Statistical Optics and Free Electron Lasers

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“It is difficult, if not impossible, to conceive a real engineering problem in optics that does not contain some element of uncertainty requiring statistical analysis...” *[J.W. Goodman, Statistical Optics]*

Input/
Initial conditions

High-gain
Amplifier

Output:
Em field

- The same situation for Free-Electron Lasers
- High-gain amplifier can be described in a deterministic way
- The initial conditions inherently include “elements of uncertainty requiring statistical analysis”
  - An external seed laser can be described as a classical deterministic field
  - An energy modulation induced by modulators can also be considered deterministic
  - Deterministic density modulation is also possible…
  - …But the shot-noise in the electron beams is always present and is very fundamental
    - ► In some cases it represents the main signal (SASE)
    - ► In others it is detrimental (seeded FELs)
  - A statistical Optics treatment of FELs must be based on
    - ► A statistical analysis of shot-noise
    - ► A model of the FEL process

Contents

- Shot noise as stochastic signal
- Deterministic part: FEL amplification
- The SASE case – FEL amplifier and stochastic process
- Correlation functions and figures of merit

Shot-noise as stochastic signal

Shot-noise description in relativistic electron beams from the LINAC

- Particle distribution – discreteness of electric charge:

$$j_z(t,\vec{r};0) = (-e)\sum_{p=1}^{N_e} \delta(t-t_p)\delta\left(\vec{r}-\vec{r}_p\right)$$

  - N_e - number of electrons per bunch (1nC ~ 6e9 electrons)
  - $t_p, \vec{r}_p$ - random variables: arrival times and position ***at the entrance of the undulator*** (z=0)

- Origin of randomness: photoemission at the cathode
  - Semiclassical theory based on
    - ► $P(1;\Delta t,\Delta A) = \alpha \Delta t\, \Delta A\, I(x,y;t)$
      - Probability of 1 photoevent in $\Delta t \ll coh.time, \Delta A \ll coh.area$
      - photocathode laser intensity $I(x,y;t)$
    - ► $P(n>1;\Delta t,\Delta A)$ negligible
    - ► Number of photoevents in Δt_1 and Δt_2 are statistically independent

- It results in Poisson space-time impulse process with probability of finding K events between t and t+τ in the area $\mathcal{A}$ given by

$$P(K;t,t+\tau) = \frac{\bar{K}}{K!}\exp(-\bar{K})$$

  - ► $\bar{K}$ is the average number of photoevents

$$\bar{K} = \alpha W = \frac{\eta W}{h\bar{\nu}}$$

  - ► η defines the quantum efficiency and

$$W = \int_{\mathcal{A}}\int_{t}^{t+\tau} d\xi dx dy I(x,y;\xi)$$

  is the integrated intensity

- More in detail, the assumptions for the semiclassical theory are based on electrons as wave-functions evolving according to the Schroedinger equation [see e.g. *Mandel and Wolf, Optical Coherence and quantum optics*]

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) - e\vec{r}\cdot\vec{E}_c(\vec{r},t)\right]\Psi(\vec{r},t)$$

  - ► m is the electron mass, $V(\vec{r})$ the atomic Coulomb potential, $\vec{E}_c = E_0\cos(\omega t - ky)\hat{z}$ the incident field (classical, deterministic); here the dipole approximation has been used
  - ► …and a calculation of the transition probability yields our starting assumptions

- Semiclassical theory is in agreement with fully quantum

- In principle, quantum uncertainty may play a role during radiation emission in terms of:
  - Quantum recoil
    - ► Negligible for a large ratio between the FEL bandwidth and the energy of a photon relative to the electron energy.
    - ► This refers to the so-called quantum FEL parameter $\bar{\rho} = \frac{2\pi\rho\gamma mc^2}{h\omega_r} \ll 1 \ in\ the\ quantum\ case$
    - ► Usually quantum recoil negligible for present projects.
    - ► FEL in the quantum regime proposed and studied elsewhere by many [*e.g. Bonifacio, Pellegrini, Piovella, Robb, Schiavi, Schroeder…*]
  - Quantum diffusion
    - ► Always present: it induces energy spread in the electron beam [*Saldin, Schneidmiller, Yurkov NIM A 381, 545 (1996)*]:

$$\frac{d\sigma_\gamma^2}{dz} = \frac{14}{15}\lambda_c^{\ 2} r_e \gamma^4 \kappa_u^2 K_{rms}^2 F(K_{rms})$$

F(K)=1.42K$_{rms}$+(1+1.50K$_{rms}$+0.95K$_{rms}$^{2})$^{-1}$(helical)

F(K)=1.70K$_{rms}$+(1+1.88K$_{rms}$+0.80K$_{rms}$^{2})$^{-1}$(linear)

$$\kappa_u = \frac{2\pi}{\lambda_u},\ \lambda_c = \frac{h}{2\pi mc},\ r_e = \frac{e^2}{mc^2}$$

    - ► Limits the shortest wavelength achievable
  - Recent work [see *Anisimov, Quantum nature of electrons in classical FELs, FEL2015*] warn that effects due to the quantum nature of single electrons (Free-space dispersion of electron wavepackets propagating along the undulator+quantum average) may yield to larger-than-previously-believed quantum effects for HXR; and especially for harmonic lasing.

  - Here we neglect these effects.

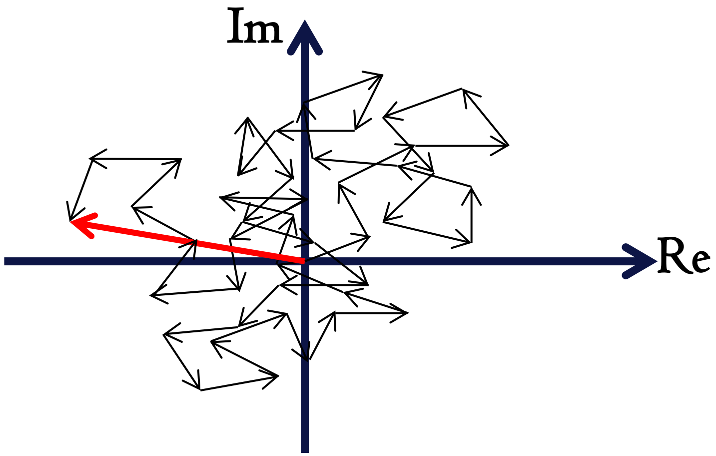
- Under these assumptions we deal with a classical current
- We stick to a classical description of the e.m. field…
- …and classical Statistical Optics is our language
- Summing up, shot-noise is a space-time random Poissonian process and current density at the entrance of the undulator given by

$$j_z(t,\vec{r};0) = (-e)\sum_{p=1}^{N_e} \delta(t-t_p)\delta\left(\vec{r}-\vec{r}_p\right)$$

- Usually, $N_e \to \infty$ is invoked to treat the process as Gaussian.
- For a 1D current in the frequency domain

$$I(t) = (-e)\sum_{p=1}^{N_e}\delta(t-t_p) \quad \longrightarrow \quad \bar{I}(\omega) = (-e)\sum_{p=1}^{N_e}\exp\left(i\omega t_p\right)$$

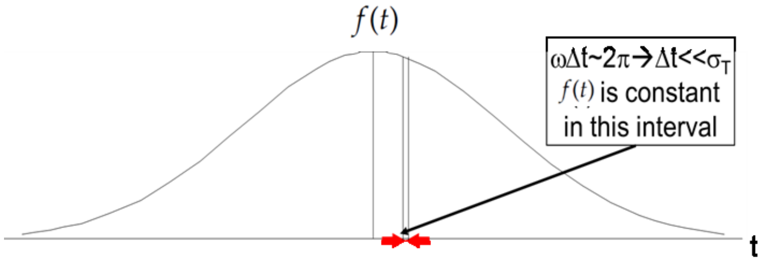
$$\bar{I}(\omega) = (-e)\sum_{p=1}^{N_e} \exp\left(i\omega t_p\right) \left\{ \begin{array}{l} \bar{I}_r = (-e)\sum_{p=1}^{N_e} \cos\left(\omega t_p\right) \\ \bar{I}_i = (-e)\sum_{p=1}^{N_e} \sin\left(\omega t_p\right) \end{array} \right.$$



$$\langle I(t)\rangle = (-e)N_e f(t)$$

$$f(t) \equiv \left\langle \delta(t - t_p)\right\rangle$$

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma_t}\exp\left(-\frac{t^2}{2\sigma_t^2}\right)$$



$\omega\Delta t \sim 2\pi \rightarrow \Delta t << \sigma_T$

$f(t)$ is constant in this interval

$$\omega\sigma_T >> 1$$

- Usually verified. E.g.

$\omega \sim 2\cdot 10^{19}\, Hz \qquad [\lambda \sim 0.1 nm]$

$\sigma_t \sim 80\, fs \qquad [\sigma_z \sim 24\,\mu m]$

- Phases uniformly distributed in (0, 2π)
- Arrival times are statistically independent of each other (discussed before)

- For $N_e \to \infty$ $\bar{I}_r$ and $\bar{I}_i$ follow Gaussian distributions… But $\bar{I}_r$ and $\bar{I}_i$ are actually ***jointly*** Gaussian, that is

$$P(\bar{I}_r, \bar{I}_i) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\bar{I}_r^{\,2} + \bar{I}_i^{\,2}}{2\sigma^2} \right)$$

- Then, $|\bar{I}|$ can be shown to follow the Rayleigh distribution

$$P(|\bar{I}|) = \frac{|\bar{I}|}{\sigma^2} \exp\left(-\frac{|\bar{I}|^2}{2\sigma^2} \right)$$

- and using the transformation $P\left(|\bar{I}|^2\right) = P\left(|\bar{I}|\right) \left| \frac{d\sqrt{|\bar{I}|^2}}{d|\bar{I}|^2} \right| = \frac{1}{2|\bar{I}|} P\left(|\bar{I}|\right)$ we obtain

$$P(|\bar{I}(\omega)|^2) = \frac{1}{\left\langle |\bar{I}(\omega)|^2 \right\rangle} \exp\left(-\frac{|\bar{I}(\omega)|^2}{\left\langle |\bar{I}(\omega)|^2 \right\rangle} \right)$$

with $\left\langle |\bar{I}(\omega)|^2 \right\rangle = 2\sigma^2$ that is the (usual) negative exponential distribution

- And in the time domain

$$P(|I(t)|^2) = \frac{1}{\left\langle |I(t)|^2 \right\rangle} \exp\left(-\frac{|I(t)|^2}{\left\langle |I(t)|^2 \right\rangle} \right)$$

- Concerning the current density

$$\hat{\tilde{j}}_z\left(\omega,\vec{k};0\right) = (-e)\sum_{p=1}^{N_e} \exp\left(i\omega t_p + i\vec{k}\cdot\vec{r}_p\right)$$

- We also have

$$P(\left|j_z(\vec{r},t)\right|^2) = \frac{1}{\left\langle\left|j_z(\vec{r},t)\right|^2\right\rangle}\exp\left(-\frac{\left|j_z(\vec{r},t)\right|^2}{\left\langle\left|j_z(\vec{r},t)\right|^2\right\rangle}\right)$$

- Any integral of the current density (in any domain!) then follows the Gamma distribution (with variance equal to the relative dispersion: more later)
- Shot-noise can be treated as a space-time Gaussian process

Deterministic part: FEL amplification

Input/
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High-gain
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- As discussed before, the FEL amplifier can be described in a deterministic way
- One way to describe the full particle distribution is to use the Klimontovich distribution

$$F\left(\psi, P, \vec{r}, \vec{p}; z\right) = \frac{2\pi}{\lambda_r n_e} \sum_{p=1}^{N_e} \delta\left(\psi - \psi_p\right) \delta\left(P - P_p\right) \delta\left(\vec{r} - \vec{r}_p\right) \delta\left(\vec{p} - \vec{p}_p\right)$$

- Here n_e is the max volume density of electrons and “standard” variables are already introduced in the longitudinal direction: $P = E - E_r$ and $\psi = k_w z + \omega(z/c - t)$
- The continuity equation holds : $\dfrac{dF\left(\psi, P, \vec{r}, \vec{p}; z\right)}{dz} = 0$
- Consider only interaction of a single electron with collective fields from the beam
- This means that we are actually invoking a Vlasov equation.

- Helical undulator, with transverse coordinates as parameters

$$\frac{\partial f}{\partial z}+\frac{\partial H}{\partial P}\frac{\partial f}{\partial \psi}-\frac{\partial H}{\partial \psi}\frac{\partial f}{\partial P}=0 \qquad H=CP+\frac{\omega}{2c\gamma_z^2\mathscr{E}_0}P^2-(U\mathrm{e}^{\mathrm{i}\psi}+U^*\mathrm{e}^{-\mathrm{i}\psi})$$

- with $C=k_\mathrm{w}-\omega/(2c\gamma_z^2)$ $U=-e\theta_\mathrm{s}\widetilde{E}(z,\vec{r}_\perp)/(2\mathrm{i})$ $\theta_s=\frac{K}{\gamma}$ $E_x+\mathrm{i}E_y=\widetilde{E}(z,\vec{r}_\perp)\exp[\mathrm{i}\omega(z/c-t)]$

- Solution in the linear regime uses the particle distribution $f=f_0+\tilde{f}_1\mathrm{e}^{\mathrm{i}\psi}+\tilde{f}_1^*\mathrm{e}^{-\mathrm{i}\psi}$ background and $f_0(\vec{r}_\perp,P)$a small (linear regime) perturbation $\tilde{f}_1(\vec{r}_\perp,P,z)$

- Given the transverse profile function S with S(0)=1 and r_0 the typical transverse beam size the current density is

$$j_z=-j_0(\vec{r}_\perp)+\tilde{j}_1\mathrm{e}^{\mathrm{i}\psi}+\mathrm{c.c.}, \quad \tilde{j}_1\simeq -ec\int\tilde{f}_1\,\mathrm{d}P \quad j_0(r)=I_0S(r/r_0)\left[2\pi\int_0^\infty rS(r/r_0)\,\mathrm{d}r\right]^{-1}$$

- Vlasov equation must be solved together with Maxwell equations

$$c^2[\nabla_\perp^2+2\mathrm{i}(\omega/c)\partial/\partial z]\widetilde{E}=-4\pi\mathrm{i}\theta_\mathrm{s}\omega\tilde{j}_1$$

- Expansion in azimuthal harmonics $\widetilde{E}(z,r,\varphi)=\sum_{n=-\infty}^{n=+\infty}\widetilde{E}^{(n)}(z,r)\mathrm{e}^{-\mathrm{i}n\varphi}$ $\tilde{j}_1(z,r,\varphi)=\sum_{n=-\infty}^{n=+\infty}\tilde{j}_1^{(n)}(z,r)\mathrm{e}^{-\mathrm{i}n\varphi}$

- Assume given starting density modulation as initial condition

$$\hat{E}^{(n)}(\hat{z},\hat{r})=\sum_j A_j^{(n)}\Phi_{nj}(\hat{r})\exp(\lambda_j^{(n)}\hat{z})$$

(Here and below I follow: *Saldin, Schneidmiller, Yurkov, Opt. Comm. 186, 2000)*

The SASE case.
FEL amplifier and stochastic process

$$\bar{j}_{\mathrm{ext}}(\vec{r}_\perp,\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} j_z(\vec{r}_\perp,z,t)\big|_{z=0} = (-e)\sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}\omega t_k}\,\delta(\vec{r}_\perp - \vec{r}_\perp^{(k)})$$

Gaussian process. And a linearly filtered Gaussian random process is also a Gaussian process.

- Therefore **in the linear regime the field inherits the statistical properties of the current**
- In particular, the intensity in the time $I(\vec{r},t)\sim|E(\vec{r},t)|^2$ or frequency domain $I(\vec{r},\omega)\sim|E(\vec{r},\omega)|^2$ (fixed z) follows a negative exponential distribution as well-known

$$P(I) = \frac{1}{\langle I\rangle}\exp\left(-\frac{I}{\langle I\rangle}\right)$$

- And any integral of *I* follows a Gamma distribution. For example given $\mho = \int d\vec{r}\,dt\,I(\vec{r},t)$ or

$\mho = \int d\vec{r}\,d\omega I(\vec{r},\omega)$ or $\mho = \int d\vec{r}\,I(\vec{r},t)$

$$P(\mho) = \frac{M^M}{\Gamma(M)}\left(\frac{\mho}{\langle\mho\rangle}\right)^{M-1}\frac{1}{\langle\mho\rangle}\exp\left(-M\frac{\mho}{\langle\mho\rangle}\right)$$

With Γ the Gamma function and $M = \frac{1}{\sigma^2} = \frac{\langle\mho\rangle^2}{\left\langle\left(\mho - \langle\mho\rangle\right)^2\right\rangle}$ the inverse relative dispersion

- M is then interpreted as the average number of modes
- The meaning of M depends on the definition of the integral.
- $\mho = \int d\vec{r}\,dt\,I(\vec{r},t)$ is the total energy in the pulse, then M is the **total** number of modes
- $\mho = \int d\vec{r}\,I(\vec{r},t)$ is the power, then M is the number of transverse modes
- $\mho = \int dt\,I(\vec{r},t)$ is the energy density at given position. Then M is the number of longitudinal modes
- And obviously considering $\mho = \int \int_{\Delta\omega} d\vec{r}\,d\omega\,I(\vec{r},\omega)$ one obtains M as the number of longitudinal modes through a monochromator.
- Degree of transverse and longitudinal coherence is inverse mode number
- Peak Brightness is defined (we will comment on this definition later!) in terms of longitudinal and transverse modes as

$$B_{ave} = \frac{4\sqrt{2}c\delta}{\lambda^3}, \delta = \dot{N}_{ph}\tau_c\zeta$$

- Any integral of I follows a Gamma distribution (seen before)...
- …Yet longitudinal and transverse modes are treated very differently by the amplification process.
- Transversely, different self-reproducing modes have different gains $\hat{E}^{(n)}(\hat{z},\hat{r}) = \sum_j A_j^{(n)} \Phi_{nj}(\hat{r}) \exp(\lambda_j^{(n)} \hat{z})$
- The well-known ‘mode-guiding’ mechanism takes place and only one mode tends to survive

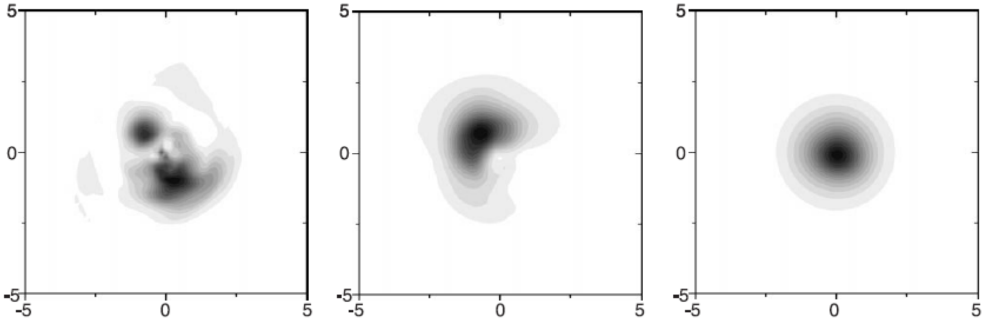


Fig. 1. Distributions of the radiation intensity across one slice of the radiation pulse at different undulator lengths, $\hat{z} = 5$, $\hat{z} = 10$, and $\hat{z} = 15$ (left, middle, and right plots, respectively). The coordinates are normalized to $2^{1/2}\sigma_r$. Here $B = 1$, $\hat{\Lambda}_p^2 \to 0$, and $\hat{\Lambda}_T^2 = 0$. Calculations have been performed with linear simulation code.

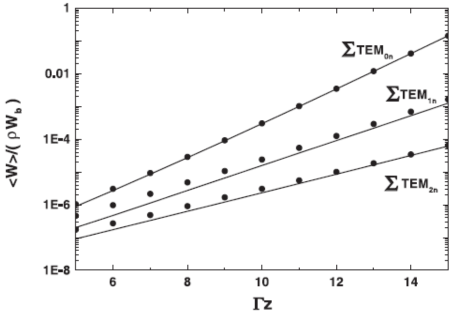
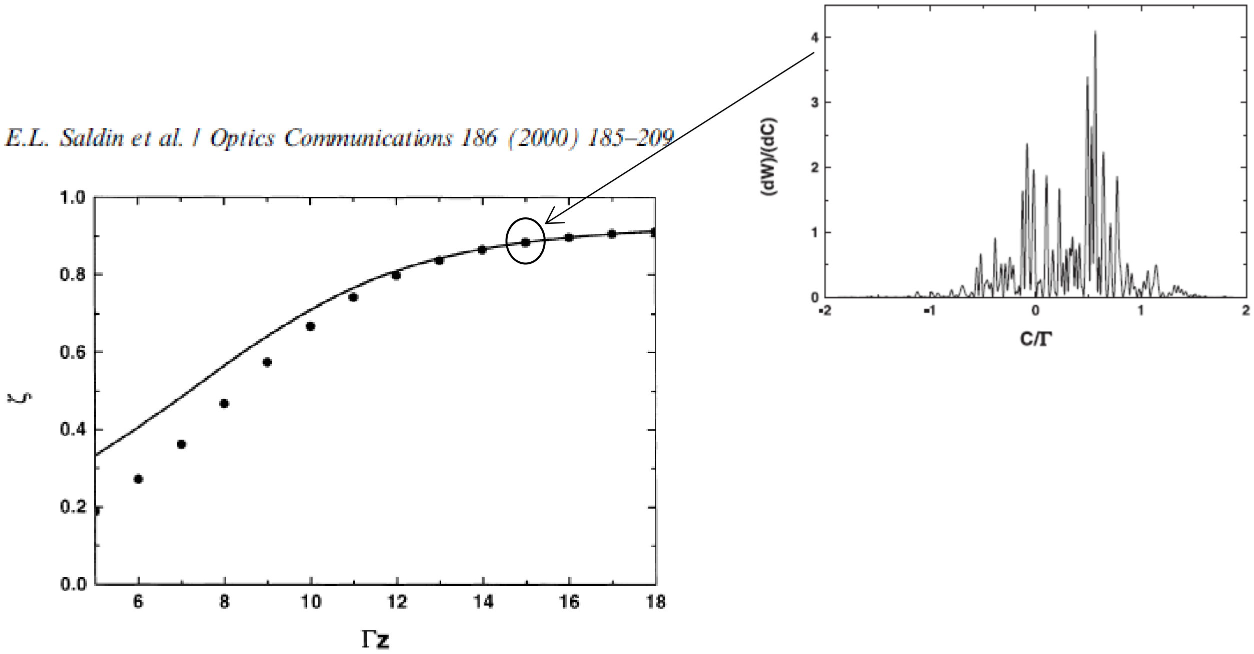


Fig. 4. Partial contributions to the total power (see Fig. 3) of three azimuthal modes with $m = 0, 1$, and 2. Here $B = 1$, $\Lambda_p^2 \to 0$, $\Lambda_T^2 = 0$, and $N_c = 7 \times 10^7$. Solid curves represent analytical results calculated with Eqs. (32) and (33) for sum of three radial modes ($n = 0, 1, 2$). The circles are the results obtained with linear simulation code FAST.

[Saldin, Schneidmiller and Yurkov, Opt. Comm 186 (2000) 185]

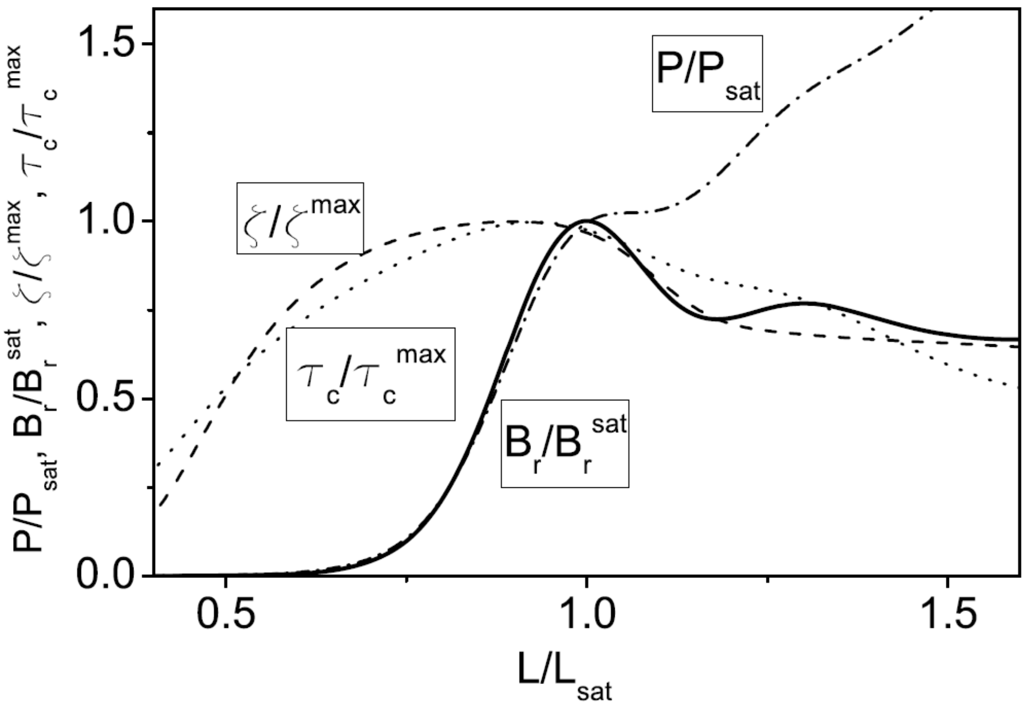
Observation

- When only one mode tends to survive one expects good transverse coherence
- This is correct. However the presence of many longitudinal modes limits the max degree achievable due to interplay between longitudinal and transverse modes: transverse modes depend on frequency



E.L. Saldin et al. / Optics Communications 186 (2000) 185–209

Fig. 9. Degree of transverse coherence of the radiation from the FEL amplifier versus the undulator length. Solid curve represents analytical results, and the circles are the results obtained with linear simulation code FAST. Here $B = 1$, $\Lambda_{\mathrm{p}}^2 \to 0$, $\Lambda_T^2 = 0$, and $N_c = 7 \times 10^7$.



[*Saldin, Schneidmiller and Yurkov, New Journal of Physics 12, 2010*]

- Behaviour of power, coherence (long. & transv.) and brightness
- Linear/Nonlinear&saturation
  - Linear regime: what discussed above holds
  - Saturation: the statistical properties of shot-noise are transformed by the non-linearity of the FEL filter

Correlation functions and figures of merit

- A statistical study of radiation properties is better done with the help of correlation functions. For instance, in the time-domain (at fixed z)

$$\Gamma(t_1,\vec{r}_1 ... t_{n+m},\vec{r}_{n+m}) = \left\langle E(t_1,\vec{r}_1)\cdot ... \cdot E(t_n,\vec{r}_n)\cdot E^*(t_{n+1},\vec{r}_{n+1})\cdot ... \cdot E^*(t_{n+m},\vec{r}_{n+m})\right\rangle$$

- And equivalently in the frequency domain

$$\overline{\Gamma}(\omega_1,\vec{r}_1 ... \omega_{n+m},\vec{r}_{n+m}) = \left\langle \overline{E}(\omega_1,\vec{r}_1)\cdot ... \cdot \overline{E}(\omega_n,\vec{r}_n)\cdot \overline{E}^*(\omega_{n+1},\vec{r}_{n+1})\cdot ... \cdot \overline{E}^*(\omega_{n+m},\vec{r}_{n+m})\right\rangle$$

- The knowledge of all order correlation functions is needed to fully characterize the stochastic process (see Goodman).

- In the case of a Gaussian process (FEL in the **linear regime**) the moment theorem applies

$$\Gamma(t_1,\vec{r}_1 ... t_{2n},\vec{r}_{2n}) = \sum_{\pi}\Gamma(t_1,\vec{r}_1,t_p,\vec{r}_p)\cdot ... \cdot \Gamma(t_n,\vec{r}_n,t_r,\vec{r}_r)$$

- This means that the basic quantity to study in these cases is

$$\Gamma(t_1,\vec{r}_1,t_2,\vec{r}_2) = \left\langle E(t_1,\vec{r}_1)\cdot E^*(t_2,\vec{r}_2)\right\rangle$$

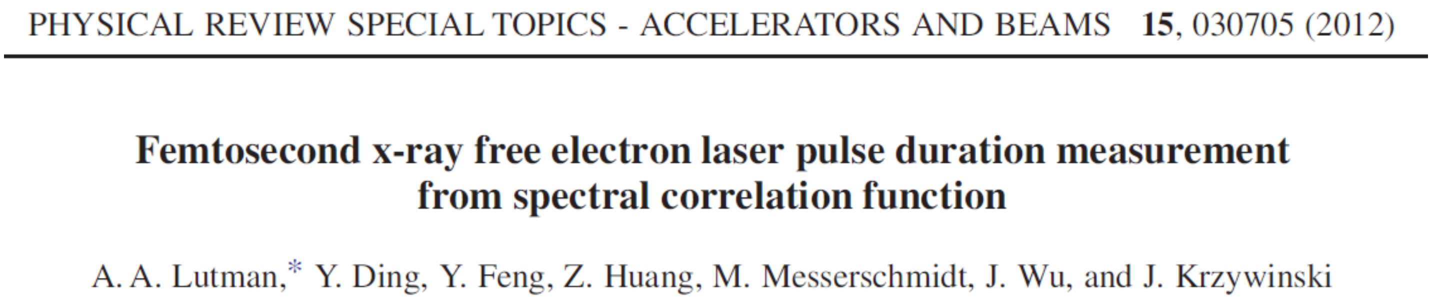
- Or equivalently the same applies in other domains for instance

$$\overline{\Gamma}(\omega_1,\vec{r}_1,\omega_2,\vec{r}_2) = \left\langle \overline{E}(\omega_1,\vec{r}_1)\cdot \overline{E}^*(\omega_2,\vec{r}_2)\right\rangle$$

- **If we are in the nonlinear regime it still makes sense to study these functions, but they do not fully characterize the process**

Longitudinal diagnostics

- Given a statistical measurement of the spectral correlation can we get the photon pulse duration?



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Femtosecond x-ray free electron laser pulse duration measurement from spectral correlation function

A. A. Lutman,* Y. Ding, Y. Feng, Z. Huang, M. Messerschmidt, J. Wu, and J. Krzywinski

- Based in weighted spectral second order correlation function

$$G_2(\delta\omega)=\int_{-\infty}^{+\infty} W(\omega)\left[\frac{\langle S(\omega-\delta\omega/2)S(\omega+\delta\omega/2)\rangle}{\langle S(\omega-\delta\omega/2)\rangle\langle S(\omega+\delta\omega/2)\rangle}-1\right]d\omega$$

$$W(\omega)=\frac{\int_{-\infty}^{+\infty}\langle S(\omega+b/2)\rangle\langle S(\omega-b/2)\rangle db}{\int_{-\infty}^{+\infty}\langle S(\omega+b/2)\rangle\langle S(\omega-b/2)\rangle db d\omega}$$

$$S(\omega)=\int_{-\infty}^{+\infty}\frac{e^{-[(\omega'-\omega)^2/2\sigma_m^2]}}{\sqrt{2\pi}\sigma_m}|\tilde{E}(\omega')|^2 d\omega',$$

(Single-shot spectrum assuming given Line for the spectrometer)

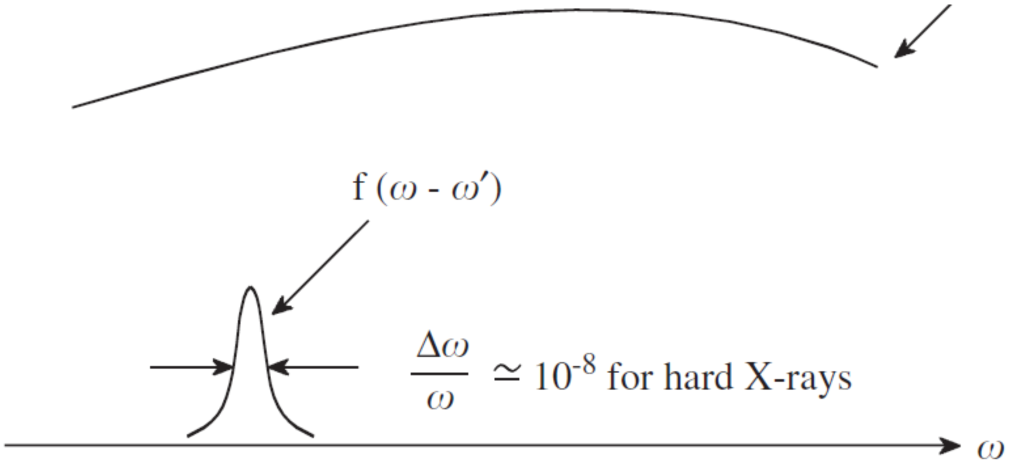
- Use a model for the amplification process to derive an analytical expression for G_2
- Fit the experimentally measured G_2 with the analytical expression to derive the pulse duration
- It works nicely!
- The model for G_2 is based on

$$g_2(\omega, \delta\omega) = 1 + |g_1(\omega, \delta\omega)|^2$$

$$g_1(\omega, \delta\omega) = \frac{\langle \tilde{E}(\omega - \delta\omega/2)\tilde{E}^*(\omega + \delta\omega/2)\rangle}{\sqrt{\langle |\tilde{E}(\omega - \delta\omega/2)|^2\rangle\langle |\tilde{E}(\omega + \delta\omega/2)|^2\rangle}},$$

$$g_2(\omega, \delta\omega) = \frac{\langle |\tilde{E}(\omega - \delta\omega/2)|^2 |\tilde{E}(\omega + \delta\omega/2)|^2\rangle}{\langle |\tilde{E}(\omega - \delta\omega/2)|^2\rangle\langle |\tilde{E}(\omega + \delta\omega/2)|^2\rangle}.$$

It works also after the linear regime, but this reasoning is strictly ok in the linear regime

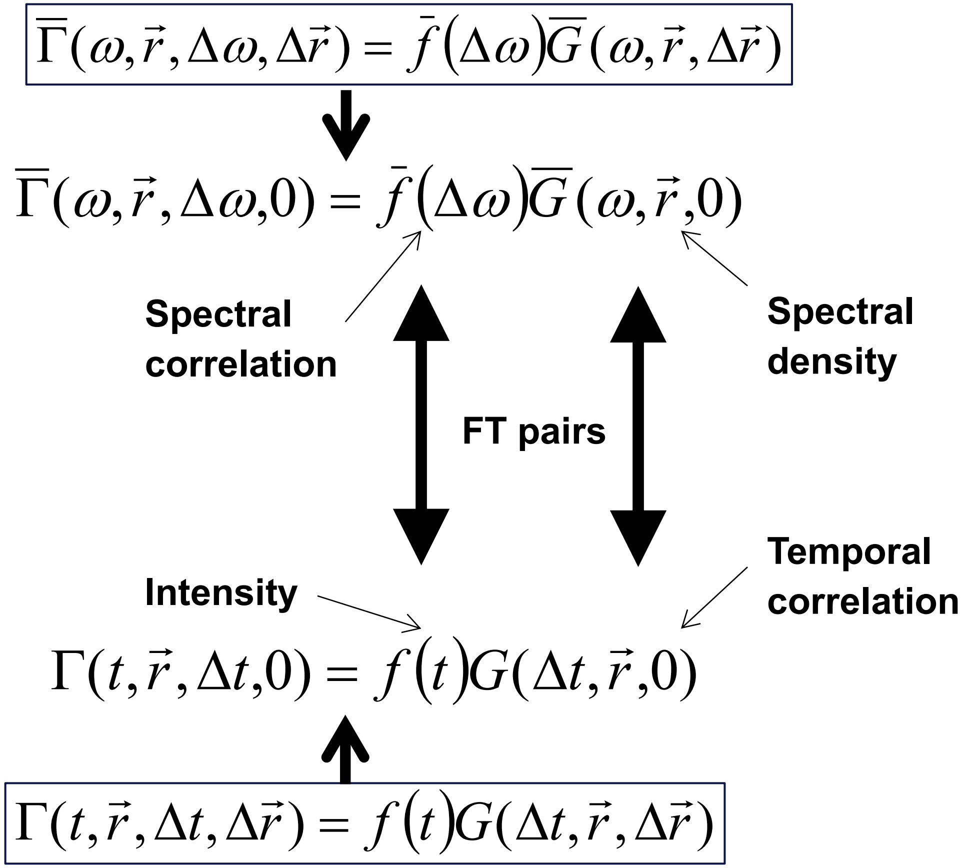


- If the inverse bunch duration is small compared to the FEL bandwidth → quasi-stationarity is a good approximation

$$\Gamma(t_1,\vec{r}_1,t_2,\vec{r}_2) = f(t)G(t_2 - t_1,\vec{r}_1,\vec{r}_2)$$

$$\bar{\Gamma}(\omega_1,\vec{r}_1,\omega_2,\vec{r}_2) = \bar{f}(\omega_1 - \omega_2)G(\omega,z,\vec{r}_1,\vec{r}_2)$$

Wiener-Khinchin Theorem
Quasi-stationarity is assumed



$$\boxed{\overline{\Gamma}(\omega,\vec{r},\Delta\omega,\Delta\vec{r}) = \bar{f}(\Delta\omega)\overline{G}(\omega,\vec{r},\Delta\vec{r})}$$

$$\overline{\Gamma}(\omega,\vec{r},\Delta\omega,0) = \bar{f}(\Delta\omega)\overline{G}(\omega,\vec{r},0)$$

Spectral correlation

Spectral density

FT pairs

Intensity

Temporal correlation

$$\Gamma(t,\vec{r},\Delta t,0) = f(t)G(\Delta t,\vec{r},0)$$

$$\boxed{\Gamma(t,\vec{r},\Delta t,\Delta\vec{r}) = f(t)G(\Delta t,\vec{r},\Delta\vec{r})}$$

$$\omega = \frac{\omega_1 + \omega_2}{2}$$
$$\Delta\omega = \omega_1 - \omega_2$$

$$\vec{r} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$
$$\Delta\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$t = \frac{t_1 + t_2}{2}$$
$$\Delta t = t_1 - t_2$$

Longitudinal Coherence

- In this way one fixes Δr = 0 and can study longitudinal coherence at fixed transverse position in the frequency domain using

$$\bar{\Gamma}(\omega,\vec{r},\Delta\omega,0)\left[=\bar{f}(\Delta\omega)\bar{G}(\omega,z,\vec{r},0)\quad if\ quasi-stationary\right]$$

- Or in the time domain using

$$\Gamma(t,\vec{r},\Delta t,0)\left[=f(t)G(\Delta t,z,\vec{r},0)\quad if\ quasi-stationary\right]$$

- Dependence on fixed position is still there, because there is no quasi-homogeneity in general (quasi-homogeneity is as quasi-stationarity in the spatial domain)

Transverse Coherence

- Study of transverse coherence is done instead by studying (slow dependence on ω if quasi-stationary) $\bar{\Gamma}(\omega,\vec{r},\Delta\omega,\Delta\vec{r})$ at $\Delta\omega=0$

$$\bar{\Gamma}(\omega,\vec{r},0,\Delta\vec{r})\sim\bar{G}(\omega,\vec{r},\Delta\vec{r})=\left\langle\tilde{E}(\omega,\vec{r}+\Delta\vec{r}/2)\cdot\tilde{E}^{*}(\omega,\vec{r}-\Delta\vec{r}/2)\right\rangle$$

- Or by studying (slow dependence on time if quasi-stationary) $\Gamma(t,\vec{r},\Delta t,\Delta\vec{r})$ at $\Delta t=0$

$$\Gamma(t,\vec{r},0,\Delta\vec{r})=\left\langle E(t,\vec{r}+\Delta\vec{r}/2)\cdot E^{*}(t,\vec{r}-\Delta\vec{r}/2)\right\rangle$$

Coherence time

$$\Gamma(t,\vec{r},\Delta t,0)\left[= f(t)G(\Delta t,\vec{r},0) \quad if\ quasi-stationary\right]$$

$$g_1(t,\vec{r},\Delta t)=\frac{\Gamma(t,\vec{r},\Delta t,0)}{\left[\Gamma(t+\Delta t/2,\vec{r},0,0)\Gamma(t-\Delta t/2,\vec{r},0,0)\right]^{1/2}}$$

$$\tau_c=\int d\Delta t\left|g_1(\cancel{t},\vec{r},\Delta t)\right|^2$$

if quasi-stationary

Degree of transverse coherence

$$\overline{G}(\omega,\vec{r},\Delta\vec{r})=\left\langle \widetilde{E}(\omega,\vec{r}+\Delta\vec{r}/2)\cdot\widetilde{E}^*(\omega,\vec{r}-\Delta\vec{r}/2)\right\rangle$$

$$\Gamma(t,\vec{r},0,\Delta\vec{r})=\left\langle E(t,\vec{r}+\Delta\vec{r}/2)\cdot E^*(t,\vec{r}-\Delta\vec{r}/2)\right\rangle$$

$$\gamma_1=\frac{\overline{G}(\omega,\vec{r},\Delta\vec{r})}{\left[\overline{G}(\omega,\vec{r}-\Delta\vec{r}/2,0)\overline{G}(\omega,z,\vec{r}+\Delta\vec{r}/2,0)\right]^{1/2}}\ or\ \frac{\Gamma(t,\vec{r},0,\Delta\vec{r})}{\left[\Gamma(t,\vec{r}-\Delta\vec{r}/2,0,0)\Gamma(t,\vec{r}+\Delta\vec{r}/2,0,0)\right]^{1/2}}$$

$$\zeta=\frac{\iint d\vec{r}\,d\Delta\vec{r}\left|\gamma_1(\vec{r},\Delta\vec{r})\right|^2\left\langle I(\vec{r}+\Delta\vec{r}/2)\right\rangle\left\langle I(\vec{r}-\Delta\vec{r}/2)\right\rangle}{\left[\int d\vec{r}\left\langle I(\vec{r})\right\rangle\right]^2}$$

It can be shown [*Saldin, Schneidmiller, Yurkov, Opt. Comm. 281 (2008)*] that in the **linear regime** ζ=1/M

Brightness

- In **Radiometry (based on geometrical optics)** the concept of **Radiance** is used
  - Radiance is Spectral photon flux per unit area per unit projection angle is the photon flux density in phase space
  - When Liouville holds (non - dissipative case) the density of system points near a given point evolving through phase - space is constant with time

$$R[\vec{r}, \vec{\theta}] = \frac{d\,\dot{N}_{ph}}{dS\, d\Omega\, (d\omega / \omega)} = \frac{dF}{dS\, d\Omega}$$

$$\int R[\vec{r}, \vec{\theta}]\, d\vec{\theta} = \frac{d\,\dot{N}_{ph}}{dS\, (d\omega / \omega)} = \frac{dF}{dS} \text{(*Spatial -spectral- flux*)}$$

$$\int R[\vec{r}, \vec{\theta}]\, d\vec{r} = \frac{d\,\dot{N}_{ph}}{d\Omega\, (d\omega / \omega)} = \frac{dF}{d\Omega} \text{(*Angular -spectral- flux*)}$$

$$\int\int R[\vec{r}, \vec{\theta}]\, d\vec{r}\, d\vec{\theta} = \frac{d\,\dot{N}_{ph}}{d\omega / \omega} = F \text{(*Total -spectral- flux*)}$$

  - Then the Brightness is the maximum flux density in phase - space, and is the theoretical max concentration of photon flux density delivered by an ideal optical imaging system

$$B = \max[R]$$

Brightness

- Beyond geometrical optics, one uses the concept of Wigner distribution (Kwang-Je Kim was the first to apply this approach to SR & FELs, see Bazarov PRSTAB 15 050703 (2012) for a review)
- Start with cross-spectral density

$$\overline{G}(\omega, z, \vec{r}, \Delta\vec{r}) = \left\langle \widetilde{E}(\omega, z, \vec{r} + \Delta\vec{r}/2) \cdot \widetilde{E}^{*}(\omega, z, \vec{r} - \Delta\vec{r}/2) \right\rangle$$

- It can be seen as the analogue of a density matrix
- Wigner distribution is the FT of the cross-spectral density

$$W\left[\vec{r}, \vec{\theta}\right] = A \int G[\vec{r}, \Delta\vec{r}]\, \mathrm{Exp}\left[-\frac{i\omega\, \vec{\theta} \cdot \Delta\vec{r}}{c}\right] d\Delta\vec{r}$$

Brightness

- The W distribution can obviously be used to express the degree of transverse coherence in an equivalent way as done with the cross-spectral density

$$\xi = \frac{\lambda^2 \int\int W^2[\vec{r}, \vec{\theta}]\, d\vec{\theta}\, d\vec{r}}{\left(\int\int W[\vec{r}, \vec{\theta}]\, d\vec{\theta}\, d\vec{r}\right)^2}$$

- The W distribution can also be used to extend the notion of Brightness in a very natural way

$$B = \max[W]$$

- Compare with the previous definition: $B_{ave} = \frac{4\sqrt{2}c\delta}{\lambda^3}, \delta = \dot{N}_{ph}\tau_c\zeta$
- The latter can be written in terms of integrals of Wigner function

$$B_{ave} = \frac{\xi}{(\lambda/2)^2}\frac{d\dot{N}_{ph}}{d\omega/\omega} = 4\frac{\int\int W^2[\vec{r}, \vec{\theta}]\, d\vec{r}\, d\vec{\theta}}{\left(\int\int W[\vec{r}, \vec{\theta}]\, d\vec{r}\, d\vec{\theta}\right)^2}\frac{d\dot{N}_{ph}}{d\omega/\omega} = 4\frac{\int\int W^2[\vec{r}, \vec{\theta}]\, d\vec{r}\, d\vec{\theta}}{\int\int W[\vec{r}, \vec{\theta}]\, d\vec{r}\, d\vec{\theta}}$$

- And does not include information on *wavefront properties, which is instead included in* $B = \max[W]$
- *…though for FELs, the fundamental mode has typically a 'nice' wavefront*

Brightness

- Note that also here we are using the transverse W function

- This might be generalized using the full correlation function

$$\overline{\Gamma}(\omega,\vec{r},\Delta\omega,\Delta\vec{r}) = \left\langle \widetilde{E}(\omega+\Delta\omega/2,z,\vec{r}+\Delta\vec{r}/2)\cdot\widetilde{E}^{*}(\omega-\Delta\omega/2,z,\vec{r}-\Delta\vec{r}/2)\right\rangle$$

- And then

$$W(\omega,t,\vec{r},\vec{\theta}) = A\int \overline{\Gamma}(\omega,\vec{r},\Delta\omega,\Delta\vec{r})\exp[-i\frac{\omega}{c}\vec{\theta}\cdot\Delta\vec{r} - i\Delta\omega t]d\Delta\vec{r}d\Delta\omega$$

- We can stick to the previous definition of Brightness

$$B = \max[W]$$