C}\varepsilon$ around x_0.

Theorems of this form are very common in numerical analysis, specially in the numerical treatment of elliptic PDE.
We want to discuss theorems of these form in the theory of quasi-periodic solutions.

Note that the a-posteriori theorems do not need to justify how you got the approximate solution.
The only thing that you need to verify are estimates on the error (and the non-degeneracy conditions).
The non-degeneracy conditions we seek do not involve global properties of the system. They only involve computations on the approximate solution considered.
In the case of quasiperiodic solutions the functional to be discussed is just

$$\mathcal{F}[K] \equiv f \circ K - K \circ T_\omega$$

The hypothesis do not require that the system is close to integrable. We recover the usual results for quasi-integrable systems. (Take the exact solution for the integrable system as an approximate solution for the quasi-integrable system).

There exist numerical techniques (interval analysis in function space) which allow one to compute upper bounds to the norm of the functional on a trig polynomial as well as upper bounds to the non-degeneracy conditions.

Roughly, one can take systematically account of all sources of error (truncation, round off).

Hence, one can obtain rigorous estimates from numerics. Applying the theorem one gets a computer assisted proof.

There are other ways of producing approximate solutions, such as formal asymptotic expansions which produce approximate solutions.
Studying the properties of the expansions one can obtain results that justify the expansions even if they are not convergent.

These a-posteriori theorems are analogues of “shadowing theorems”. They tell us that near computed solutions, there are true solutions.
The standard shadowing theorems are proved under “hyperbolicity conditions”.
Here we use the opposite: (The systems involve rotations whose separations does not grow in time).
The results we stablish are better that the hyperbolic shadowing theorems, since they also include statemnts about the measure of phase space occupied by the shadowing solutions.

These *a-posteriori* theorems are often proved by describing an iterative method and giving conditions that ensures that it converges to a solution. We will mainly use:

- Newton method
- Newton method + smoothing (Nash-Moser methods).

Of course, a precise formulation requires some assumptions on invertibility and tameness.

The Newton method is more efficient numerically than a contraction mapping principle close to breakdown.

The iterative method can be implemented as a numerical method.

Having an a-posteriori theorem allows to compute with confidence even close to breakdown.

Note that when we have an a-posteriori theorem, we can easily justify any method that produces approximate solutions on which we can justify the non-degeneracy conditions.

- Calculate with confidence very close to breakdown.
- Numerical calculations $\Longrightarrow$ rigorous results.
- One can deduce results from singular perturbation expansion, justify results of bifurcations.
- Tori with extra properties (adjusting parameters). Notably *non-twist tori* which are design goals.

The a-posteriori format also has some theoretical consequences (many of them pointed out in Moser 67).

- differentiability with respect to parameters
- analyticity (taking complex parameters), hence convergence of formal series
- A-posteriori results for analytic systems imply a-posteriori results for finitely differentiable systems. (regularity is not optimal using standard methods).
- Monogenic properties; Whitney differentiabity when parameters (e.g. the set of frequencies) has empty interior)

- In many cases, one can prove local uniqueness by using just interpolation inequalities
- One can prove bootstrap of regularity.
  All solutions of a critical regularity ar as regular as allowed by the regularity of the problem.

In summeary:

- We will describe a (very efficient) algorithm to improve solutions.
- Give sufficient conditions that guarantee that it converges adn that the true solution is close to the computed one.

This allows to perform rigorous continuation till breakdown:

- Start from something you know.
- Change parameters
- Iterate the method till you can verify that you have a solution for the new parameter.
- Change parameters again and take the solution you have as an approximate solution. (Repeat)

The usual caveats of continuation methods: If the change method does not succeed in a few iterations, repeat with a smaller change of parameters. Give up when you do not succeed even with small changes. Note that, we have a completely reliable criterion to declare that our solution is correct. No need for reruns or ad-hoc checks.

Three different stages:

1. Prove an a-posteriori theorem.
2. Implement numerical algorithms, run them and interpret the results.
3. Implement formal expansions, analyze them, interpret the results.

To formulate properly a theorem, we need to make precise the norms we will use:

Definition

Given $\rho > 0$, *we denote by* $\mathbb{T}^n_\rho$ *the set*

$$\mathbb{T}^n_\rho = \{z = x + iy \in \mathbb{C}^n/\mathbb{Z}^n : \ x \in \mathbb{T}^n, \ |y_j| \le \rho \ , \quad j = 1, ..., n\} \ .$$

Given $\rho > 0$, *we denote by* $\mathcal{A}_\rho$ *the set of functions which are analytic in* $\mathrm{Interior}(\mathbb{T}^n_\rho)$ *and continuous on the* $\mathbb{T}^n_\rho$*.*
We endow $\mathcal{A}_\rho$ *with the norm*

$$\|f\|_{\mathcal{A}_\rho} = \sup_{z \in \mathbb{T}^n_\rho} |f(z)| \ . \tag{2}$$

Definition

Given $m > 0$ *and denoting the Fourier series of a function* $f = f(z)$ *as* $f(z) = \sum_{k\in\mathbb{Z}} \widehat{f}_k \exp(2\pi ikz)$*, we define the space* H^m *as*

$$H^m = \left\{ f : \mathbb{T}^n \to \mathbb{C} \;:\; \|f\|_m \equiv \left(\sum_{k\in\mathbb{Z}^n} |\,\widehat{f}_k\,|^2 (1 + |k|^2)^m \right)^{1/2} < \infty \right\} . \quad (3)$$

Diophantine numbers

We denote by $\omega \in \mathbb{R}^n$ the *frequency* of motion, which we assume to satisfy the Diophantine condition

$$|\omega \cdot q - p| \ \geq \ \nu |q|^{-\tau} \ , \qquad p \in \mathbb{Z} \ , \quad q \in \mathbb{Z}^n \backslash \{0\} \ , \tag{4}$$

for suitable positive real constants $\nu \leq 1$, $\tau \geq 1$. The corresponding set of Diophantine vectors is denoted by $\mathcal{D}_n(\nu, \tau)$. If the dimension of the space is obvious, we will omit the subindex n.

The a-posteriori theorem is proved by describing a Newton method which starting with an approximate solution produces a much more approximate one.
The method will not use transformation theory. Only use the embedding and geometric identities.

- We only need functions of as many dimensions as the dimension of the torus, not the phase space. (*curse of dimensionality*).
- The functions we consider do not need to enforce any geometric constraint.
- The step will be quadratically convergent.
- If the function is discretized by N numbers, A step requires
  - Storage $O(N)$
  - Operations $O(N \ln(N))$.
- The basic step admits very easy implementations (about 50 lines in Matlab). Of course, much more work – and theory! – is needed to assess errors, get results close to boundary.

Conformaly symplectic systems

A concrete case where all of this has been implemented.
Given a symplectic manifold M with symplectic form Ω we say that $f : M \to M$ is conformally symplectic when:

$$f^*\Omega = \lambda\Omega \qquad \lambda \in \mathbb{R}$$

Conformally symplectic maps appear in:

- Differential geometry: Banyaga, Agrachev.
- Mechanical systems with friction proportional to the velocity (e.g. spin-orbit problem in Celestial Mechanics).
- Euler-Lagrange equations for discounted functionals

$$\mathcal{S}(u) = \sum_{i=0}^{\infty} L(u_i, u_{i+1}) e^{-ri}$$

  when L satisfies the Legendre condition. $\partial_1 \partial_2 L$ invertible.
  Similarly for continuos time models

$$\int_0^{\infty} L(x(t), x'(t)) e^{-rt}\, dt$$

- Gaussian thermostats
- Noose-Hoover dynamics
- Hamilton-Jacobi equations with zero order terms

$$H(x, Du) + u = 0$$

- The assumption that the system is conformally symplectic is much stronger than the system is dissipative (most of dissipative systems are not conformally symplectic).
- The results we will obtain for the more restrictive hypothesis are stronger than those obtained for general dissipative systems
  See Moser 67, Broer-Huitema-Sevryuk for the role of other geometric assumptions.

- The case $\lambda = 1$ is the symplectic case.
  It is, however a very singular limit and it has surprises. Study this singular limit will be a good test case of the philosophy (validating singular expansions).
  In the financial formulation is the limit of zero discount (studied recently by Iturriaga, Fathi, Siconolfi).
  Note that conformally symplectic systems is a graded Lie algebra (grading is λ) This algebraic structure changes completely the parameter count and it is at the root of unexpected phenomena even in the near integrable case (Moser 67, Broer Huitema Sevryuk Lie algebra approach to near integrable KAM).

- Conformally symplectic maps have a variational principle, a weak KAM theory and a converse KAM theory.
- In many cases, they are maps deviating the vertical.
- They are very rigid. No Birkhoff invariants in neighborhood of Lagrangian tori.
- Lagrangian tori have a very interesting interaction with the theory of normally hyperbolic manifolds. The Lyapunov exponents are locked (in particular, bounded away from zero). So, the only way that normal hyperbolcity is lost is that the angles between tangent and stable bundles go to zero.
- Empirically, it is found that the breakdown is ruled by scaling laws Mathematical theory to be developed completely. Pioneering work by D. Rand

Main idea for the iterative algorithm

Device a (quasi) Newton method to study (1).
This Newton method will:

- Take advantage of some geometric properties that lead to some *"magic"* cancellations in the linearized equation.
- Adjust parameters for the map. (The parameters will be part of the unknowns of equation).
- It will depend on some non-degeneracy conditions to deal with zero divisors.

Statement of the main theorem, assumptions

Assume

H1 Let $\omega \in \mathcal{D}_n(\nu, \tau)$
H2 Let f_μ be a family of conformally symplectic mappings with respect to a symplectic form Ω, that is $f_\mu^*\Omega = \lambda\Omega$ with λ constant.

Statement of the main theorem, assumptions

approximate solution Let $K_0 : \mathbb{T}^n \to \mathcal{M}$, $\mu_0 \in \mathbb{R}^n$ and define E, such that

$$f_{\mu_0} \circ K_0 - K_0 \circ T_\omega = E \; .$$

H3 Assume that the following non–degeneracy condition holds:

$$\det \begin{pmatrix} \overline{S} & \overline{S(B_b)^0} + \overline{\widetilde{A}_1} \\ (\lambda - 1)\mathrm{Id} & \overline{\widetilde{A}_2} \end{pmatrix} \neq 0 \; , \tag{5}$$

where S is an algebraic expression involving derivatives of K_0 written explicitly in the paper. $\widetilde{A}_1$, $\widetilde{A}_2$ denote the first and second n columns of the $2n \times n$ matrix $\tilde{A} = M^{-1} \circ T_\omega D_{\mu_0} f_{\mu_0} \circ K_0$, $(B_b)^0$ is the solution of $\lambda(B_b)^0 - (B_b)^0 \circ T_\omega = -(\widetilde{A}_2)^0$. We denote by

$$\mathcal{T} \equiv \left\| \begin{pmatrix} \overline{S} & \overline{S(B_b)^0} + \overline{\widetilde{A}_1} \\ (\lambda - 1)\mathrm{Id} & \overline{\widetilde{A}_2} \end{pmatrix}^{-1} \right\|$$

Statement of the main theorem, assumptions in analytic case

A) Analytic case:
Assume **H1–H3** and that $K_0 \in \mathcal{A}_\rho$ for some $\rho > 0$. Assume furthermore that for $\mu \in \Lambda$, Λ being an open set in $\mathbb{R}^n$, we have that f_μ is a C^1–family of analytic functions on a domain – open connected set – $\mathcal{C} \subset \mathbb{C}^n/\mathbb{Z}^n \times \mathbb{C}^n$ with the following assumption on the domain.
H4 There exists a $\zeta > 0$, so that

$$\operatorname{dist}(\mu_0, \partial\Lambda) \geq \zeta \tag{6}$$

$$\operatorname{dist}(K_0(\mathbb{T}^n_\rho), \partial\mathcal{C}) \geq \zeta \tag{7}$$

Statement of the main theorem, analytic case

Furthermore, assume that the solution is sufficiently approximate in the following sense.

H5 We assume that, for some $0 < \delta < \rho/2$, E satisfies the inequality

$$||E||_{\mathcal{A}_\rho} \leq C\,\nu^{2\ell}\,\delta^{2\ell\tau}\ ;$$

here and below C denotes a constant that can depend on τ, n, $\mathcal{T}$, $\|DK_0\|_\rho$, $\|N\|_\rho$, $\|M\|_\rho$ $\|M^{-1}\|_\rho$, as well as on ζ entering in **H4**. In such a case, ℓ takes the value 2. If we allow C to depend on λ with $|\lambda| \neq 1$, we can take $\ell = 1$.

Then, there exists μ_e, K_e such that

$$f_{\mu_e} \circ K_e - K_e \circ T_\omega = 0\ . \tag{8}$$

The quantities K_e, μ_e satisfy

$$\begin{aligned} ||K_e - K_0||_{\mathcal{A}_{\rho-\ell\delta}} &\leq C\,\nu^{-\ell}\,\delta^{-\ell\tau}\,||E||_{\mathcal{A}_\rho} \\ |\mu_e - \mu_0| &\leq C\,||E||_{\mathcal{A}_\rho}\ . \end{aligned} \tag{9}$$

The argument also leads to

- Local Uniqueness
- Existence of Lindstedt series

Some easy corollaries:

- Bootstrap of regularity: For an analytic map, all solutions with high enough Sobolev regularity are analytic.
- Smooth dependence on parameters.
- Study of the limit $\lambda \to 1$. Including (Gevrey asymptotics).
- A numerical continuation algorithm
- A criterion for breakdown of analytic circles: At breakdown, either some of the non-degeneracy conditions fails or the Sobolev norms blow up.

Idea of the proof

We start with

$$f_\mu \circ K - K \circ T_\omega = E \tag{10}$$

The Newton method would require to solve

$$Df_\mu \circ K\Delta - \Delta \circ T_\omega + \partial_\mu f_\mu \circ K\delta = -E$$

The key is that there is a frame of reference near the approximate torus that makes this equation into constant coefficient equations.

Taking derivatives of the initial approximation, we obtain

$$Df_\mu \circ K DK = DK \circ T_\omega + DE$$

Using the geometric structure, one gets that the complementary directions also transform well.

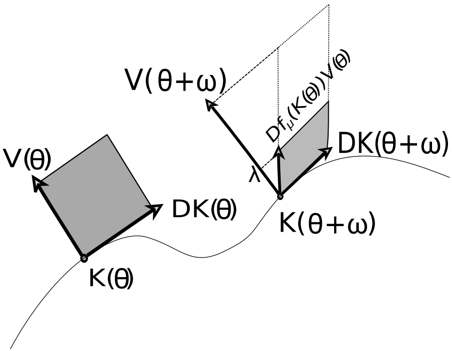


Figure: The preservation of a geometric structure leads to a frame of coordinates which transforms well

One therefore, obtains an *explicit* matrix M [1] which satisfies

$$Df \circ K(\theta)M(\theta) = M(\theta+\omega)\begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} + E_R$$

Where $E_R = O(E)$.
The checking only involves algebraic operations, and the fact that approximately invariant parameterizations are approximately Lagrangian.

[1] Formed out of derivatives of K, the symplectic matrix, derivatives of f using algebraic manipulations

If we change the unknown of the Newton equation to

$$\Delta = MW$$

the Newton equation becomes

$$(Df_\mu) \circ K\, MW - M \circ T_\omega\, W \circ T_\omega + (\partial_\mu f_\mu) \circ K\, \delta = -E \qquad (11)$$

This is equivalent to

$$M \circ T_\omega \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} W + E_R W + M \circ T_\omega W \circ T_\omega + \partial_\mu f_\mu \circ K\delta = -E$$

The equation

$$M \circ T_\omega \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} W + M \circ T_\omega W \circ T_\omega + \partial_\mu f_\mu \circ K \delta = -E$$

can be reduced to solving difference equations of the standard form in KAM theory.

Write $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$.

Then W_2 satisfies a cohomology equation, choose δ so that the average is fixed. Choose the avertage of W_2 so that the equation for w_1 is solvable.

The constant coefficients "cohomology equations"

$$W(\theta) - \lambda W(\theta + \omega) = A(\theta)$$

Can be solved using Fourier coefficients

$$\hat{W}_k = \hat{A}_k/(1 - \lambda e^{2\pi i k\cdot\omega})$$

Note that $\lambda - 1$ is special becase $k = 0$ can only be solved if $\hat{A}_0 = 0$. This is where Diophantine conditions enter.

Algorithm

Given $K : \mathbb{T}^n \to \mathcal{M}$, $\mu \in \mathbb{R}^n$, *we denote by* $\lambda \in \mathbb{R}$ *the conformal factor for* f_μ. *We perform the following computations:*

1) $E \leftarrow f_\mu \circ K - K \circ T_\omega$
2) $\alpha \leftarrow DK$
3) $N \leftarrow [\alpha^T \alpha]^{-1}$
4) $M \leftarrow [\alpha, J^{-1} \circ K \alpha N]$
5) $\beta \leftarrow M^{-1} \circ T_\omega$
6) $\widetilde{E} \leftarrow \beta E$
7) $P \leftarrow \alpha N$

   $A \leftarrow \lambda \, \mathrm{Id}$

   $\gamma \leftarrow \alpha^T J^{-1} \circ K \alpha$

   $S \leftarrow (P \circ T_\omega)^T Df_\mu \circ K \; J^{-1} \circ KP - (N \circ T_\omega)^T (\gamma \circ T_\omega)(N \circ T_\omega) A$

$\tilde{A} \leftarrow M^{-1} \circ T_\omega \; D_\mu f_\mu \circ K$

8) $(B_a)^0$ solves $\quad \lambda (B_a)^0 - (B_a)^0 \circ T_\omega = -(\widetilde{E}_2)^0$

$(B_b)^0$ solves $\quad \lambda (B_b)^0 - (B_b)^0 \circ T_\omega = -(\widetilde{A}_2)^0$

9) Find $\overline{W}_2$, σ solving

$$0 = -\overline{S}\,\overline{W_2} - \overline{S(B_a)^0} - \overline{S(B_b)^0}\sigma - \overline{\widetilde{E}_1} - \overline{\widetilde{A}_1}\sigma \tag{12}$$

$$(\lambda - 1)\overline{W_2} = -\overline{\widetilde{E_2}} - \overline{\widetilde{A}_2}\sigma \;. \tag{13}$$

10) $(W_2)^0 = (B_a)^0 + \sigma (B_b)^0$

11) $W_2 = (W_2)^0 + \overline{W}_2$

12) $(W_1)^0$ solves $(W_1)^0 - (W_1)^0 \circ T_\omega = -(SW_2)^0 - (\widetilde{E}_1)^0 - (\widetilde{A}_1)^0\sigma$

13) $K \leftarrow K + MW$

$\mu \leftarrow \mu + \sigma \;.$

One can prove local uniqueness easily. After some normalizations the solutions of the linearzed equation are unique. (Note that if K, μ is a solution of invariance, we have that K_σ, μ is also a solution. We need to get rid of that.
One can also show that the solutions of the linearized equation are unique up to addition of a constant to W_1, which is fixed by the normalizations.

So we conclude that if there are two solutions $\mathcal{F}(K_1, \mu_1) = \mathcal{F}(K_2, \mu_2) = 0$. Using Taylor's formula we have

$$\begin{aligned} 0 &= \mathcal{F}(K_2, \mu_2) = F(K_1, \mu_1) + D\mathcal{F}(K_1, \mu_1)(K_2 - K_1, \mu_2 - \mu_1) + R \\ &= D\mathcal{F}(K_1, \mu_1)(K_2 - K_1, \mu_2 - \mu_1) + R \end{aligned}$$

where R is quadratic.

Using the uniquess of the normalized solutions of the linearized equation and the estimates for the solutions and Hadamard's three circle theorem, we obtain.

$$
\begin{aligned}
||K_1 - K_2||_{\rho-\delta} + |\mu_1 - \mu_2| &\le C\delta^{-A}(||K_1 - K_2||_{\rho} + |\mu_1 - \mu_2|)^2 \\
&\le C\delta^{-A}(||K_1 - K_2||_{\rho+\delta} + |\mu_1 - \mu_2|)(||K_1 - K_2||_{\rho-\delta} + |\mu_1 - \mu
\end{aligned}
$$

Hence if

$$
C\delta^{-A}(||K_1 - K_2||_{\rho+\delta} + |\mu_1 - \mu_2|) < 1
$$

we obtain $K_1 = K_2; \mu_1 = \mu_2$.
In the strictly dissipative case, there is a simpler dynamical argument.

It is important to observe:

- The algorithm involves only manipulation of functions with as many variables as the torus.
- We are performing:
  - compositions, multiplications, additions, inverting matrices
  - shifting derivatives, solving cohomology equations with constant coefficients

• These operations require $O(N)$ operations in Discretizations in a grid Fourier discretization

• These operations work as well or even better for complex numbers **and complex frequencies**.

The theorem and its proof do not require any change for complex frequencies.

The hypothesis of the theorem (approximate solution) + non-degeneracy can be readily verified in a concrete system given a trig. polynomial numerical approximation in a concrete system.
The tesis of D. Rana (20 years ago, contained upper and lower bounds for Siegel disks, differing in 10^{-3} and for standard maps differing by 3×10^{-2}.
Compare with upper bounds by McKay-Percival, Jungreis for twist mappings (1-D), M. Muldoon in 4-D symplectic mappings.

For whiskered tori, J.-L. Figueras, A. Haro get proofs which are 10^{-6} rigorous estimates close to the expected values.

A recent paper J. L. Figueras, A. Haro, A Luque in FOCM, gets

- For the standard map, golden mean, upper and lower bounds differering by less than 10^{-4}.
- Lower bounds for 2-D maps (no comparable upper bounds).
- Lower bounds in maps which are not twist

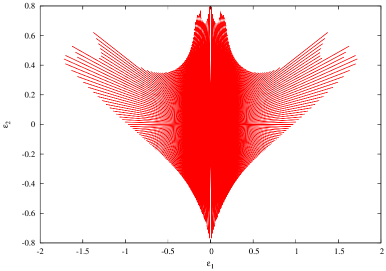
For conformal symplectic mappings (Calleja, Celletti, R.L) reasonable lower bounds (no interval) for some conformally symplectic systems.

Some numerical studies of domain of existence, based on criteria of blow up after Calleja-Celletti and Calleja-Figueras

Study of the domain of parameters for which there is a golden mean invariant circle for the map

$$
\begin{aligned}
p' &= p + \frac{\epsilon_1}{2\pi}\sin(2\pi q) + \frac{\epsilon_2}{4\pi}\sin(4\pi q) \\
q' &= q + \lambda p' + \mu
\end{aligned}
$$

lambda = 0.9



0.8
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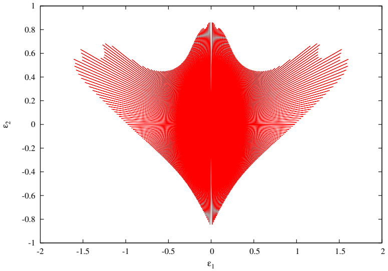
2
ε_1

$lambda = 0.5$



1
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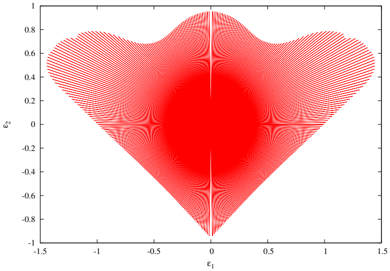
$lambda = 0.1$



Some surprises that demand new mathematics

- Loss of hyperbolicity

At breakdown, the circle remains $C^{1+\delta}$.
The Lyapunov transverse exponent is λ.
The loss of hyperbolicity can only come from the fact that the stable bundle becomes tangent to the invariant manifold.
(The circle is analytic up to breakdown, but the analyticity domain collapses and indeed many derivatives blow up.)

Numerical observations waiting for proofs:

- The distance between the bundles $\approx A(\epsilon_c - \epsilon)$ (Similar results in other contexts proved by K. Bjerklov, M. Saprykina. Their deep method seems to be rather generalizable.)
- $$\lambda(\epsilon) \approx \lambda(\epsilon_c) + A(\epsilon_c - \epsilon)^{1/2}$$
- $$||D^r K_\epsilon||_{L^2} \approx (\epsilon_c - \epsilon)^{\alpha - \beta r}$$
- In multiparameter families, the boundary includes smooth manifolds.
- In systems with other geometries, Haro and Figueras have identified many other mechanisms.

It seems that it will be interesting to look for new more challenging applications **(Please, add more, give feedback!!)**

New Areas:

- Systems with more degrees of freedom, including PDE's.
  - Evolutionary PDE
  - Elliptic PDE
  - Lattice systems.
  - State dependent delay equations.
- Chemistry is full of systems that have high degeneracies and resonances
- Solid State Physics problems with non-local interactions, pinning/depinning phenomena.
- Multiscale systems.

New Mathematics:

- Other objects (See the recent book of A. Haro, M. Canadell, J. L. Figueras, J. M. Mondelo
- Other geometrical problems (systems with constraints, non-holonomies)
- Study bifurcations, singular perturbations,

New Mathematics (cont.)

- Use more design parameters to get tori with more desirable properties (non-twist tori).
- Better converse KAM theory in higher dimensions.
  (some a-priori bounds of Aubry-Mather theory are false).
- Describe rigorously:
  - Effect of external noise in dynamics
  - Errors in parameters
  - Non-autonomous pulses
  - Incorporate temperature in variational methods
- Tori with different topologies.
  - Systematic numerics
  - Asymptotics in the area covered
  - Statistics in multiplarticle systems (Conjecture. In multiparticle systems secondary tori are much more abundant.)
- Relations with homogenization theory.
  For periodic media, the cell problem is very similar to the invariance equation. For quasi-periodic or random media, many bounds are false.
- Phenomena at breakdown

Computer issues

- Better representations of functions (Sparse representations, other function basis, other norms).
- Standarize the tools. Cooperation with other standard packages leading to rigorous mathematics such as CAPD.
- Higher level languages
- Taking advantage of GPU other architectures.
- Take advantage of new algorithms (frequency maps, fast calculations of rotation numbers.)

A Commercial

We are looking forward to further discussion.
MSRI Program in 2018 (Berkeley)

August 13, 2018 - December 14, 2018

Hamiltonian systems, from topology to applications through analysis

Organizers: Rafael de la Llave (Georgia Institute of Technology), LEAD Albert Fathi (École Normale Supérieure de Lyon), Vadim Kaloshin (University of Maryland), Robert Littlejohn (University of California, Berkeley), Philip Morrison (University of Texas at Austin), Tere M. Seara (Universitat Politècnica de Catalunya), Serge Tabachnikov (Pennsylvania State University), Amie Wilkinson (University of Chicago)

The interdisciplinary nature of Hamiltonian systems is deeply ingrained in its history. Therefore the program will bring together the communities of mathematicians with the community of practitioners, mainly engineers, physicists, and theoretical chemists who use Hamiltonian systems daily. The program will cover not only the mathematical aspects of Hamiltonian systems but also their applications, mainly in space mechanics, physics and chemistry.

The mathematical aspects comprise celestial mechanics, variational methods, relations with PDE, Arnold diffusion and computation. The applications concern celestial mechanics, astrodynamics, motion of satellites, plasma physics, accelerator physics, theoretical chemistry, and atomic physics.

The goal of the program is to bring to the forefront both the theoretical aspects and the applications, by making available for applications the latest theoretical developments, and also by nurturing the theoretical mathematical aspects with new problems that come from concrete problems of applications.

Updated On Jul 25, 2016 04:30 PM PDT