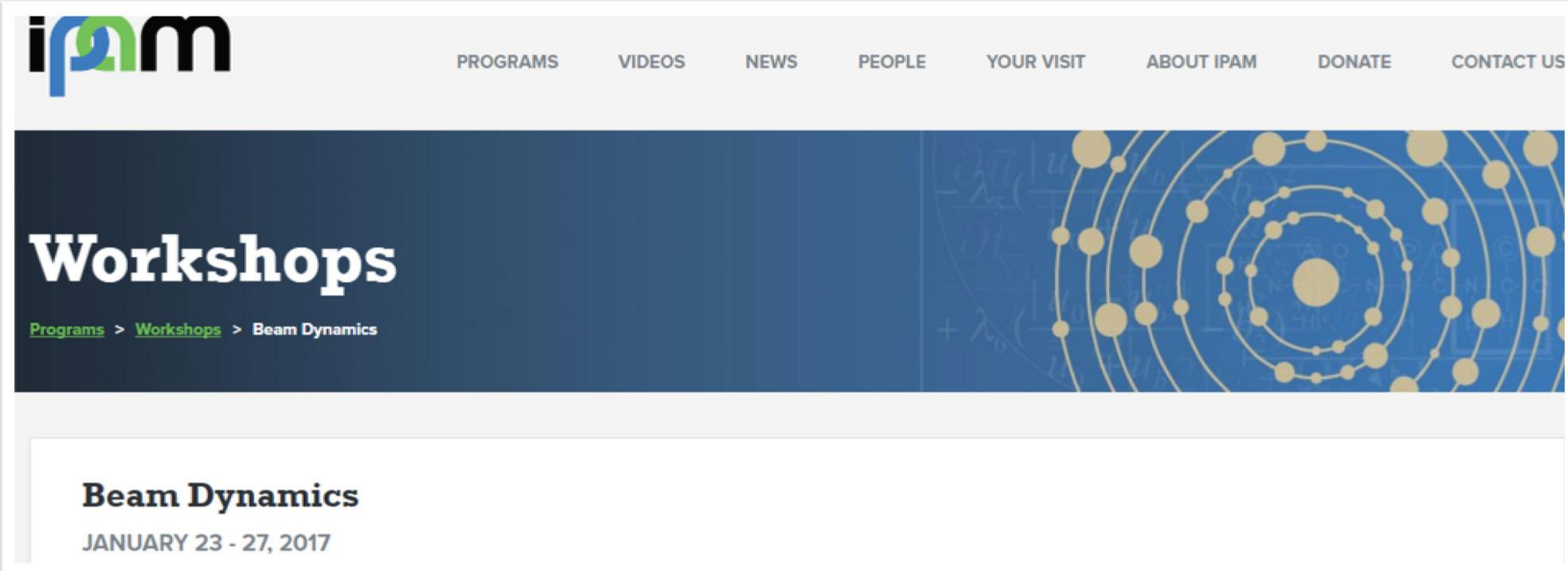


Solving High-Gain FEL systems using Van Kampen's Normal Mode Expansion

Zhirong Huang

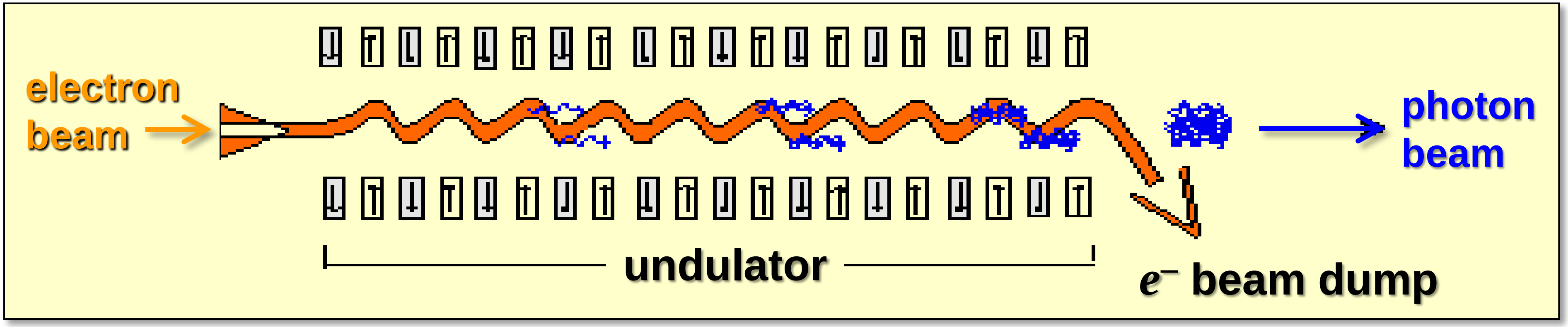
Stanford University and SLAC

Outline

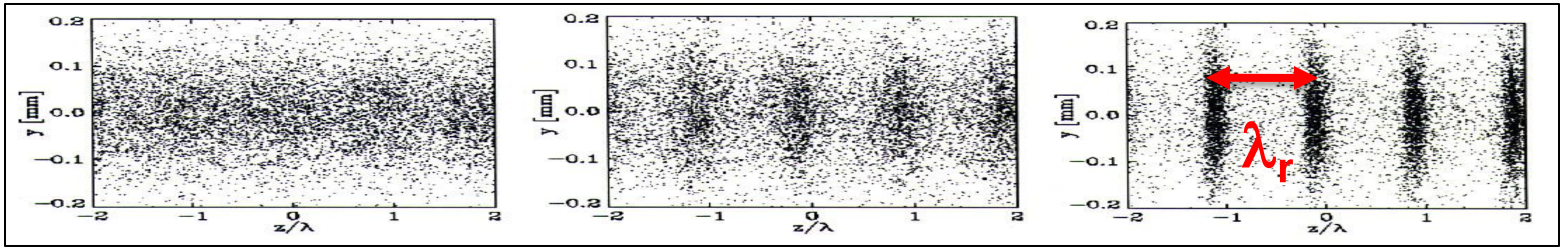
- Introduction
- 3D high-gain theory
- Variable parameter FEL (wakefield+taper)
- Summary

Free Electron Lasers

- Produced by **resonant interaction** of a relativistic electron beam with EM radiation in an undulator (J. Madey, 1971)

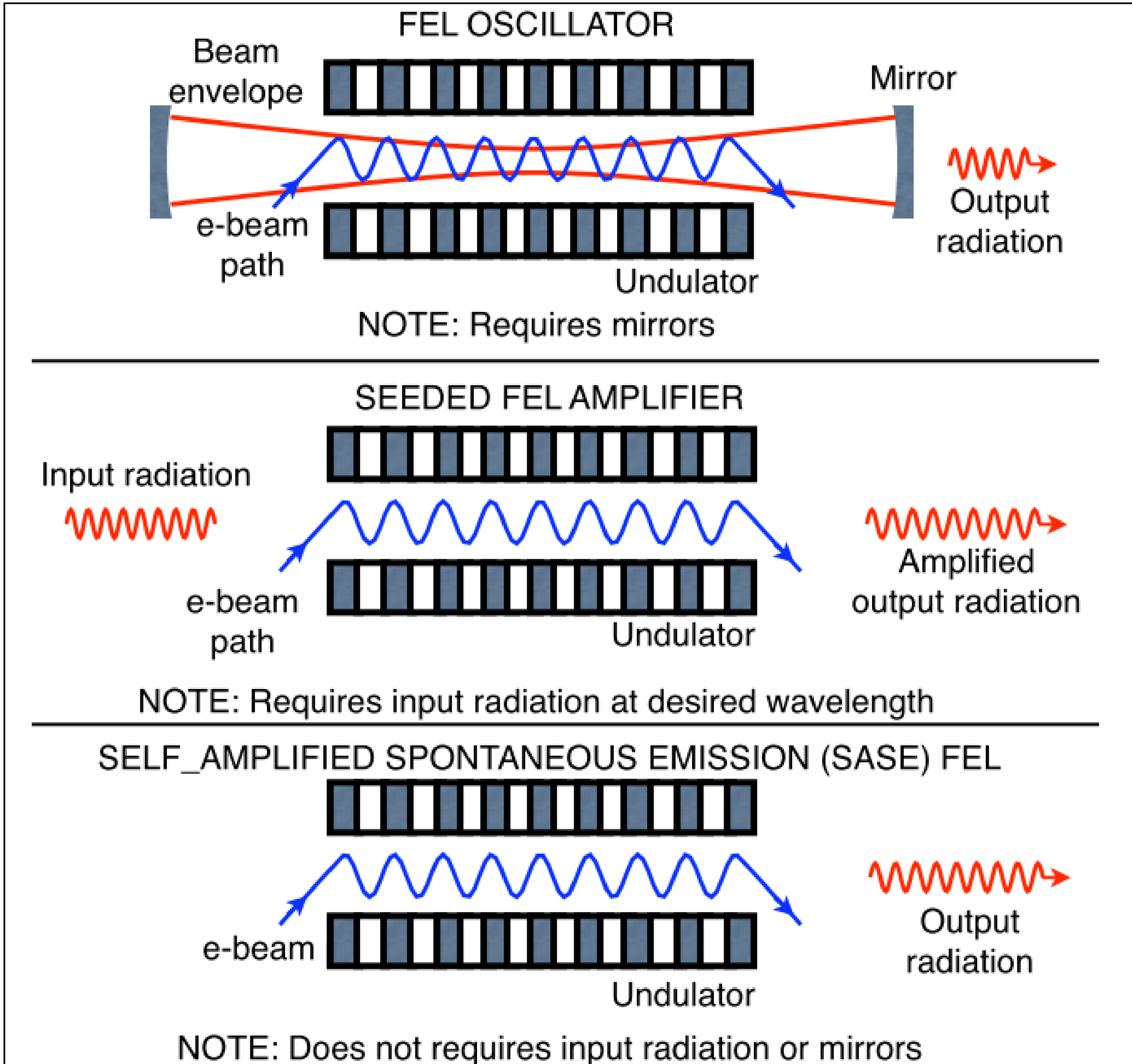


$$\lambda_r = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$



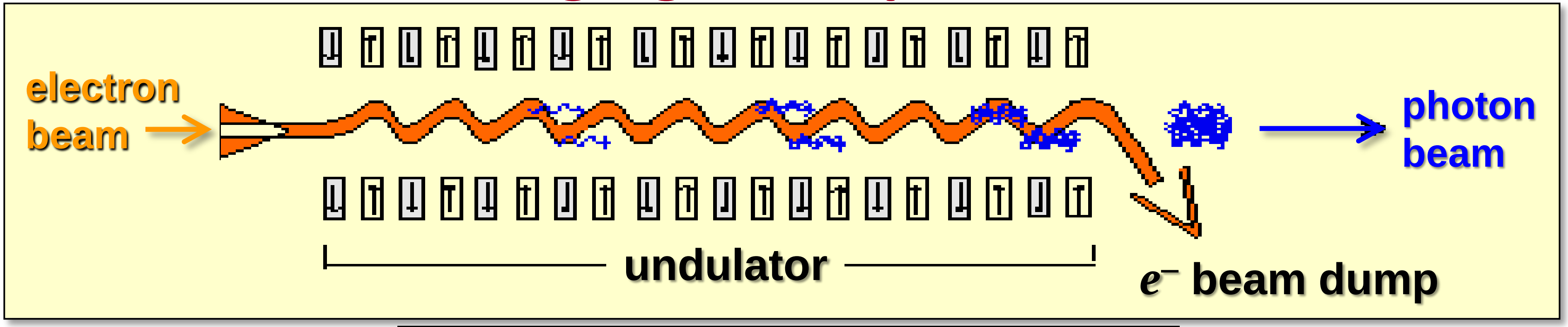
- **Radiation intensity ∝ N^2**
- **Tunable, Powerful, Coherent radiation sources**

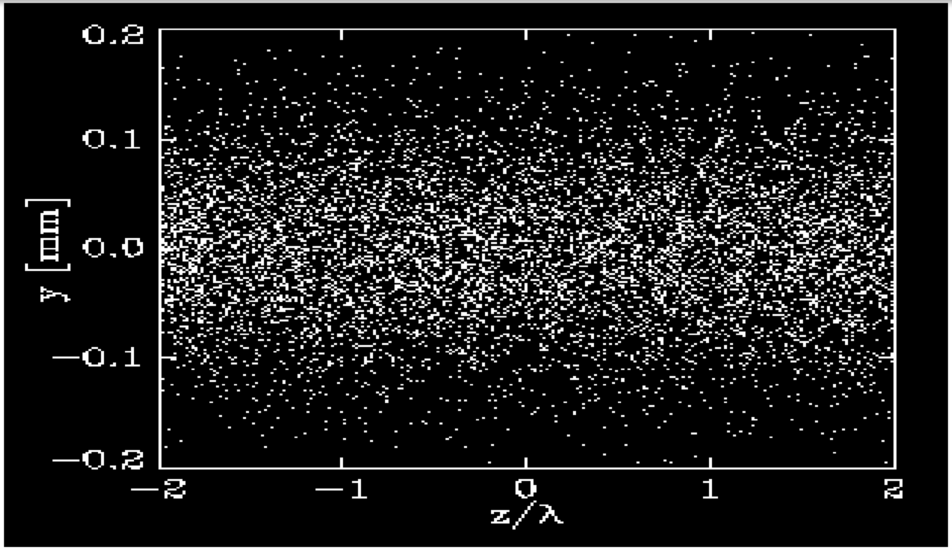
Three FEL modes



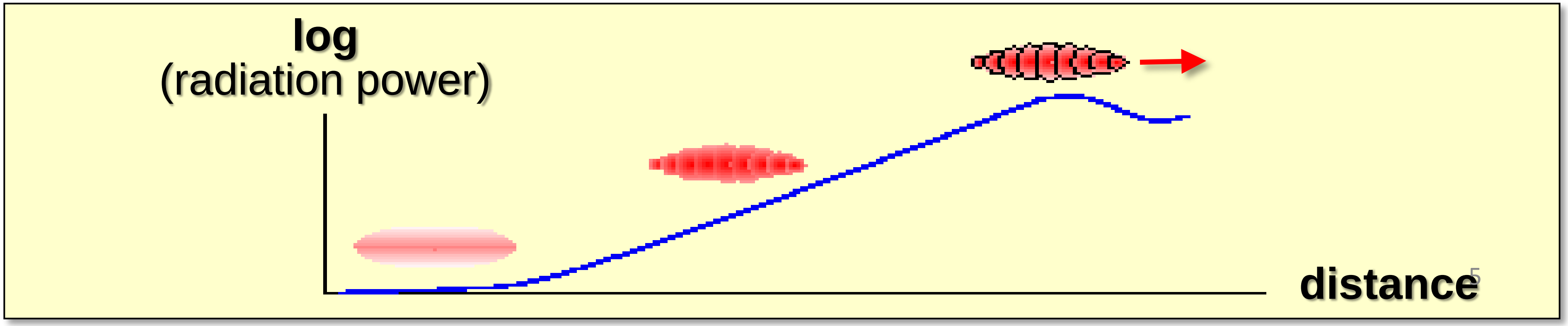
Ryan's talk next

High-gain amplifier

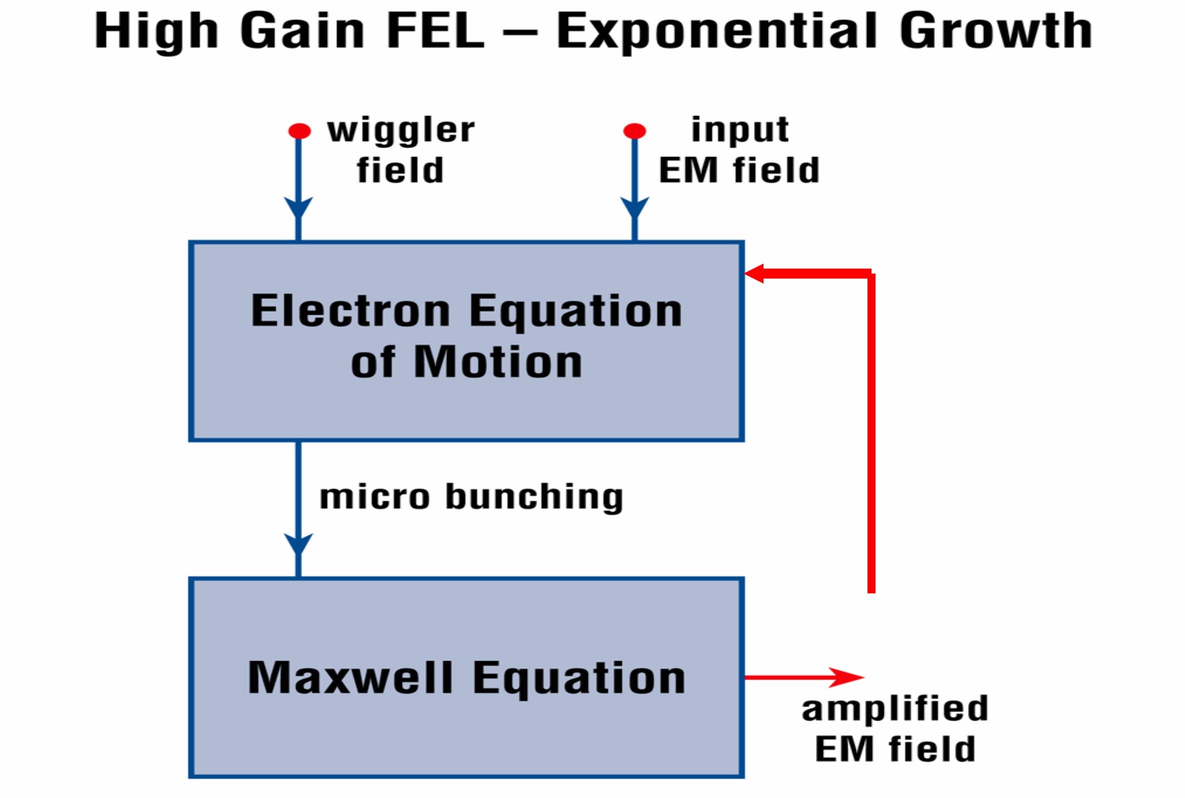




S. Reiche



What a wonderful instability



SASE

1D Theory: exponential growth, proposes SASE

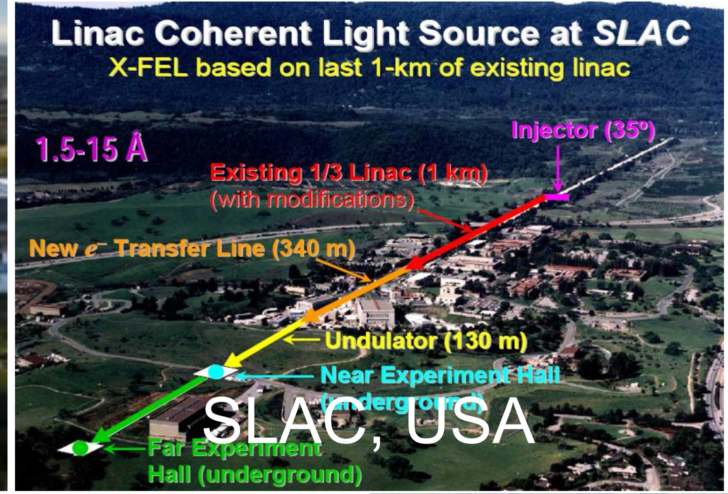
Self Amplified Spontaneous Emission

Saldin et al. (1980)
Bonifacio, Pellegrini et al. (1984)

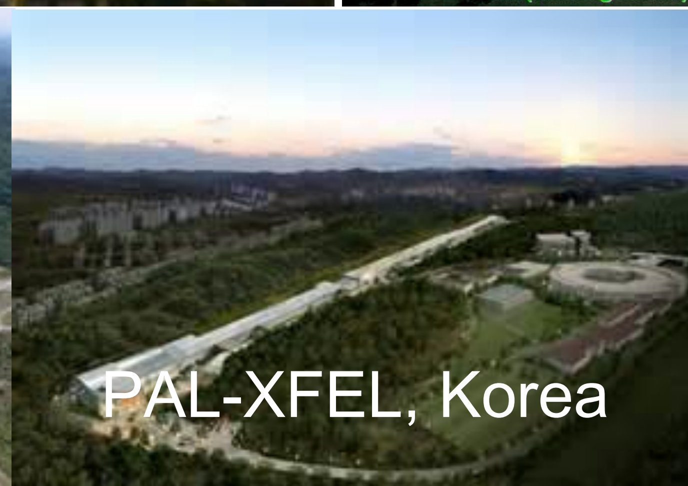
Claudio Pellegrini receives the Fermi Award (2015)

Let me be clear Mr. President: it may be 'noisy' and its not really 'free', but it is a high-gain single-pass device for converting electrons into a lot of x rays, and it is really quite useful…


DESY, Germany


Linac Coherent Light Source at SLAC
X-FEL based on last 1-km of existing linac
1.5-15 Å
Injector (35º)
Existing 1/3 Linac (1 km)
(with modifications)
New e^- Transfer Line (340 m)
Undulator (130 m)
Near Experiment Hall
Far Experiment
Hall (underground)
SLAC, USA


SACLA
Spring-8, Japan


PAL-XFEL, Korea


SwissFEL


FERMI, Italy


SINAP, China

1D FEL Pendulum equation

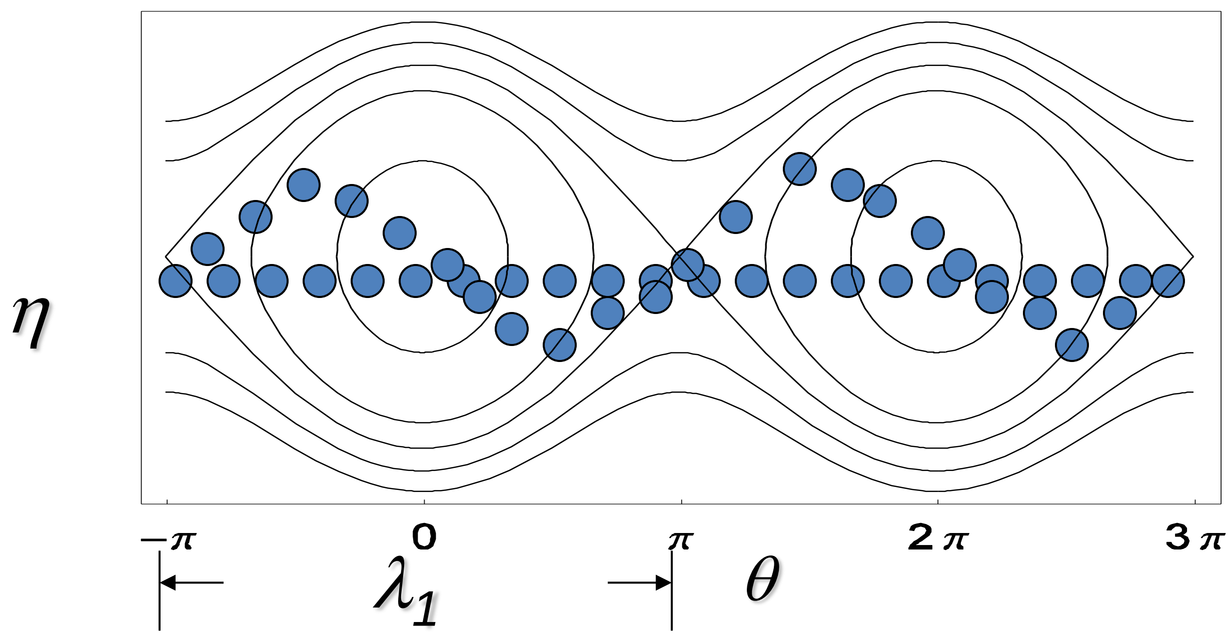
- Longitudinal electron motion in combined undulator and radiation fields described by pendulum equations

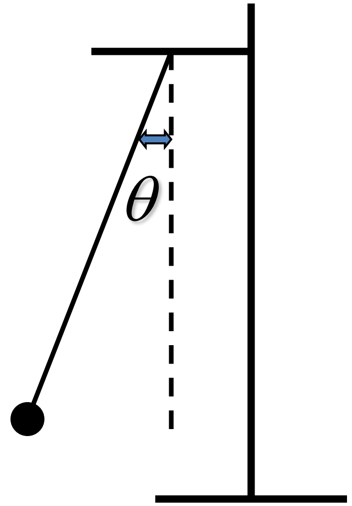
$$\frac{d\theta}{dz} = 2\eta k_u, \qquad \frac{d\eta}{dz} = \chi_1(\tilde{E}e^{i\theta} + \tilde{E}^* e^{-i\theta})$$

$$\chi_1 = \frac{eK[\mathrm{JJ}]}{(2\gamma_0^2 mc^2)}$$

$$[\mathrm{JJ}] = J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)$$ for planar undulator

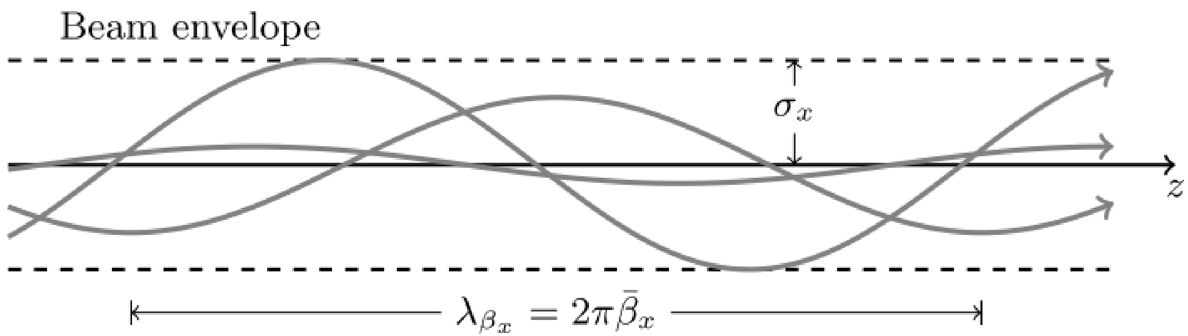
=1 for helical undulator





3D Equation of motion

- In the transverse plane, the electrons perform betatron oscillations, which can be described in the context of the smooth approximation.



- In the longitudinal dimension, one obtains the 3D generalization of the 1D pendulum equations.

$$\frac{d\theta}{dz} = 2k_u\eta - \frac{k_1}{2}(p^2 + k_\beta^2 x^2),$$

$$\frac{d\eta}{dz} = \chi_1 \int d\nu \; e^{i\nu\theta} E_\nu(x; z) + c.c.,$$

$$\frac{dx}{dz} = p, \qquad \frac{dp}{dz} = -k_\beta^2 x,$$

Vlasov-Maxwell formalism I

- The interaction between the electron beam and the FEL radiation can be described in the framework of the Vlasov-Maxwell equations.

- The e-beam is described in terms of a distribution function $F = F(\theta, \eta, \boldsymbol{x}, \boldsymbol{p}; z)$ in 6D-phase space. In view of the importance of stochastic effects such as shot noise, we use the Klimontovich distribution:

$$F(\theta, \eta, \boldsymbol{x}, \boldsymbol{p}; z) = \frac{k_1}{n_e} \sum_{j=1}^{N_e} \delta[\theta - \theta_j(z)] \delta[\eta - \eta_j(z)] \times \delta[\boldsymbol{x} - \boldsymbol{x}_j(z)] \delta[\boldsymbol{p} - \boldsymbol{p}_j(z)],$$

n_e: on-axis electron number density

- The distribution function is governed by the Vlasov equation

$$\frac{\partial F}{\partial z} + \frac{d\theta}{dz}\frac{\partial F}{\partial \theta} + \frac{d\eta}{dz}\frac{\partial F}{\partial \eta} + \frac{d\boldsymbol{x}}{dz} \cdot \frac{\partial F}{\partial \boldsymbol{x}} + \frac{d\boldsymbol{p}}{dz} \cdot \frac{\partial F}{\partial \boldsymbol{p}} = 0,$$

K.-J. Kim, PRL 57, 1871 (1986)

K.-J. Kim, Z. Huang, R. Lindberg, Synchrotron Radiation and FELs (Cambridge Press, 2017)

Vlasov-Maxwell formalism II

- In the linear, exponential-gain regime, a **perturbation approach** is applicable. This process involves:

  - Decomposing the distribution function into a background distribution function $\bar{F}$ and a small perturbation δF i.e. $F = \bar{F} + \delta F$. We then introduce the Fourier amplitude F_ν through $F_\nu = (1/2\pi)\int d\theta(\delta F)e^{-i\nu\theta}$ and $\delta F = \int d\nu F_\nu e^{i\nu\theta}$.

  - Treating F_ν and E_ν as first order (small) quantities.

- After some manipulation (which involves using the equations of motion), we obtain a linearized Vlasov equation:

$$\left\{\frac{\partial}{\partial z} + p\cdot\frac{\partial}{\partial x} - k_\beta^2 x\cdot\frac{\partial}{\partial p} + i\nu\left[2\eta k_u - \frac{k_1}{2}(p^2 + k_\beta^2 x^2)\right]\right\} F_\nu = -\chi_1 E_\nu \frac{\partial}{\partial \eta}\bar{F}$$

Vlasov-Maxwell formalism III

- We also use a driven paraxial wave equation for the radiation field:

$$\left(\frac{\partial}{\partial z} + i\Delta\nu k_u + \frac{\nabla_\perp^2}{2ik_1}\right) E_\nu(x;z) = -\chi_2 \frac{k_1}{2\pi} \sum_{j=1}^{N_e} e^{-i\nu\theta_j(z)} \delta[x - x_j(z)]$$

extra 3D term due to radiation diffraction

$$\chi_2 \equiv eK[\mathrm{JJ}]/2\varepsilon_0\gamma_r$$

- In terms of the distribution function amplitude, the paraxial becomes

$$\left(\frac{\partial}{\partial z} + i\Delta\nu k_u + \frac{\nabla_\perp^2}{2ik_1}\right) E_\nu = -\chi_2 n_e \int dp d\eta \, F_\nu$$

current term now includes momentum integration

- These linearized Vlasov-Maxwell equations accurately describe the FEL operation up to the onset of nonlinear, saturation effects.

Scaled equations

- We introduce a set of convenient scaled quantities

$$\hat{z} = 2\rho k_u z \qquad \hat{\eta} = \frac{\eta}{\rho}, \qquad a_\nu = \frac{\chi_1}{2k_u\rho^2}E_\nu = \frac{eK[\mathrm{JJ}]}{4\gamma_r^2 mc^2 k_u \rho^2}E_\nu,$$

$$\hat{x} = x\sqrt{2k_1 k_u \rho} \qquad \hat{p} = p\sqrt{\frac{k_1}{2k_u\rho}}. \qquad f_\nu = \frac{2k_u\rho^2}{k_1}F_\nu. \qquad \hat{k}_\beta = k_\beta/(2k_u\rho)$$

Pierce-or FEL-parameter

$$\rho = \left[\frac{n_e\chi_1\chi_2}{(2k_u)^2}\right]^{1/3} = \left(\frac{e^2K^2[\mathrm{JJ}]^2 n_e}{32\epsilon_0\gamma_r^3 mc^2 k_u^2}\right)^{1/3} = \left[\frac{1}{8\pi}\frac{I}{I_A}\left(\frac{K[\mathrm{JJ}]}{1+K^2/2}\right)^2\frac{\gamma\lambda_1^2}{2\pi\sigma_x^2}\right]^{1/3}$$

- The linearized FEL equations become

$$\left(\frac{\partial}{\partial\hat{z}} + i\frac{\Delta\nu}{2\rho} + \frac{\hat{\nabla}_\perp^2}{2i}\right)a_\nu(\hat{x};\hat{z}) = -\int d\hat{\eta}d\hat{p}\; f_\nu(\hat{\eta},\hat{x},\hat{p};\hat{z})$$

$$\left(\frac{\partial}{\partial\hat{z}} + i\dot{\theta} + \hat{p}\cdot\frac{\partial}{\partial\hat{x}} - \hat{k}_\beta^2\hat{x}\frac{\partial}{\partial\hat{p}}\right)f_\nu = -a_\nu\frac{\partial\bar{f}_0}{\partial\hat{\eta}},$$

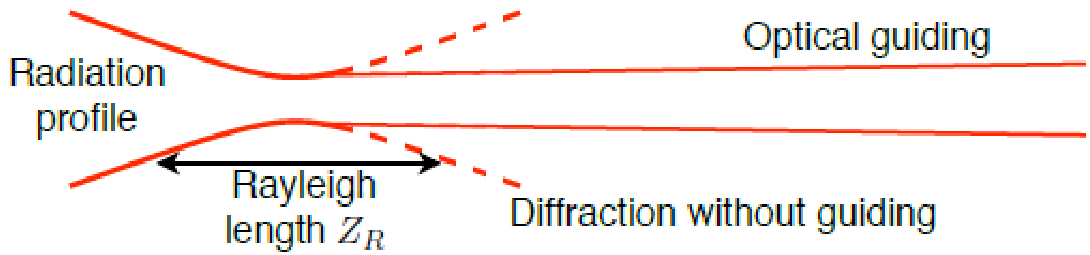
phase derivative $\dot{\theta} = \frac{d\theta}{d\hat{z}} = \hat{\eta} - \frac{\hat{p}^2 + \hat{k}_\beta^2\hat{x}^2}{2}$

Van Kampen's normal mode expansion I

- For such a z-independent case, we seek the self-similar, guided eigenmodes of the FEL. These are solutions of the form:

$$\Psi = \begin{bmatrix} a_\nu(\hat{x};\hat{z}) \\ f_\nu(\hat{\eta},\hat{x},\hat{p},\hat{z}) \end{bmatrix} = e^{-i\mu_\ell \hat{z}} \begin{bmatrix} \mathcal{A}_\ell(\hat{x}) \\ \mathcal{F}_\ell(\hat{x},\hat{p},\hat{\eta}) \end{bmatrix}$$

- They are characterized by a constant growth rate μ_l and a z-independent radiation/density mode profile A_l/F_l (**Optical guiding**)



- Substituting into the Vlasov-Maxwell (FEL) equations, we obtain two coupled relations for the growth rate and the mode amplitudes:

$$\begin{bmatrix} \mu_\ell \mathcal{A}_\ell + \left(-\frac{\Delta\nu}{2\rho} + \frac{1}{2}\hat{\nabla}_\perp^2\right)\mathcal{A}_\ell + i\int d\hat{p}d\hat{\eta}\;\mathcal{F}_\ell \\ \mu_\ell \mathcal{F}_\ell + i\mathcal{A}_\ell \frac{\partial \bar{f}_0}{\partial \hat{\eta}} + \left\{-\nu\dot{\theta} + i\left(\hat{p}\cdot\frac{\partial}{\partial \hat{x}} - \hat{k}_\beta^2 \hat{x}\cdot\frac{\partial}{\partial \hat{p}}\right)\right\}\mathcal{F}_\ell \end{bmatrix} = 0.$$

Van Kampen's normal mode expansion II

- Eigenmode equation $(\mu_\ell + \mathbf{M})\Psi_\ell = 0$
- The matrix operator ***M*** is not Hermitian, eigenvalue μ_ℓ can be complex and associated eigenvectors are not orthogonal.
- Van Kampen's method construct such an orthogonal set using the adjoint eigenvalue equation

$$\left(\mu_\ell^\dagger + \mathbf{M}^\dagger\right)\Psi_\ell^\dagger = 0.$$

such that $\left(\Psi_k^\dagger, \Psi_l\right) = \delta_{k,\ell}\left(\Psi_l^\dagger, \Psi_\ell\right)$

K. M. Case, plasma oscillations, annals of physics 7, 349 (1959)

- Eliminate electron distribution $\mathcal{F}_\ell$, a dispersion relation emerges

$$\left(\mu_\ell - \frac{\Delta\nu}{2\rho} + \frac{1}{2}\hat{\boldsymbol{\nabla}}_\perp^2\right)\mathcal{A}_\ell(\hat{x}) - i\int d\hat{p}\,d\hat{\eta}\int_{-\infty}^{0} d\tau\; e^{i(\nu\dot{\theta}-\mu_\ell)\tau}\frac{d\bar{f}_0}{d\hat{\eta}}\,\mathcal{A}_\ell(\hat{x}_+) = 0.$$

3D solution

- Using a specific $\overline{f}_0$, we obtain an explicit dispersion relation:

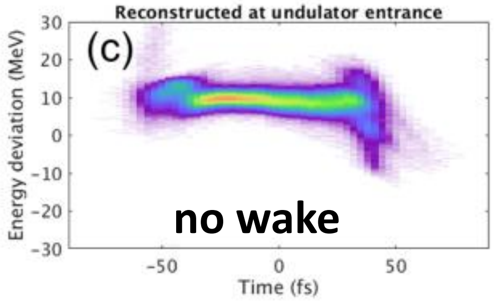
$$\left(\mu - \frac{\Delta\nu}{2\rho} + \frac{1}{2}\hat{\nabla}_\perp^2\right) A(\hat{x}) - \frac{1}{2\pi \hat{k}_\beta^2 \hat{\sigma}_x^2} \int_{-\infty}^{0} d\tau\, \tau e^{-\hat{\sigma}_\eta^2 \tau^2/2 - i\mu\tau}$$
$$\times \int d\hat{p}\, A[\hat{x}_+(\hat{x},\hat{p},\tau)] \exp\left[-\frac{1 + i\tau \hat{k}_\beta^2 \hat{\sigma}_x^2}{2\hat{k}_\beta^2 \hat{\sigma}_x^2}\left(\hat{p}^2 + \hat{k}_\beta^2 \hat{x}^2\right)\right] = 0.$$

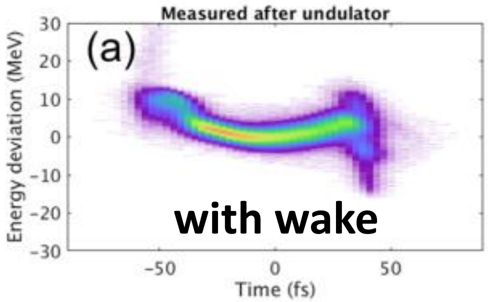
- There are four dimensionless parameters that affect the growth rate:
  - $\hat{\sigma}_x$ is a quantitative measure of the diffraction effect
  - $\hat{\sigma}_x \hat{k}_\beta$ is a measure of the emittance effect
  - $\hat{\sigma}_\eta$ represents the energy spread effect
  - $\Delta\nu/(2\rho)$ is scaled frequency detuning

- Ming Xie and others used a vibrational technique to obtain a fitting formula that captures all these effects for FEL designs

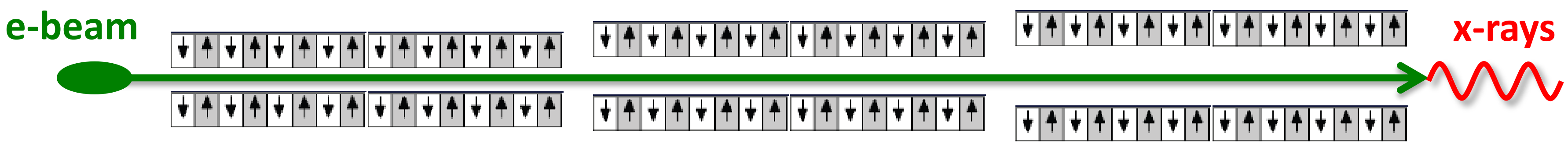
Applications to undulator wakefield and tapering

- Undulator wakefield is an important source of time-dependent energy loss

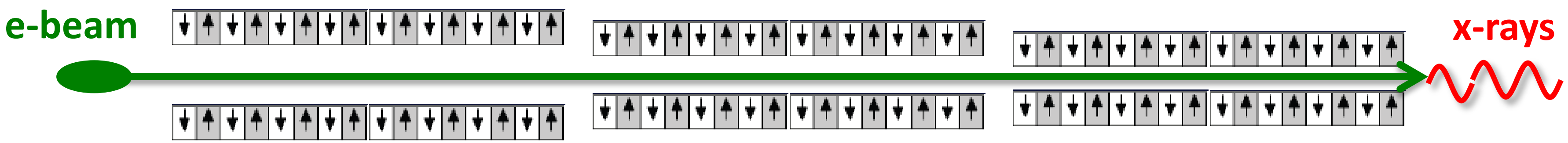




- Compensate the average energy loss by tapering undulator
- Tapered undulator keeps FEL resonance and increase power



- Reverse taper (increasing K) is very useful for certain applications



FEL with slowly varying beam and undulator parameters

- E-beam energy $\gamma_c(z)$, undulator parameter K(z)
- Initial resonant wavelength $\lambda_0 = \frac{2\pi}{k_0} = \frac{\lambda_u}{2\gamma_c(0)^2}\left[1 + \frac{K(0)^2}{2}\right]$
- Resonant energy $\gamma_r(z) = \sqrt{\frac{\lambda_u}{2\lambda_0}\left[1 + \frac{K(z)^2}{2}\right]}$
- Longitudinal motion is described by

$\theta = (k_0 + k_u)z - ck_0 t^*$ (ponderomotive phase)

$\eta = \frac{\gamma(z)-\gamma_c(z)}{\rho\gamma_c(0)}$ (normalized energy, change only due to FEL)

$$\frac{d\theta}{dz} = 2k_u \frac{\gamma(z)-\gamma_r(z)}{\gamma_c(0)} = 2k_u\rho\left[\eta + \frac{\gamma_c(z)-\gamma_r(z)}{\rho\gamma_c(0)}\right]$$

$\frac{d\eta}{dz} \propto E\cos(\theta + \phi)$ (E and ϕ are radiation field and phase)

- *Z. Huang, G. Stupakov, Phys. Rev. ST Accel. Beams 8, 040702 (2005)*

WKB approximation

- Well-known technique in QM for slowly-varying potential
- FEL is characterized by ρ: the relative gain bandwidth is a few ρ, and radiation field gain length ~ $\lambda_u/(4\pi\rho)$
- Relative change in beam energy w.r.t resonant energy

$$\delta(z) = \frac{1}{\rho}\frac{\gamma_c(z) - \gamma_r(z)}{\gamma_c(0)}$$

Normalized to ρ

- Apply WKB technique if the relative energy change per field gain length is smaller than ρ, i.e.,

$$\left|\frac{d\delta}{d\tau}\right| < 1\,, \quad \tau = 2\rho k_u z = \frac{z}{\lambda_u/(4\pi\rho)}$$

- Satisfied for wakefields induced energy change ~ a few ρ over saturation distance (~ 10 field gain length)

Modified Maxwell-Vlasov equations

- Radiation field $a_\nu(\tau)$ at frequency detune $\nu=(\omega-\omega_0)/(2\rho\omega_0)$

$$\frac{da_\nu}{d\tau} + i\nu a_\nu = \underbrace{\int_{-\infty}^{\infty} d\eta f_\nu(\eta;\tau)}_{\text{source current at frequency } \nu}$$

- Fourier component of the distribution function $f_\nu(\eta;\tau)$ satisfies the linearized Vlasov equation following pend. Eqs.

$$\frac{df_\nu}{d\tau} + i[\eta + \delta(\tau)]f_\nu - a_\nu(\tau)\frac{dV}{d\eta} = 0$$

not small — $\delta(\tau)$

$V(\eta)$: initial energy distribution

- In matrix notation $$\frac{d\Gamma}{d\tau} = iM\Gamma$$

$$\Gamma = \begin{pmatrix} a_\nu(\tau) \\ f_\nu(\eta;\tau) \end{pmatrix}, \quad M = \begin{pmatrix} -\nu & -i\int_{-\infty}^{\infty} d\eta \\ -i\frac{dV}{d\eta} & -[\eta + \delta(\tau)] \end{pmatrix}$$

0^{th} order solution

- Seek a solution of the form

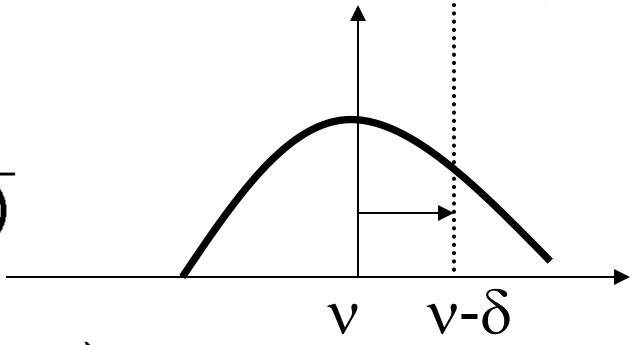
$$\Gamma \approx \exp\left[-i \int_0^{\tau} \mu_0(\tau')d\tau'\right] \Psi_0 \,, \;\; \Psi_0 = \begin{pmatrix} A_0 \\ F_0(\eta;\tau) \end{pmatrix}$$

➔ $-i\mu_0 \Psi_0 = iM \Psi_0$ since $d\Psi_0/d\tau$ is 1^{st} order

- Complex growth rate $\mu=\mu_0(\tau)-\delta(\tau)$ satisfies same 1-D dispersion relation as a constant-parameter FEL if we define

$\nu'(\tau) = \nu - \delta(\tau)$ ← changes with energy

Growth rate $\mathrm{Im}[\mu_0]$



$$\mu(\tau) - \nu'(\tau) = \int d\eta \frac{dV/d\eta}{\eta-\mu(\tau)}$$

- Eigenvector $\Psi_0 \propto \begin{pmatrix} 1 \\ \frac{i}{\mu_0-[\eta+\delta(\tau)]}\frac{dV}{d\eta} \end{pmatrix}$

1st order correction

- To account for τ dependency, introduce corrections

$$\Gamma \approx \exp\left[-i\int_0^{\tau}[\mu_0(\tau)+\mu_1(\tau)]d\tau'\right](\Psi_0+\Psi_1)$$

➔ (μ_1 and Ψ_1) << (μ_0 and Ψ_0), but not $\int_0^{\tau}\mu_1(\tau')d\tau'$

- Insert into $\frac{d\Gamma}{d\tau} = iM\Gamma$

$$\rightarrow \quad \frac{d\Psi_0}{d\tau} - i\mu_1\Psi_0 = i(\mu_0+M)\Psi_1$$

- To find $\mu_1(\tau)$, use Van Kampen's adjoint eigenvector Φ_0

$$\mu_1 = -i\frac{(\Phi_0, d\Psi_0/d\tau)}{(\Phi_0,\Psi_0)}$$

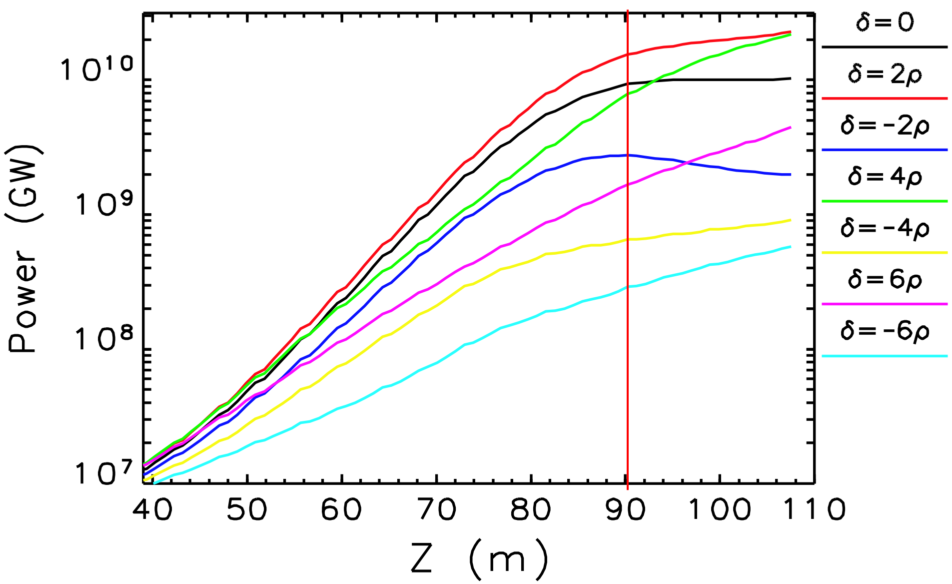
Comparison w/ simulations

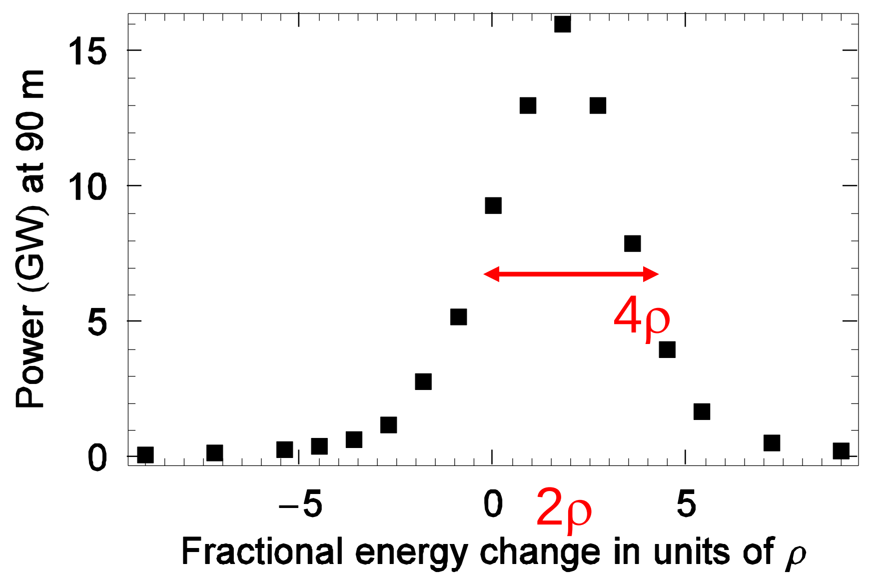
- Radiation power dependence on δ is a gaussian

$$P(\delta; z) = P_m(z)\exp\left[-\frac{1}{2}\left(\frac{\delta(z)-\delta_m(z)}{\sqrt{3}\sigma_\omega/\rho}\right)^2\right]$$

- *GENESIS* simulation of LCLS power vs. δ,

➔ Power enhancement ~ 2 when energy gain 2ρ at saturation



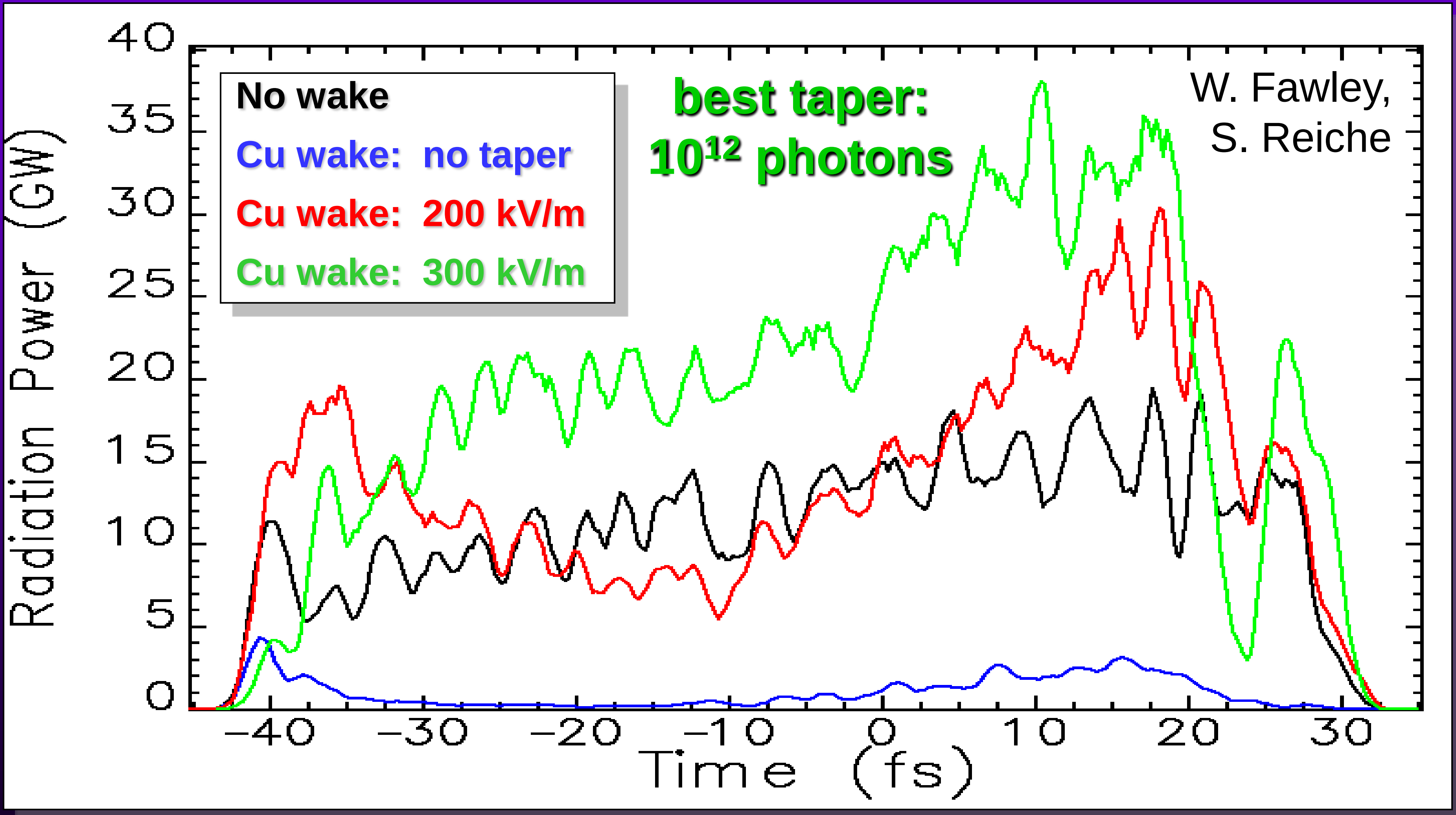


- Power vs. δρ has RMS $\sqrt{3}\sigma_\omega$

FWHM $4\sigma_\omega$ (~4ρ at saturation)

0.2-nC FEL Simulations with Taper

P. Emma's talk at PAC05



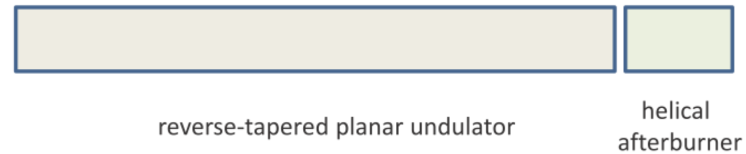
This study led to abandoning 1-nC LCLS

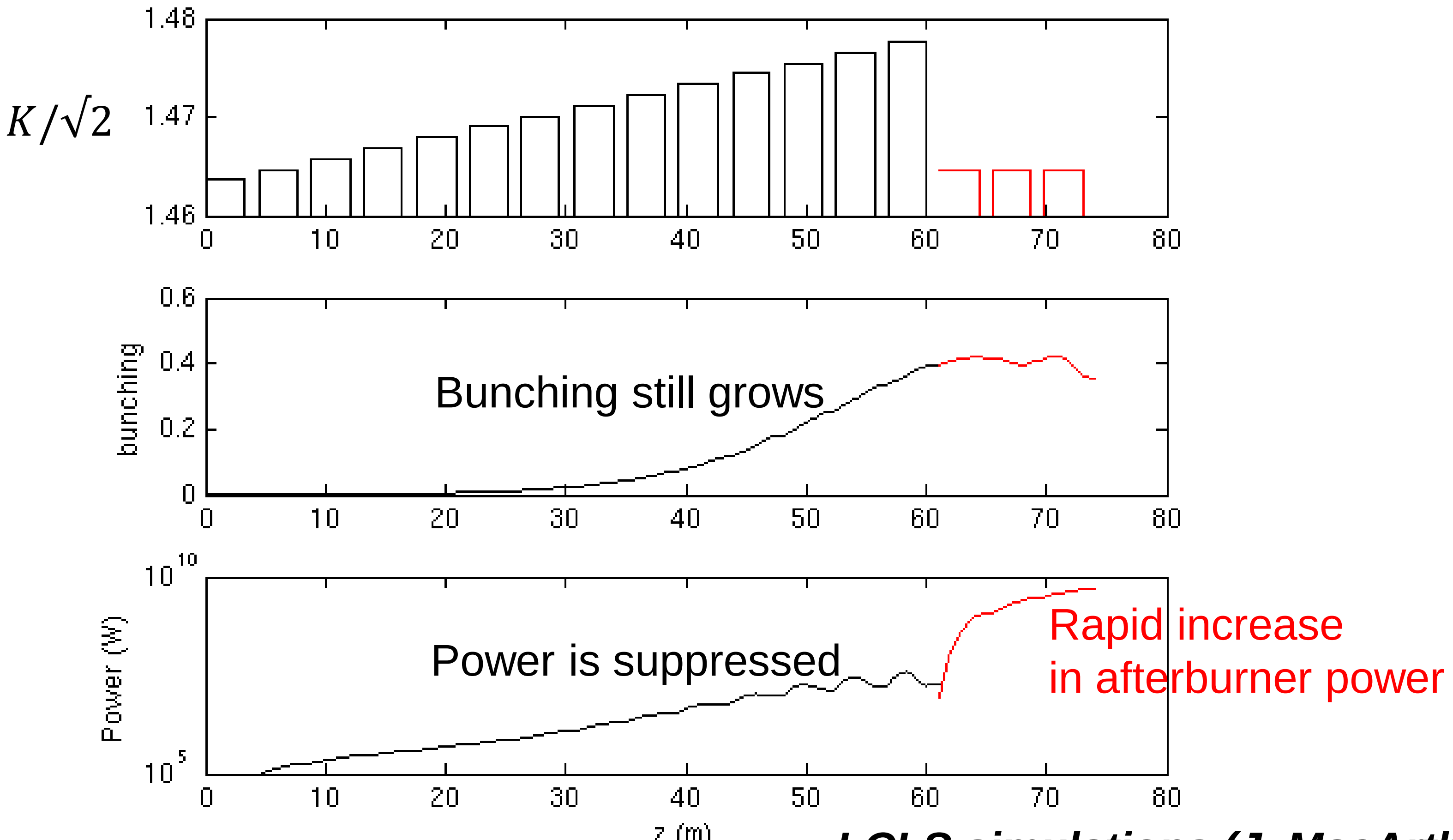
REVERSE UNDULATOR TAPERING FOR POLARIZATION CONTROL AT XFELS

E. A. Schneidmiller and M.V. Yurkov DESY, Hamburg, Germany

Abstract

The standard approach to obtaining variable polarization at X-ray FELs is to use a short afterburner with controlled field polarization behind a baseline planar undulator. A method for suppression of an intense linearly polarized





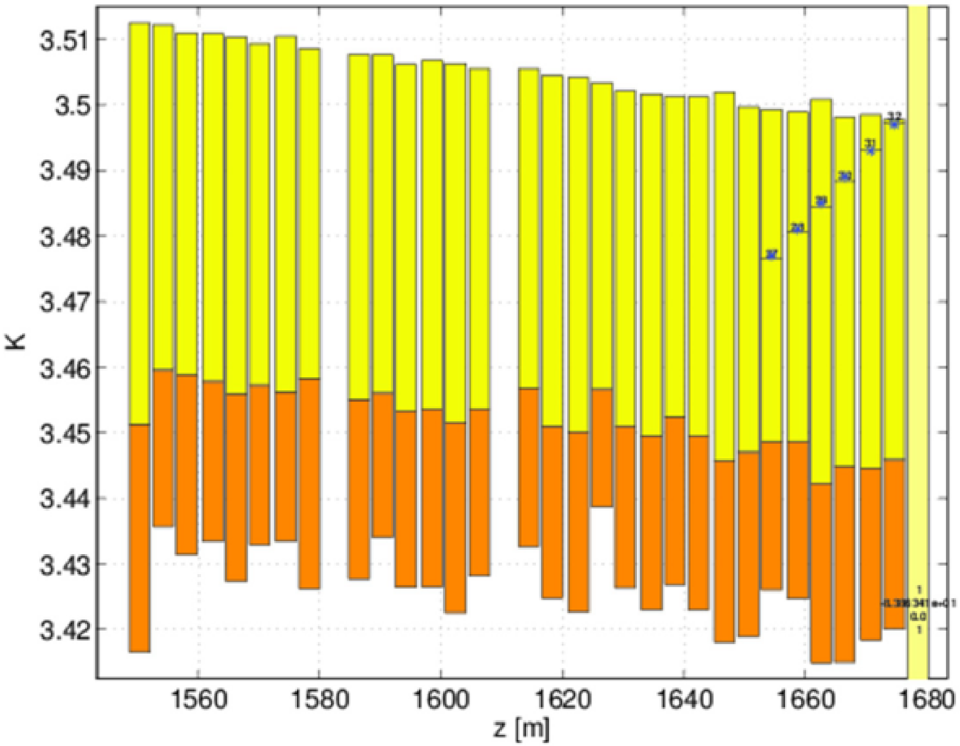
LCLS simulations (J. MacArthur)

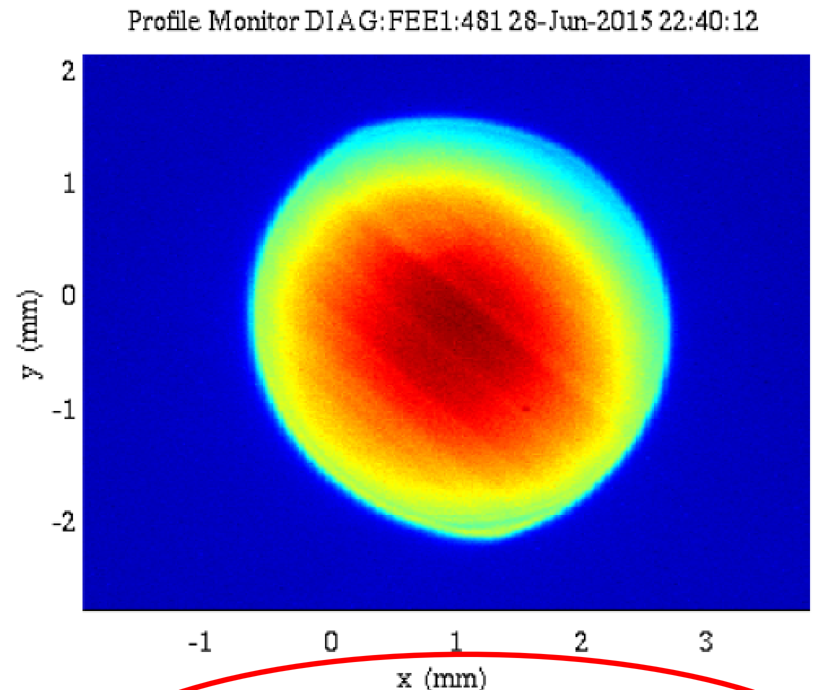
Delta in Enhanced Afterburner Configuration at 710 eV

SLAC

Reverse Taper

E.A. Schneidmiller, M.V. Yurkov, “Obtaining high degree of circular polarization at X-ray FELs via a reverse undulator taper”, arXiv:1308.3342 [physics.acc-ph]





- X-ray growth suppressed during reverse taper

- 30 μJ with Delta off
- 510 μJ with Delta on

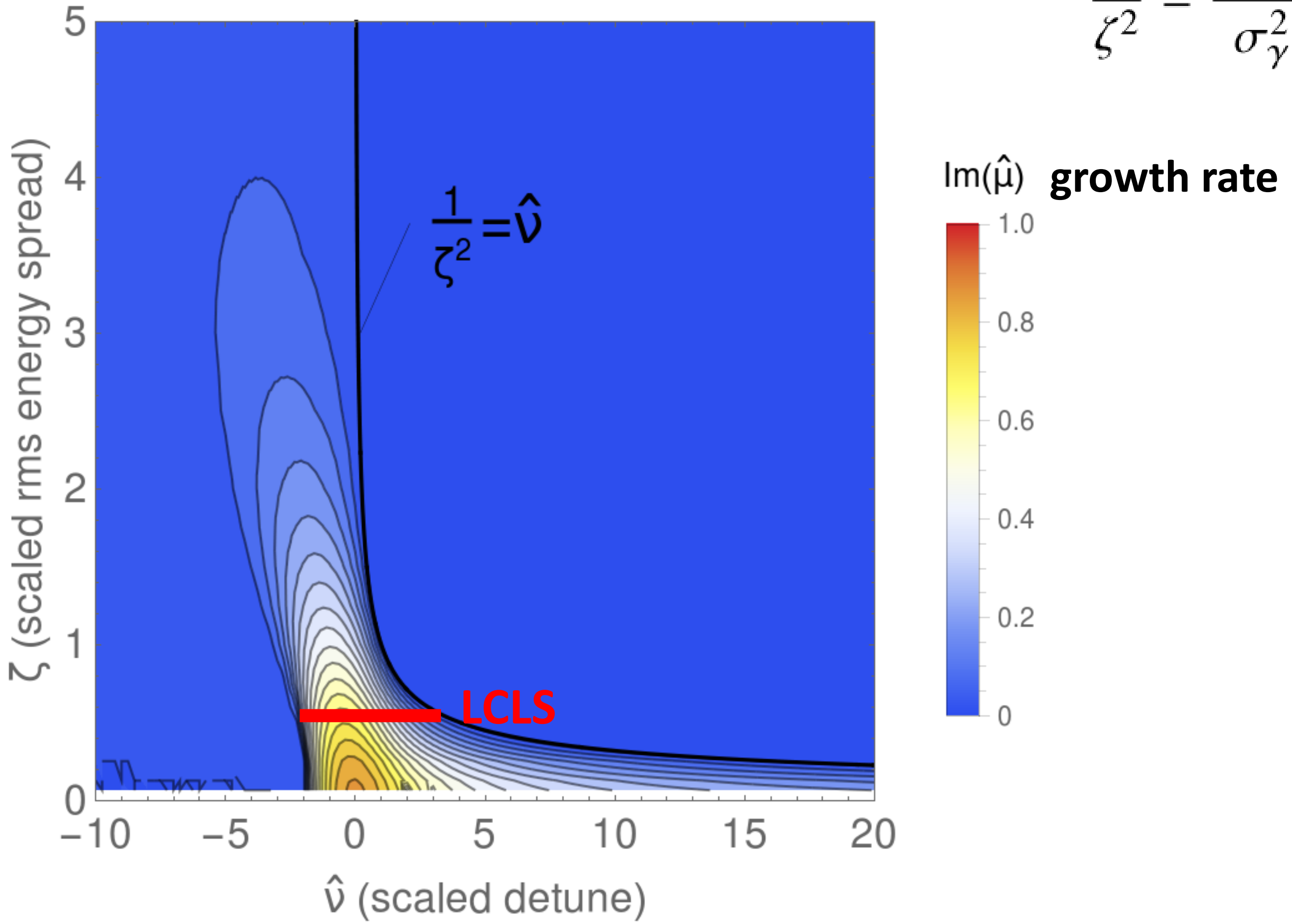
Peak Current increased above 4 kA

James MacArthur WEP004

Reverse taper sensitivity to e-beam energy spread

- Why did we only observe a factor of 20 contrast ratio (circular polarization vs. linear polarization)?
- Using the above theory, we find the maximum detune (taper) depends strongly on relative energy spread

$$\frac{1}{\zeta^2} = \frac{\rho^2 \gamma_0^2}{\sigma_\gamma^2},$$



- *J. MacArthur, A. Marinelli, A. Lutman, H. Nuhn, Z. Huang, FEL2015 proceedings*

Summary

- **Development of FEL theory is one of the most successful beam dynamic stories that directly guided X-ray FEL sources.**
- **Van Kampen's methods are critical to solve 3D theory FEL equations.**
- **FEL theory can be extended via WKB approximation to slowly varying beam and undulator parameters.**
- **Can be applied to tapered (and reverse-tapered) undulators (before saturation) for power enhancement and special operating modes.**