Integrable Dynamical Systems in Particle Accelerators

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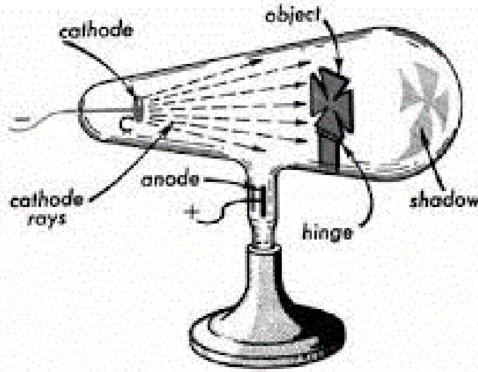
IPAM 2017 Beam Dynamics

Electric Charge in the Field of a Magnetic Pole

- Magnetic pole – “end” of a semi-infinite solenoid
- In 1896, Birkeland reported studies of cathode rays in a Crookes tube when a strong, straight electromagnet was placed outside and to the left.

William Crookes
(1832-1919)



- The nature of cathode rays was not yet understood

Kristian Birkeland

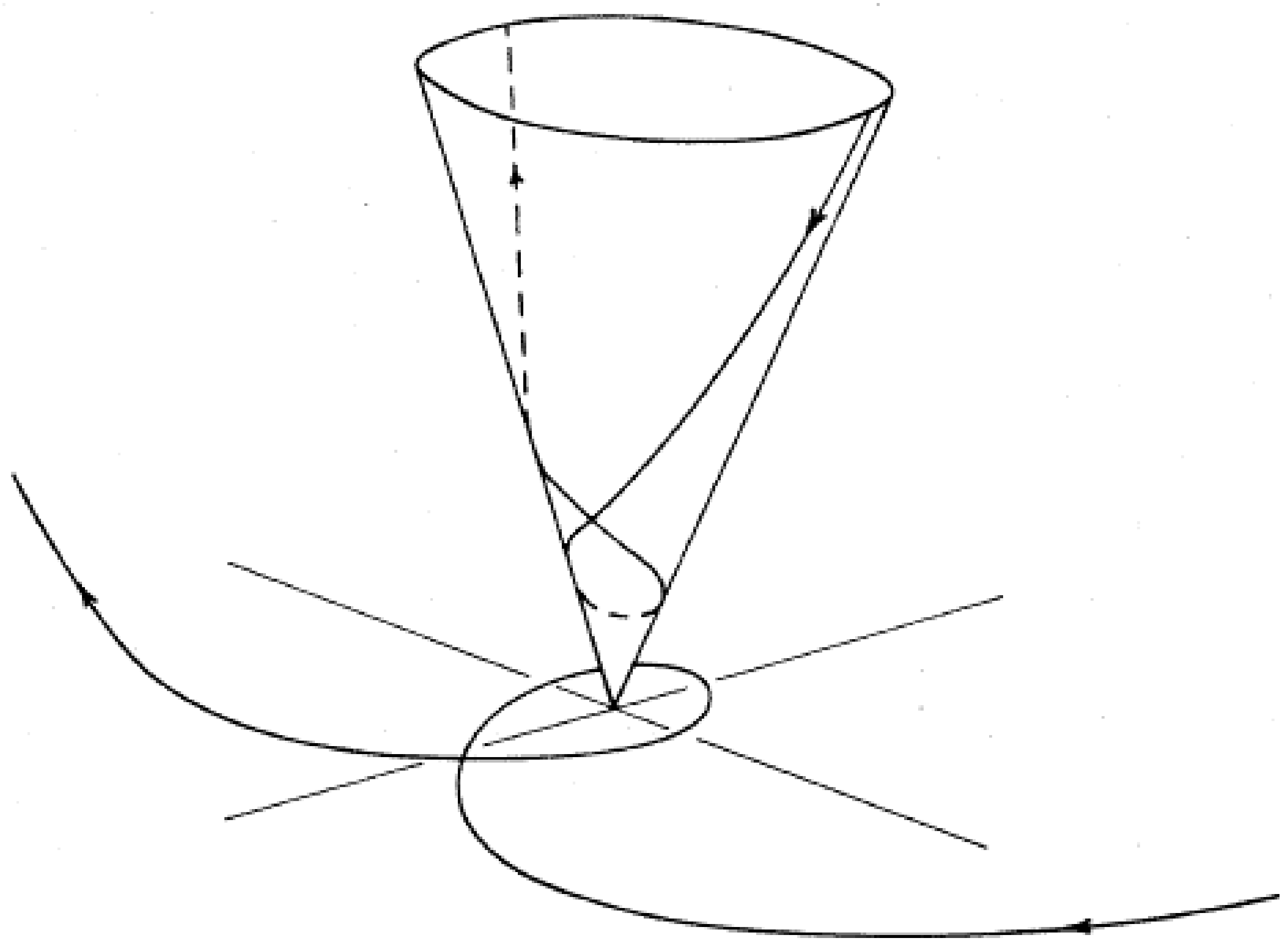
- Birkeland’s scientific efforts are honored on the 200-kroner Norwegian banknote.
  - In 1896 his major interest was Aurora Borealis.
- He was one of Poincare’s students in 1892

Magnetic monopole

- The nature of cathode rays was not understood in 1896, which were “discovered” to be electrons by J.J. Thomson in 1897 (in experiments with Crookes tubes and magnets).
- In 1896, before the Thomson’s discovery, Poincare has suggested that Birkeland’s experiment can be explained by “cathode rays being charges moving in the field of a magnetic monopole”
  – He wrote a brilliant paper in 1896, proving that charge motion in the field of magnetic monopole is fully integrable (but unbounded).

$$\mathbf{B} = \frac{k\mathbf{r}}{r^3}$$

Motion is on the cone surface



Aurora Borealis

- Beginning in 1904, a younger colleague, C. Størmer, was inspired by Birkeland to make extensive modeling of the trajectories of electrons in the Earth's magnetic field, approximated as that of a magnetic dipole.
  - Størmer studied under Darboux and Poincare in 1898-1900

Størmer and Birkeland in 1910

Two magnetic monopoles

- One can imagine that the motion of an electric charge between two magnetic monopoles (of opposite polarity) would be integrable, but it is not.
  - Only approximate "adiabatic" integrals exist, when poles are far apart (as compared to the Larmour radius)
  - This is the principle of a magnetic "bottle" trap; also, the principle of "weak focusing in accelerators".
- Non-integrability in this case is somewhat surprising because the motion in the field of two Coulomb centers is integrable.
  - This has been know since Euler and was Poincare's starting point for the 3-body problem quest.

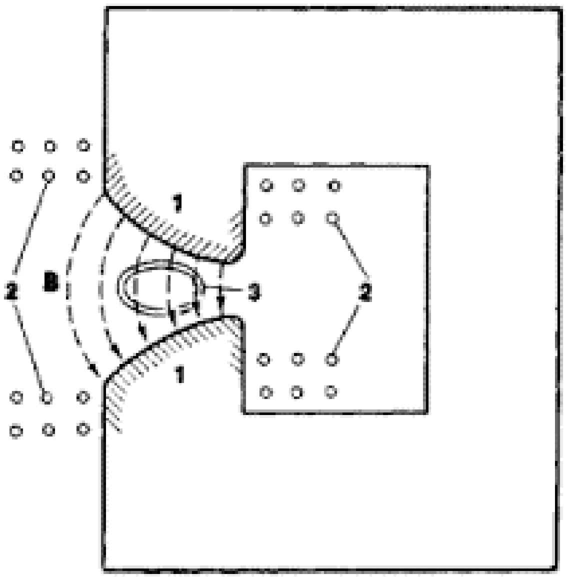
Particle motion in static magnetic fields

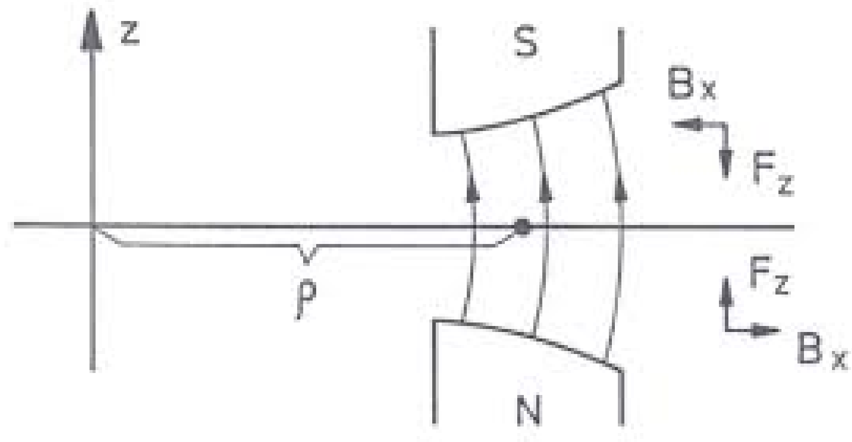
- For accelerators, there are no useful exactly integrable systems for axially symmetric magnetic fields in vacuum:

$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{r} - \frac{eA_\theta(r,z)}{c}\right)^2$$

- Until 1959, all circular accelerators relied on approximate (adiabatic) integrability.
  - These are the so-called weakly-focusing accelerators
  - Required large magnets and vacuum chambers to confine particles;

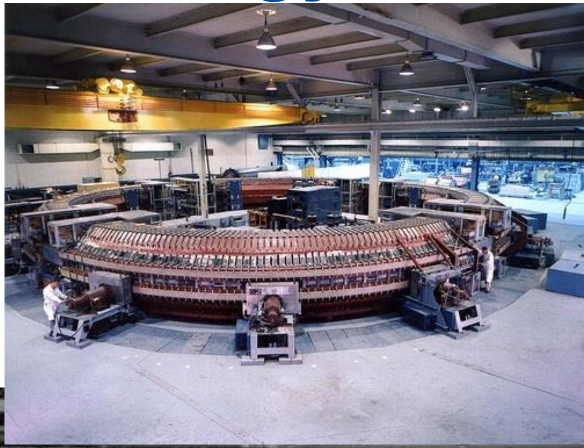
Weak focusing

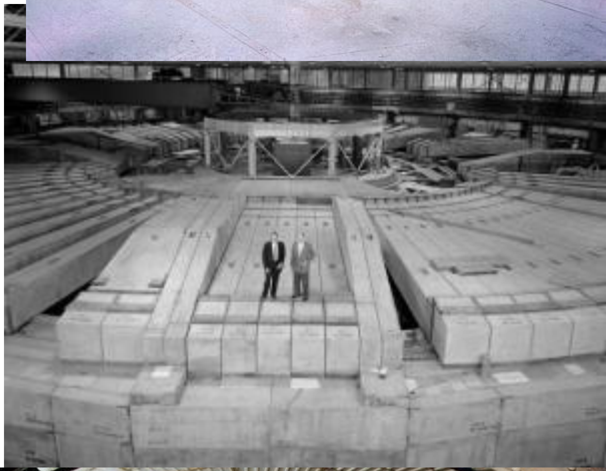




- The magnetic fields can be approximated by the field of two magnetic monopoles of opposite polarity

The race for highest beam energy





- Cosmotron (BNL, 1953-66): 3.3 GeV
  - Produced “cosmic rays” in the lab
  - Diam: 22.5 m, 2,000 ton
- Bevatron (Berkeley, 1954): 6.3 GeV
  - Discovery of antiprotons and antineutrons: 1955
  - Magnet: 10,000 ton
- Synchrophasatron (Dubna,1957): 10 GeV
  - Diam: 60 m, 36,000 ton
  - Highest beam energy until 1959

Strong Focusing

PHYSICAL REVIEW VOLUME 88, NUMBER 5 DECEMBER 1, 1952

The Strong-Focusing Synchroton—A New High Energy Accelerator*

ERNEST D. COURANT, M. STANLEY LIVINGSTON,† AND HARTLAND S. SNYDER
Brookhaven National Laboratory, Upton, New York

(Received August 21, 1952)

Strong focusing forces result from the alternation of large positive and negative *n*-values in successive sectors of the magnetic guide field in a synchrotron. This sequence of alternately converging and diverging magnetic lenses of equal strength is itself converging, and leads to significant reductions in oscillation amplitude, both for radial and axial displacements. The mechanism of phase-stable synchronous acceleration still applies, with a large reduction in the amplitude of the associated radial synchronous oscillations. To illustrate, a design is proposed for a 30-Bev proton accelerator with an orbit radius of 300 ft, and with a small magnet having an aperture of 1×2 inches. Tolerances on nearly all design parameters are less critical than for the equivalent uniform-*n* machine. A generalization of this focusing principle leads to small, efficient focusing magnets for ion and electron beams. Relations for the focal length of a double-focusing magnet are presented, from which the design parameters for such linear systems can be determined.

BETATRON OSCILLATIONS

RESTORING forces due to radially-decreasing magnetic fields lead to stable "betatron" and "synchrotron" oscillations in synchrotrons. The amplitudes of these oscillations are due to deviations from the equilibrium orbit caused by angular and energy spread in the injected beam, scattering by the residual gas, magnetic inhomogeneities, and frequency errors. The strength of the restoring forces is limited by the

* Work done under the auspices of the AEC.
† Massachusetts Institute of Technology, Cambridge, Massachusetts.

CERN Proton Synchrotron

- In Nov 1959 a 28-GeV Proton Synchrotron started to operate at CERN
  - 3 times longer than the Synchrophasatron but its magnets (together) are 10 times smaller (by weight)
  - Since then, all accelerators have strong focusing

Strong focusing

Specifics of accelerator focusing:

- Focusing fields must satisfy Maxwell equations in vacuum

$$\Delta\varphi(x, y, z) = 0$$

- For stationary fields: focusing in one plane while defocusing in another
  - quadrupole: $$\varphi(x, y) \propto x^2 - y^2$$
  - However, alternating quadrupoles results in effective focusing in both planes

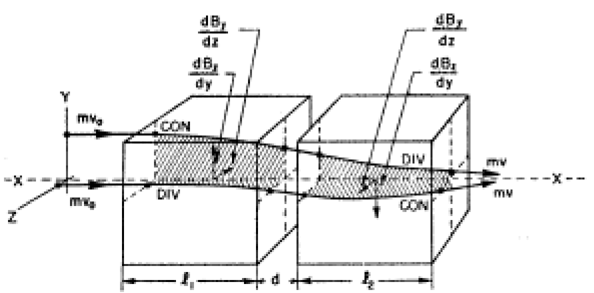


FIG. 8. Illustration of double-focusing in two magnetic lenses with field gradients in opposite directions, showing the alternately convergent and divergent forces and the net convergence of the system.

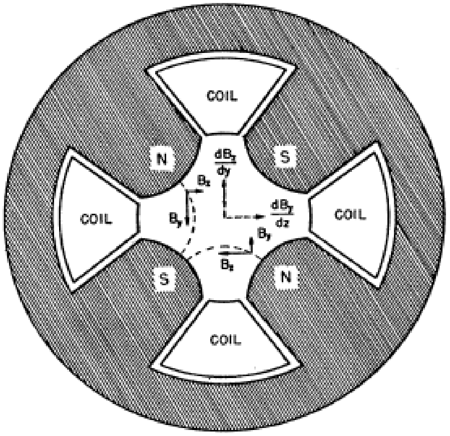
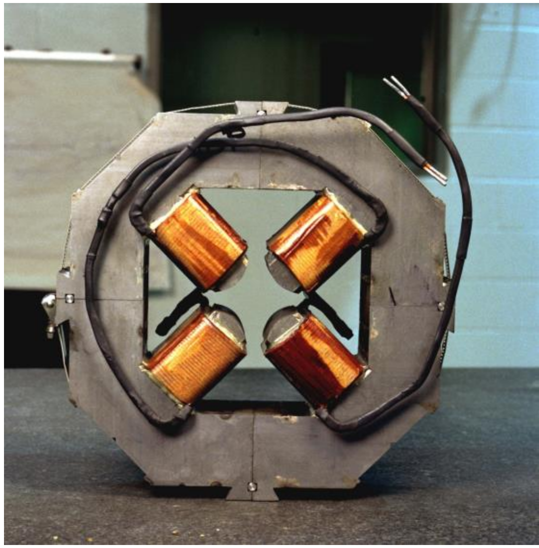


FIG. 9. Cross section of a 4-pole magnet with hyperbolic pole faces to produce uniform and equal field gradients dB_z/dy and dB_y/dz.



The accelerator Hamiltonian

$$H = c\left[m^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A} \right)^2 \right]^{\frac{1}{2}}$$

- After some canonical transformations and in a small-angle approximation

$$H' = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{x^2}{2\rho} + \frac{K}{2}\left(x^2 - y^2\right) - \frac{x\delta}{\rho} + \ldots$$

where δ is the relative momentum deviation. For δ << 1:

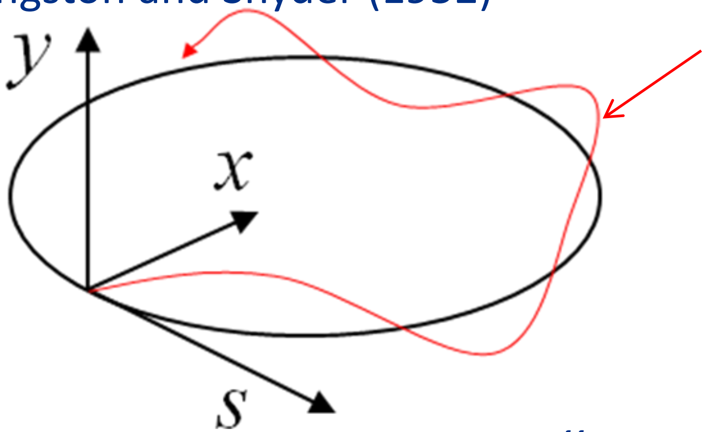
$$H' \approx \frac{p_x^2 + p_y^2}{2} + \frac{K_x(s)x^2}{2} + \frac{K_y(s)y^2}{2}$$

For a pure quadrupole magnet: $K_x(s) = -K_y(s)$

This Hamiltonian is separable!

Strong Focusing – Our Standard Approach Since 1952

Christofilos (1949); Courant, Livingston and Snyder (1952)



Particle undergoes betatron oscillations

$$\begin{cases} x'' + K_x(s)x = 0 \\ y'' + K_y(s)y = 0 \end{cases}$$

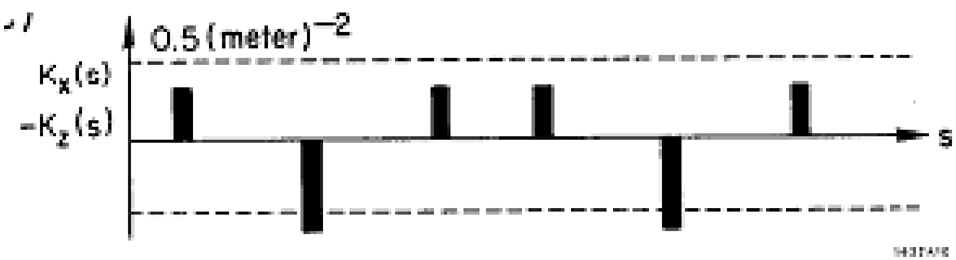
s is "time"

$$K_{x,y}(s + C) = K_{x,y}(s)$$

-- piecewise constant alternating-sign functions

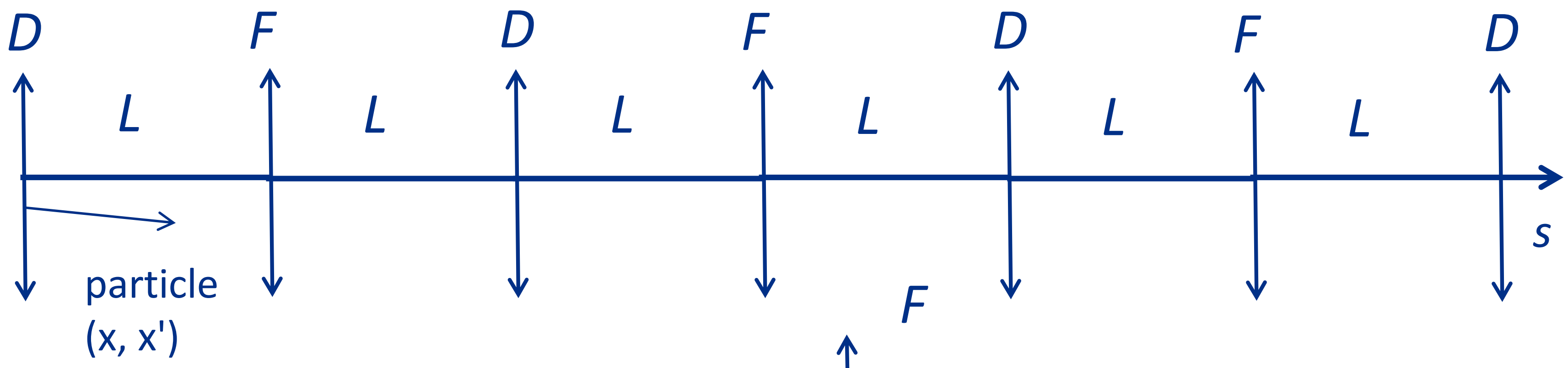
Focusing is provided by *linear* elements, dipoles and quadrupoles

$$H = \frac{1}{2}p_n^2 + \frac{1}{2}x_n^2$$

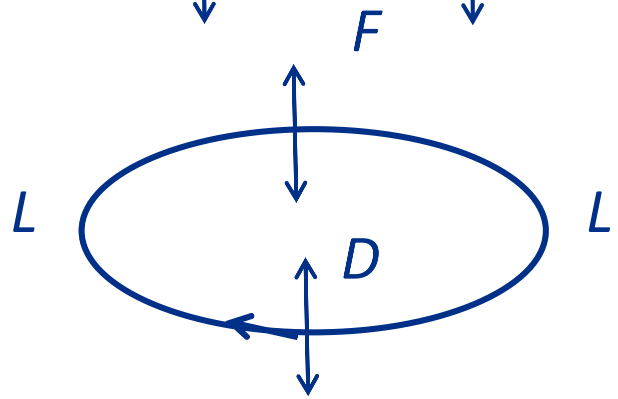


--Magnet lattice and focussing functions in the normal cells of a particular guide field.

A simple periodic focusing channel (FODO)



...Equivalent to:



- Thin alternating lenses and drift spaces

- Let's launch a particle with initial conditions *x* and x'

Simplest accelerator elements

- A drift space: *L* – length
$$\begin{pmatrix} x \\ x' \end{pmatrix}_L = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

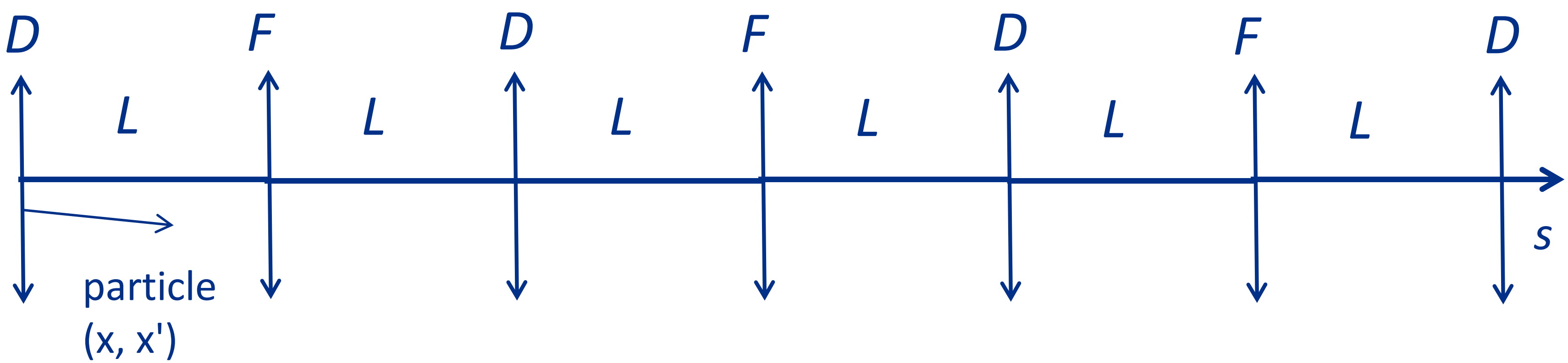
- A thin focusing lens:
$$\begin{pmatrix} x \\ x' \end{pmatrix}_F = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

- A thin defocusing lens:
$$\begin{pmatrix} x \\ x' \end{pmatrix}_D = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Particle stability in a simple channel

Particle motion is stable only for certain *L* and *F* $\boxed{0 < \frac{L}{F} < 2}$

When the motion is stable, it is periodic.



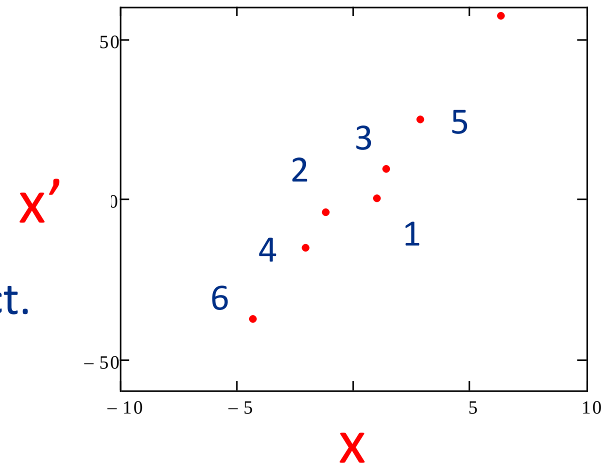
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \underbrace{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ F^{-1} & 1 \end{pmatrix}}\begin{pmatrix} x \\ x' \end{pmatrix}_{i}$$

Stability: $\left|Trace(M)\right| < 2$

Phase space trajectories

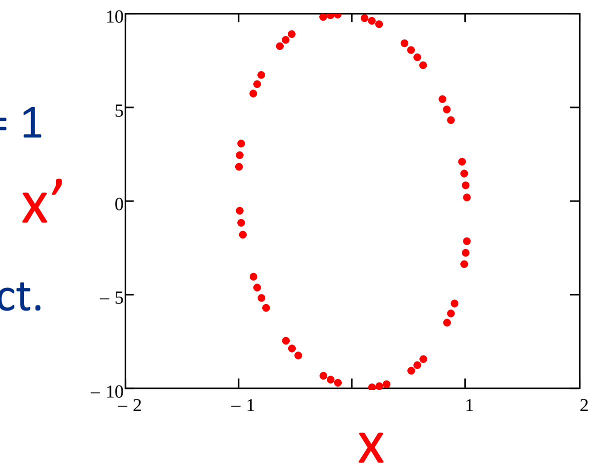
F = 0.49, *L* = 1

7 periods,
unstable traject.



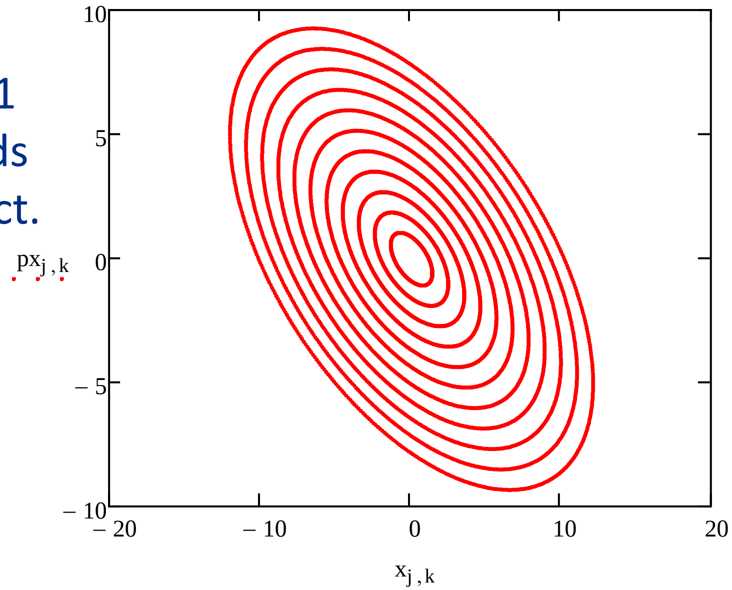
F = 0.51, *L* = 1

50 periods,
stable traject.



When this simple focusing channel is stable, it is stable for ALL initial conditions !

F = 1.2, *L* = 1
1000 periods
stable traject.



All trajectories are periodic and, ALL particles are isochronous: they oscillate with the same frequency (betatron tune)!

$$2\pi\nu = \operatorname{acos}\left(\frac{1}{2}\left|Trace(M)\right|\right)$$

Courant-Snyder invariant

Equation of motion for betatron oscillations

$$z''+K(s)z=0,$$

$$z = x \text{ or } y$$

$$I_z=\frac{1}{2\beta(s)}\left(z^2+\left(\frac{\beta'(s)}{2}z-\beta(s)z'\right)^2\right)$$

Invariant (integral) of motion, a conserved qty.

where $\left(\sqrt{\beta}\right)''+K(s)\sqrt{\beta}=\frac{1}{\sqrt{\beta^3}}$

Action-angle variables

$$H = \frac{p^2}{2} + \frac{K(s)z^2}{2}$$

$$F_1(z,\psi) = -\frac{z^2}{2\beta}\left[\tan\psi - \frac{\beta'}{2}\right]$$

$$I_z = -\frac{\partial F_1}{\partial \psi} = \frac{1}{2\beta(s)}\left(z^2 + \left(\frac{\beta'(s)}{2}z - \beta(s)z'\right)^2\right)$$

$$z = \sqrt{2I_z\beta}\cos\psi$$

$$p = -\sqrt{2I_z/\beta}\left(\sin\psi - \frac{\beta'}{2}\cos\psi\right)$$

$$H_1 = H + \frac{\partial F_1}{\partial s} = \frac{I_z}{\beta(s)}$$

- We can further remove the *s*-dependence by transforming the time variable, *s*.

The accelerator Hamiltonian

$$H = \frac{I_x}{\beta_x(s)} + \frac{I_y}{\beta_y(s)}$$

$$\text{where} \quad \left(\sqrt{\beta_{x,y}}\right)'' + K_{x,y}(s)\sqrt{\beta_{x,y}} = \frac{1}{\sqrt{\beta_{x,y}}^{\,3}}$$

- The time (s) dependence can be transformed out , but only after separating the Hamiltonian into the “x” and “y” parts.
- New ‘time” variable: $d\psi_{x,y} = \dfrac{ds}{\beta_{x,y}}$ → $\tilde{H}_{x,y} = I_{x,y}$

S. Nagaitsev, Jan 23, 2017

Non-linear focusing

- It became obvious very early on (~1960), that the use of nonlinear focusing elements in accelerators is necessary and some nonlinearities are **unavoidable** (magnet aberrations, space-charge forces, beam-beam forces)
  - Sexupoles appeared in 1960s for chromaticity corrections
  - Octupoles were installed in CERN PS in 1959 but not used until 1968. For example, the LHC has ~350 octupoles for Landau damping.
- It was also understood at the same time, that nonlinear focusing elements have both beneficial and detrimental effects, such as:
  - They drive nonlinear resonances (resulting in particle losses) and decrease the dynamic aperture (also particle losses).

Example: electron storage ring light sources

- Low beam emittance (size) is vital to light sources
  - Requires Strong Focusing
  - Strong Focusing leads to strong chromatic aberrations
  - To correct Chromatic Aberrations special nonlinear magnets (sextupoles) are added

dynamic aperture limitations lead to reduced beam lifetime

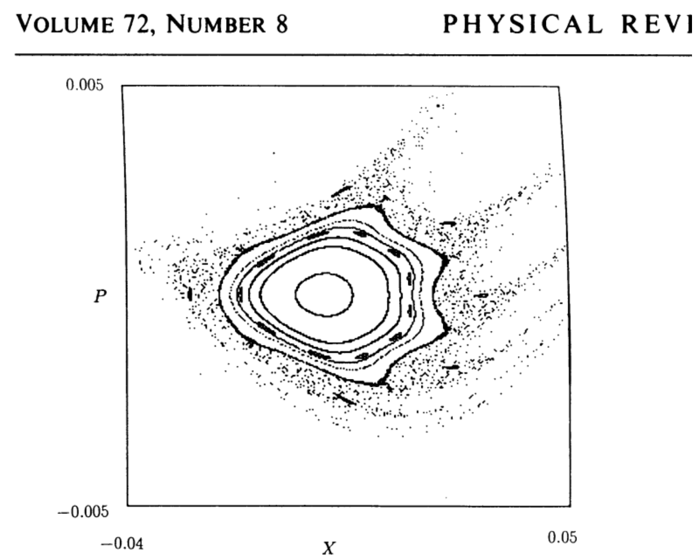


FIG. 1. Surface of section for the ALS.

Example: Landau damping

COLLIDING BEAMS: PRESENT STATUS; AND THE SLAC PROJECT*

B. Richter

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

The discovery in the early '60's at the Princeton-Stanford ring of what was thought to be the resistive wall instability brought the realization that circular accelerators are fundamentally unstable devices because of the interaction of the beam with its environment. Stability is achieved only through Landau damping and/or some external damping system.

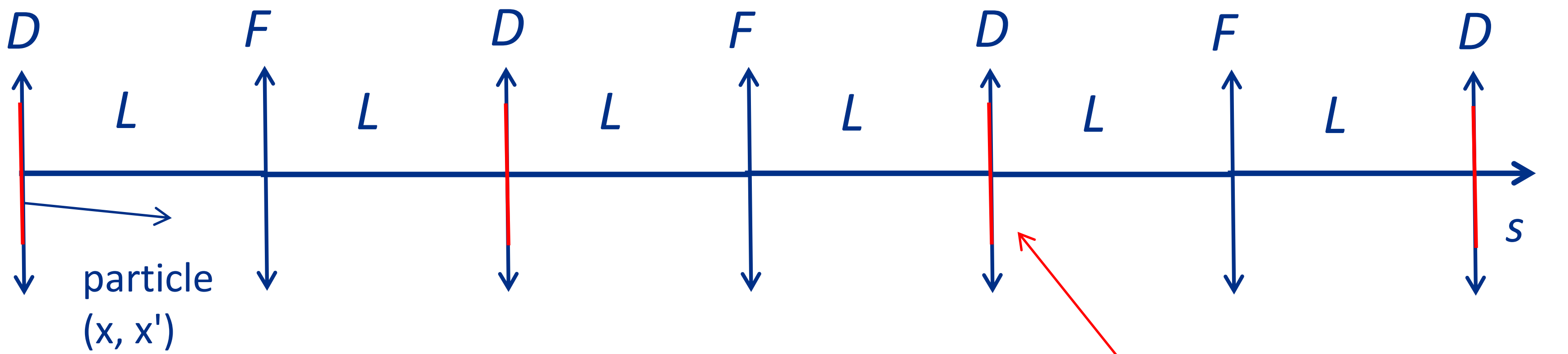
Report at HEAC 1971

- Landau damping – the beam's "immune system". It is related to the spread of betatron oscillation frequencies. The larger the spread, the more stable the beam is against collective instabilities.
  - The spread is achieved by adding special magnets -- octupoles
- External damping (feed-back) system – presently the most commonly used mechanism to keep the beam stable.

Most accelerators rely on both

- LHC:
  - Has a transverse feedback system
  - Has 336 Landau Damping Octupoles
- Octupoles (an 8-pole magnet):
  - Potential: $$\varphi(x,y) \propto x^4 + y^4 - 6x^2y^2$$
  - Results in a cubic nonlinearity (in force)

Let's add a cubic nonlinearity…



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F^{-1} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' - \alpha x^3 \end{pmatrix}_i$$

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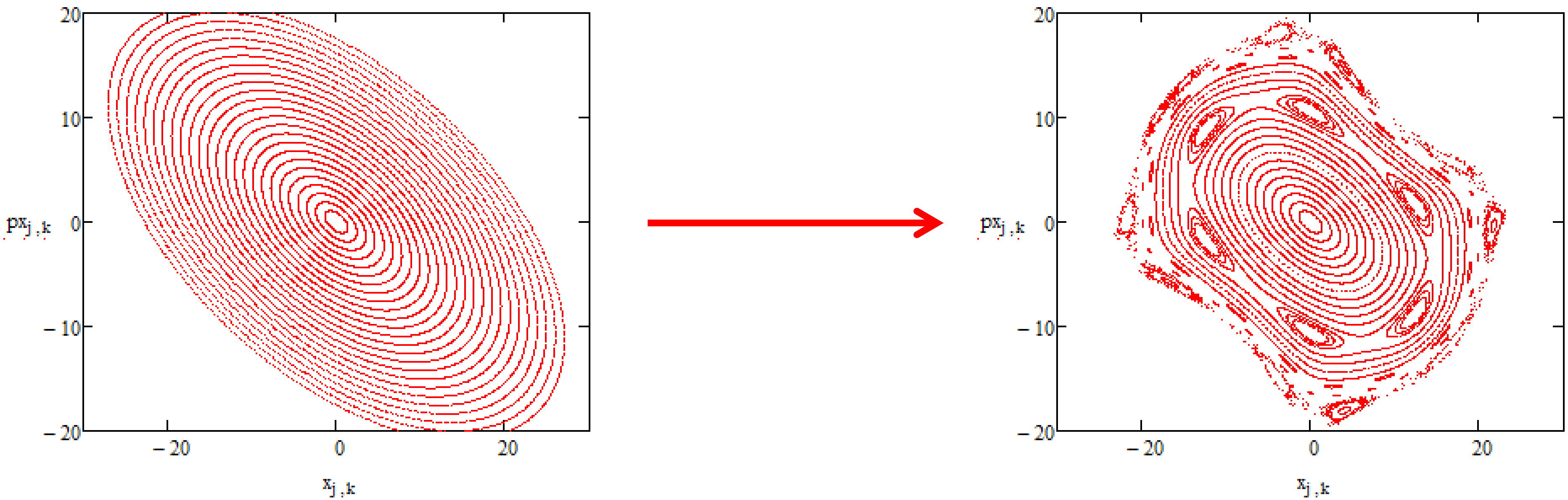
Jan 23, 2017

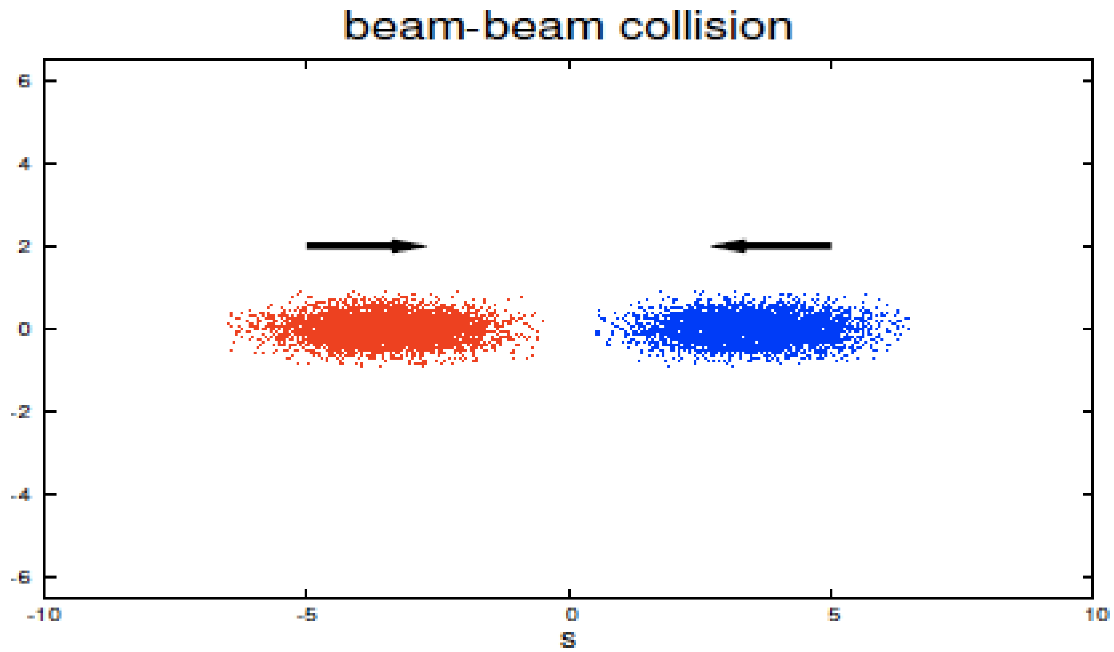
The result of this nonlinearity:

- Betatron oscillations are no longer isochronous:
  - The frequency depends on particle amplitude (initial conditions)
- Stability depends on initial conditions
  - Regular trajectories for small amplitudes
  - Resonant islands (for larger amplitudes)
  - Chaos and loss of stability (for even larger amplitudes)



Example: beam-beam effects

- Beams are made of relativistic charged particles and represent an electromagnetic potential for other charges



- Typically:
  - 0.001% (or less) of particles collide
  - 99.999% (or more) of particles are distorted

Beam-beam effects

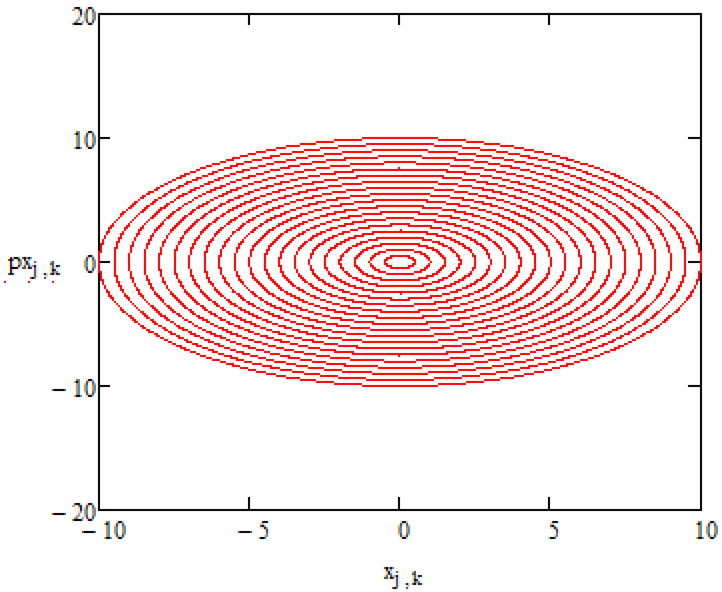
- One of most important limitations of all past, present and future colliders

Luminosity

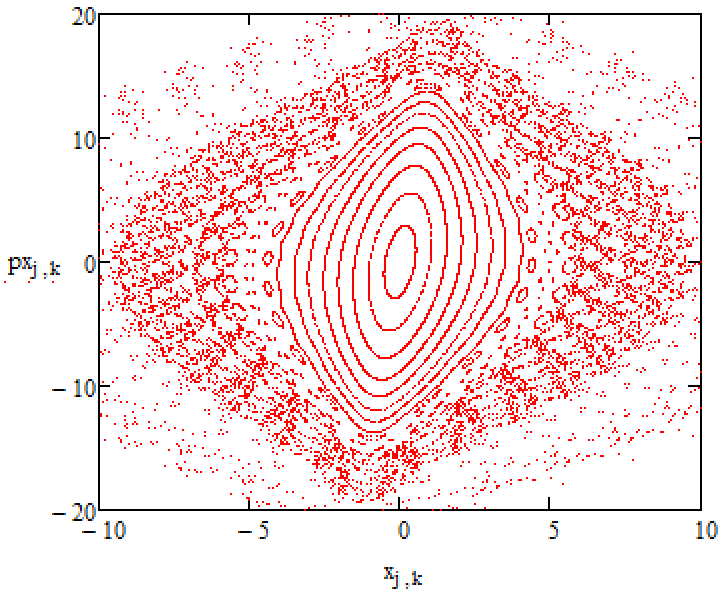
$$\mathcal{L} \propto \frac{N_p^2}{\sigma_x \sigma_y} \cdot n_b$$

Beam-beam Force

$$F \propto \frac{N_p}{\sigma} \cdot \frac{1}{r} \cdot \left[1 - e^{-\frac{r^2}{2\sigma^2}} \right]$$



beam-beam



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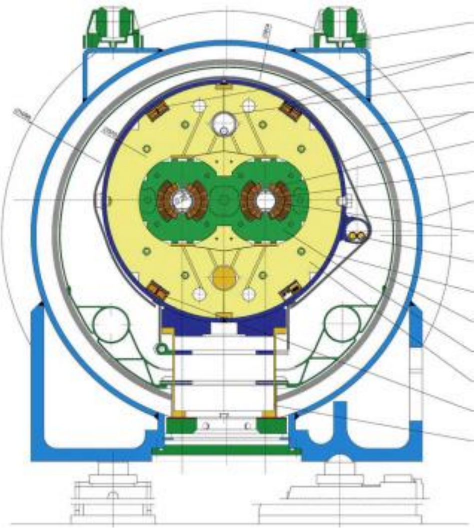
Challenges of modern accelerators (the LHC case)

- LHC: 27 km, 7 TeV per beam
  - The total energy stored in the magnets is HUGE: 10 GJ (2,400 kilograms of TNT)
  - The total energy carried by the two beams reaches 700 MJ (173 kilograms of TNT)
  - Loss of only one ten-millionth part (10^{-7}) of the beam is sufficient to quench a superconducting magnet
- LHC vacuum chamber diameter : ~40 mm
- LHC average rms beam size (at 7 TeV): 0.14 mm
- LHC average rms beam angle spread: 2 μrad
  - Very large ratio of forward to transverse momentum
- LHC typical cycle duration: 10 hrs = 4×10^8 revolutions
- Kinetic energy of a typical semi truck at 60 mph: ~7 MJ

What keeps particles stable in an accelerator?



- Particles are confined (focused) by static magnetic fields in vacuum.
  - Magnetic fields conserve the total energy
- An ideal focusing system in all modern accelerators is nearly integrable
  - There exist 3 conserved quantities (integrals of motion); the integrals are "simple" – polynomial in momentum.
  - The particle motion is confined by these integrals.

$$H \approx \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3$$

$$J = \frac{1}{2\pi} \oint p dq$$ -- particle's action

S. Nagaitsev, Jan 23, 2017

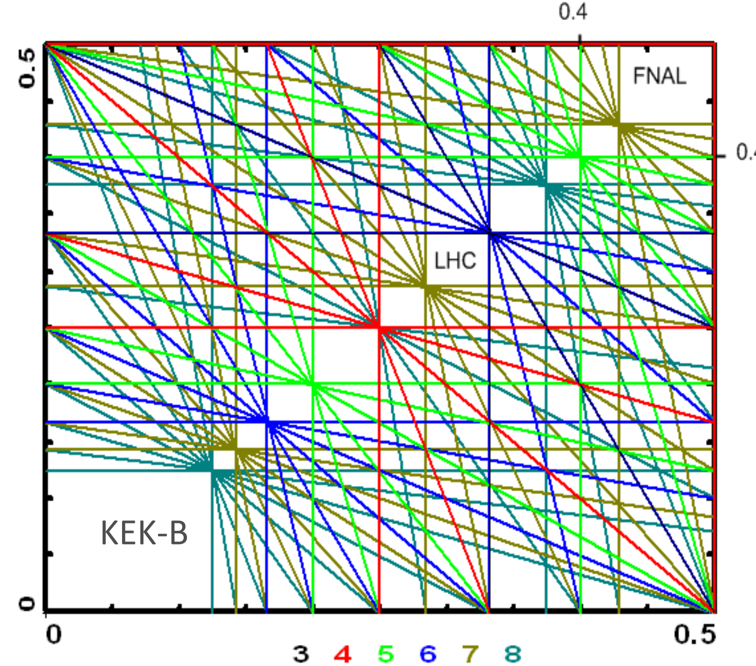
Summary so far

- Chaotic and unstable particle motion appears even in simplest examples of accelerator focusing systems with nonlinearities
  - The nonlinearity shifts the particle betatron frequency to a resonance ($n\omega_x + m\omega_y = k$)
  - The same nonlinearity introduces a time-dependent resonant kick to a resonant particle, making it unstable.
    - The nonlinearity is both the driving term and the source of resonances simultaneously

Integrability in Accelerators

- All present machines are designed to be integrable: drifts, quadrupoles, dipoles-- can all be accommodated in the Courant-Snyder invariants.
  - These are all examples of linear systems (equivalent to a harmonic oscillator)
- The addition of nonlinear focusing elements to accelerators breaks the integrability, …but this additions are necessary and unavoidable in all modern machines – for chromatic corrections, Landau damping, strong beam-beam effects, space-charge, etc

KAM theory



- Developed by Kolmogorov, Arnold, Moser (1954-63).
- Explains why we can operate accelerators away from resonances.
- The KAM theory states that if the system is subjected to a weak nonlinear perturbation, some of periodic orbits survive, while others are destroyed. The ones that survive are those that have “sufficiently irrational” frequencies (this is known as the non-resonance condition).
- Does not explain how to get rid of resonances
  - Obviously, for accelerators, making ALL nonlinearities to be ZERO would reduce (or eliminate) resonances
  - However, nonlinearities are necessary and unavoidable.

Nonlinear Integrable Systems

- **Are there "magic" nonlinearities with zero resonance strength?**
- **The answer is – *yes* (we call them "*integrable*")**
- Need two integrals of motion for transverse focusing (a 2-d system)
  - Strong focusing is a linear integrable system; two integrals of motion are the Courant-Snyder invariants
- There many integrable dynamical systems, but we know only a handful suitable for accelerators
- What we are looking for is a non-linear equivalent to Courant-Snyder invariants, for example

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{\alpha}{4}\left(x^4 + y^4\right)$$

Specifics of accelerator focusing

- The transverse focusing system is effectively time-dependent
  - In a linear system (strong focusing), the time dependence can be transformed away by introducing a new “time” variable (the betatron phase advance). Thus, we have the Courant-Snyder invariant.
- The focusing elements we use in accelerator must satisfy:
  - The Laplace equation (for static fields in vacuum)
  - The Poisson equation (for devices based on charge distributions, such as electron lenses or beam-beam interaction)

Non-linear elements

$$H_0 = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{K_x(s)x^2}{2} + \frac{K_y(s)y^2}{2}$$

The linear part

$$H_1 \approx H_0 + \frac{B''(s)}{6B\rho}\left(-3y^2x + x^3\right)$$

The sextupole part

$$H_2 \approx H_0 + \frac{B'''(s)}{24B\rho}\left(x^4 - 6y^2x^2 + y^4\right)$$

The octupole part

The addition of these nonlinear elements to accelerator focusing (almost always) makes it non-integrable.

--Time-dependent Henon-Heiles system

Accelerator research areas, where integrability would help

- Single particle dynamics:
  1. How to make the dynamical aperture larger? (light sources, colliders)
  2. How to make the tune spread larger? (Landau damping in high-intensity rings)
  3. How to reduce beam halo?
- Multi-particle dynamics:
  1. How to reduce detrimental beam-beam effects?
  2. How to compensate space-charge effects?
  3. How to suppress instabilities?
  4. How to reduce beam halo?

Integrable nonlinearities

- So, far we were able to find 2 classes of nonlinear accelerator-suitable systems;

1. Systems, where we are able to remove the time dependence, thus making it effectively autonomous.
   – This requires for the “time” variable to be the same in x and y. And then we can find some simple examples of autonomous “useful” integrable systems.
   – We know only a handful of examples in 4D
2. Systems, that are discrete integrable nonlinear mappings
   – This class originates from Edwin McMillan (the McMillan mapping).
   – We know only one example in 4D.

Topics for this workshop

1. Some nonlinear integrable systems are better than others. Which ones are most suitable for accelerators?
   – Nekhoroshev’s theory may be important here

2. We need more examples of accelerator-suitable 4D integrable mappings.

3. How to “correct” the existing nonlinearities in a ring to improve integrability?

4. How to compensate a distributed nonlinear force from space charge of the beam itself with a localized nonlinear element?

Nikolay Nekhoroshev

Russian Math. Surveys 32:6 (1977), 1–65
From Uspekhi Mat. Nauk 32:6 (1977), 5–66

AN EXPONENTIAL ESTIMATE OF THE
TIME OF STABILITY OF NEARLY-INTEGRABLE
HAMILTONIAN SYSTEMS

- Nekhoroshev's theory: a step beyond KAM
- Introduced the concept of "steepness".
  - The steep Hamiltonians are most stable
  - A linear Hamiltonian is not steep

Example 1

- Conceptually, we (at Fermilab) know now how to make a focusing system (with quadrupoles and thin octupoles), which results in the following 2D integrable nonlinear Hamiltonian

$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + \frac{\alpha}{4}\left(x_n^4 + y_n^4\right)$$

OR

$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + \frac{\alpha}{4}\left(x_n^2 + y_n^2\right)^2$$

In normalized variables

$$x_n = \frac{x}{\sqrt{\beta(s)}},$$

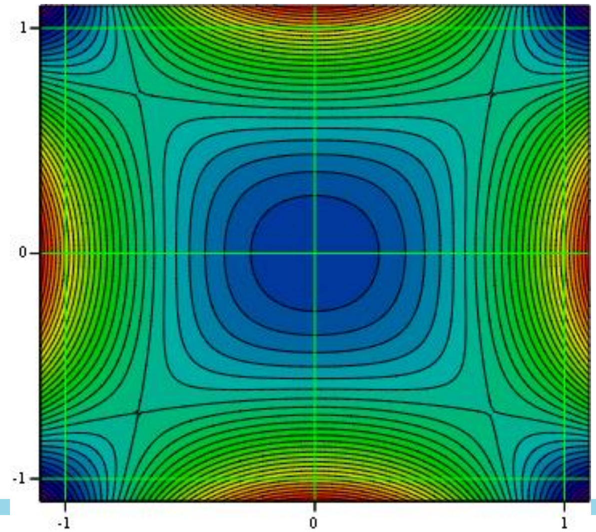
$$p_n = p\sqrt{\beta(s)} - \frac{\beta'(s)x}{2\sqrt{\beta(s)}},$$

- This concept we found is highly impractical but very important as it may serve as a model for modeling studies.

Example 2

- A nonlinear partially-integrable focusing system with one integral of motion. Can be implemented in practice (with octupoles). This is one of the systems we are planning to test at Fermilab.
- A Henon-Heiles type system

$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + \frac{\alpha}{4}\left(x_n^4 + y_n^4 - 6x_n^2 y_n^2\right)$$



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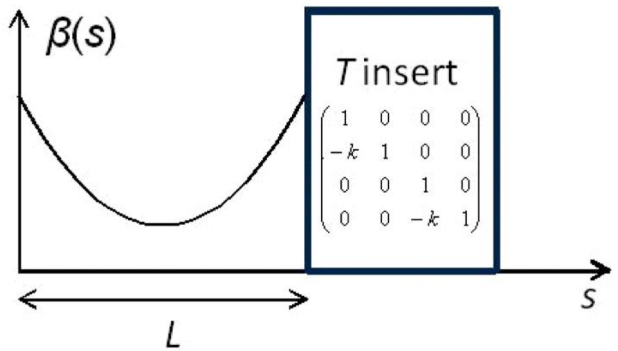
Implementation

1 Start with a round axially-symmetric *linear* lattice (FOFO) with the element of periodicity consisting of

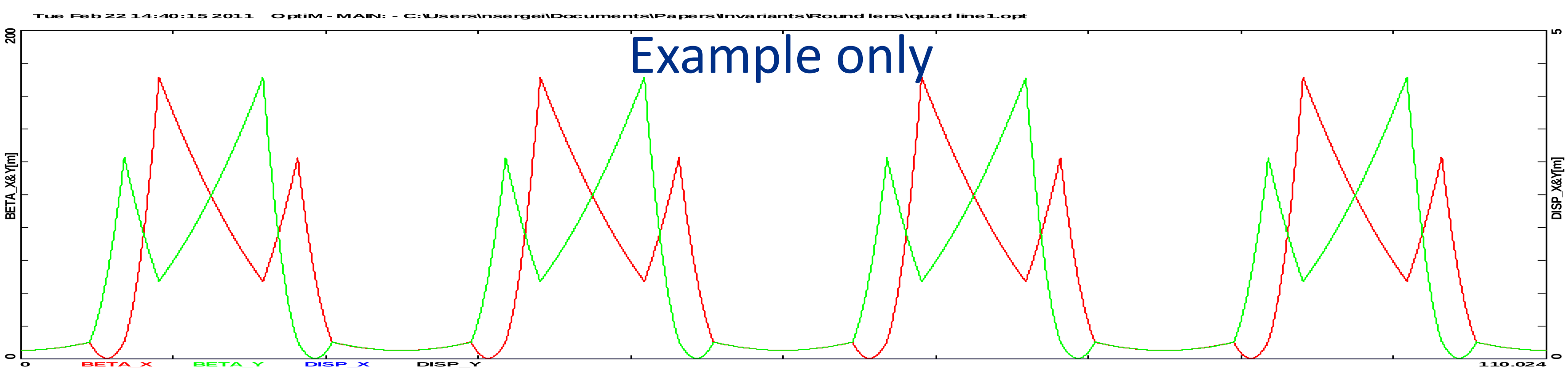
  a. Drift L

  b. Axially-symmetric focusing block “T-insert” with phase advance n × π

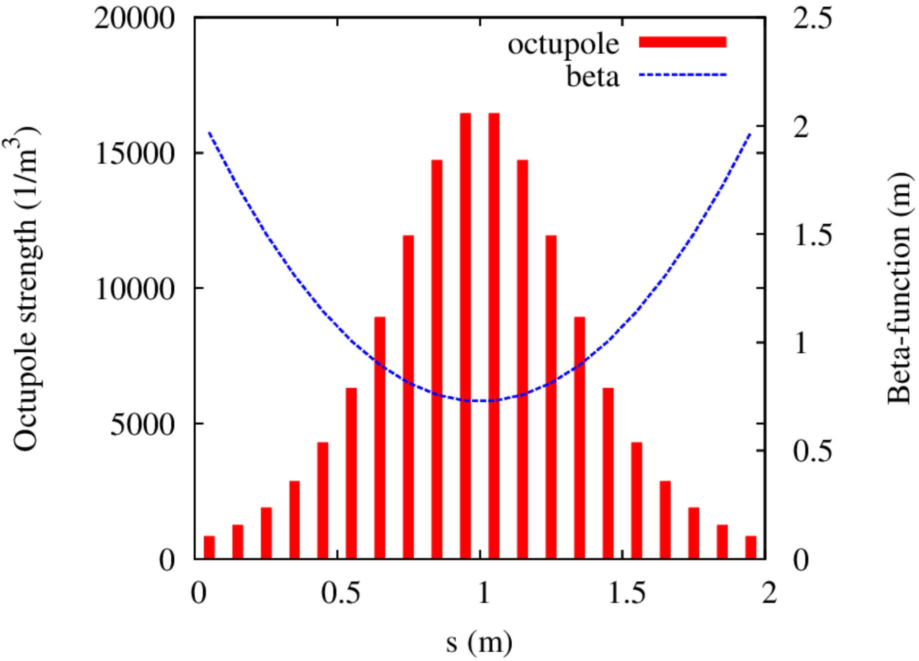


2 Add special nonlinear potential *V(x,y,s)* in the drift such that

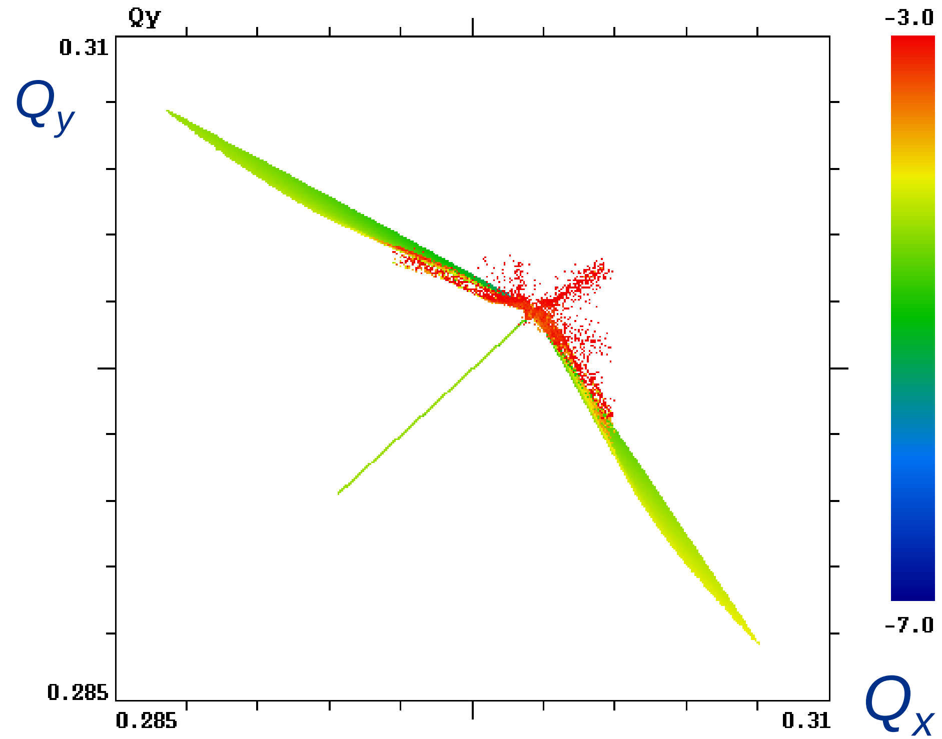
$$\Delta V(x, y, s) \approx \Delta V(x, y) = 0$$

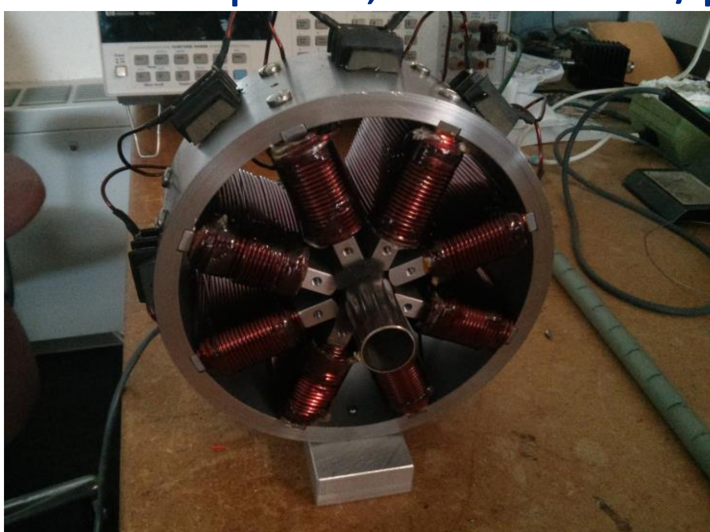


Octupoles



20 octupoles, scaled as $1/\beta(s)^3$





- While the dynamic aperture is limited, the attainable tune spread is large ~0.03 – compare to 0.001 created by LHC octupoles

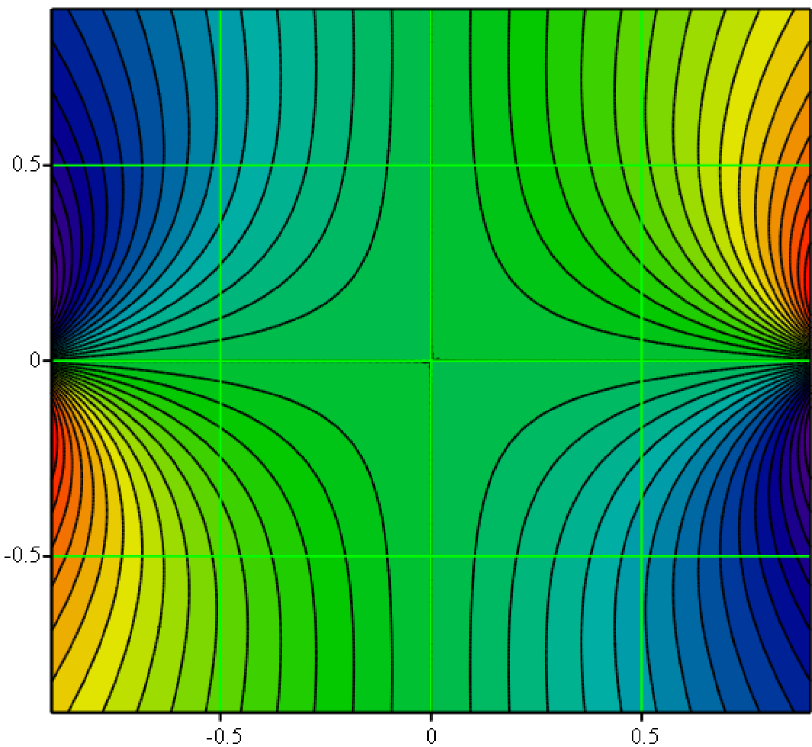
Example 3 (Phys. Rev. Accel. Beams 13, 084002)

- An integrable nonlinear system with a special Darboux potential (separable in elliptic coordinates).

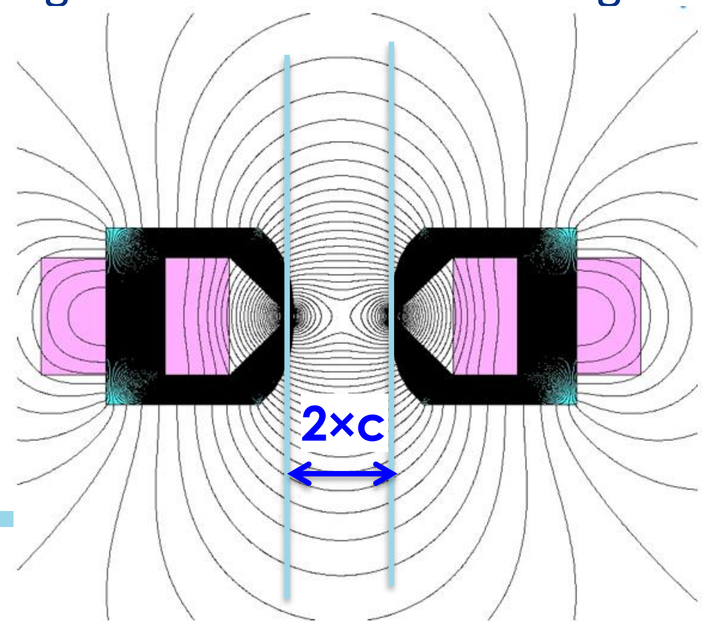
$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + U(x_n, y_n)$$

$$U(x,y) \approx \frac{t}{c^2}\,\mathrm{Im}\left((x+iy)^2 + \frac{2}{3c^2}(x+iy)^4 + \frac{8}{15c^4}(x+iy)^6 + \frac{16}{35c^6}(x+iy)^8 + \ldots \right)$$

For $|z| < c$



This potential has two adjustable parameters: t – strength and c – location of singularities



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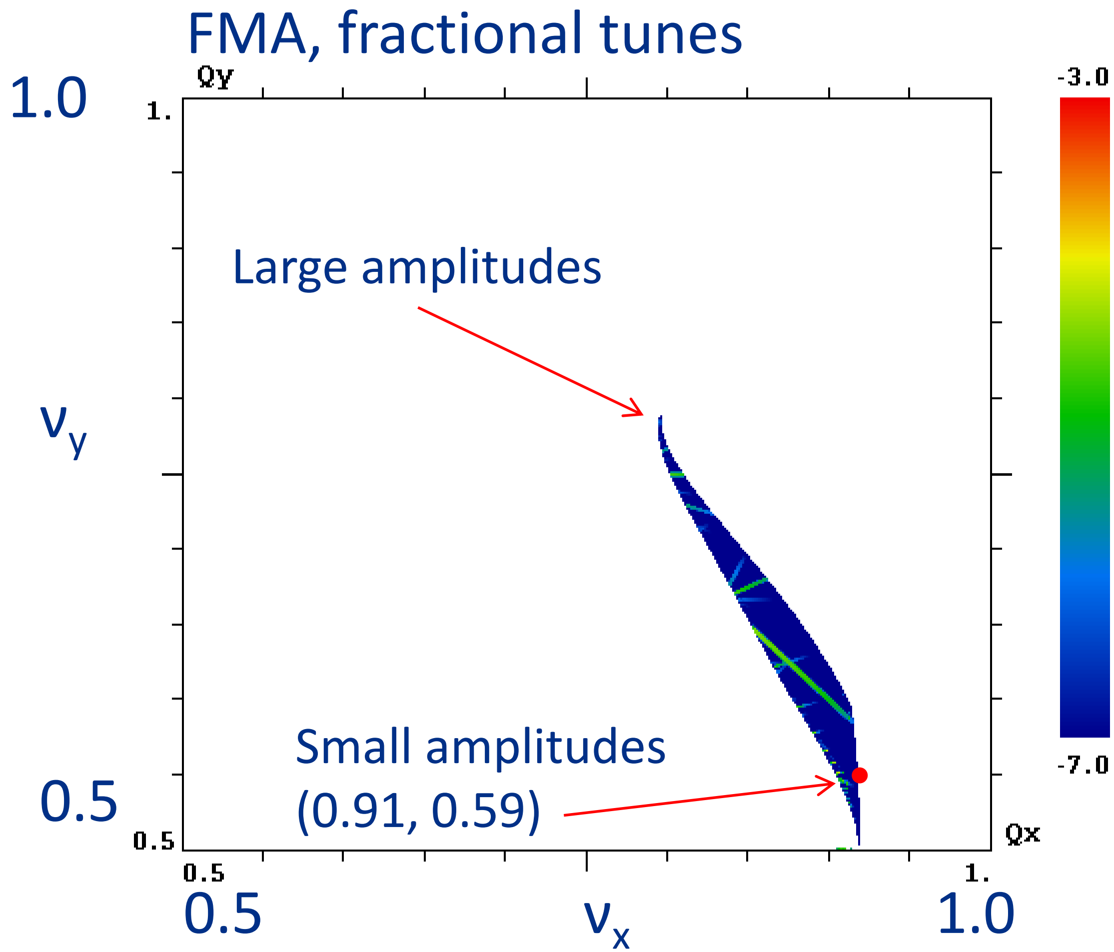
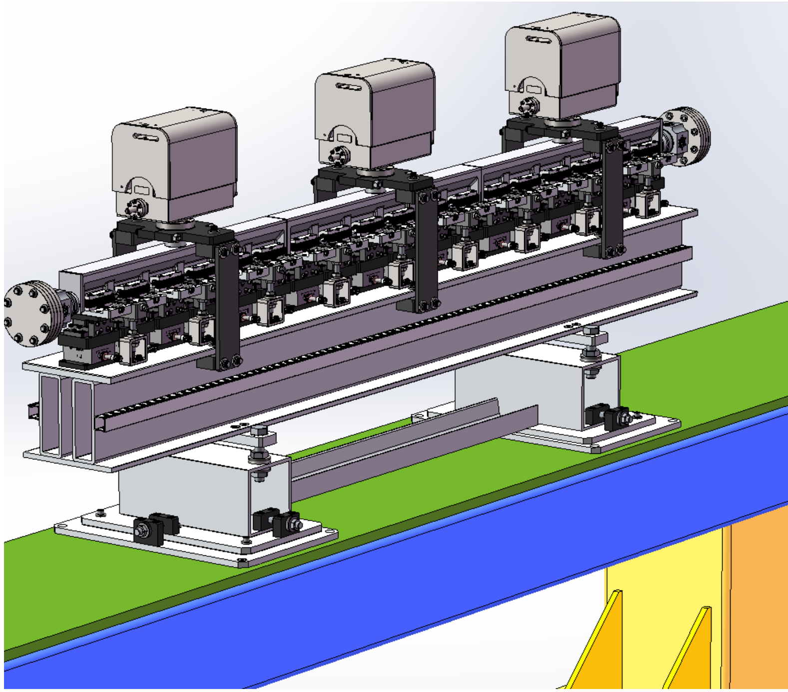
- A single 2-m long nonlinear lens creates a tune spread of ~0.25.





1.8-m long magnet to be delivered in 2016

Example 4: McMillan mapping

- In 1967 E. McMillan published a paper

SOME THOUGHTS ON STABILITY
IN NONLINEAR PERIODIC FOCUSING SYSTEMS

Edwin M. McMillan

September 5, 1967

- Final report in 1971. This is what later became known as the "McMillan mapping":

$$\boxed{\begin{aligned} x_i &= p_{i-1} \\ p_i &= -x_{i-1} + f(x_i) \end{aligned}}$$

$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

$$Ax^2p^2 + B\left(x^2p + xp^2\right) + C\left(x^2 + p^2\right) + Dxp = \text{const}$$

If $A = B = 0$ one obtains the Courant-Snyder invariant

McMillan 1D mapping

- At small *x*: $f(x) \to -\frac{D}{C}x$

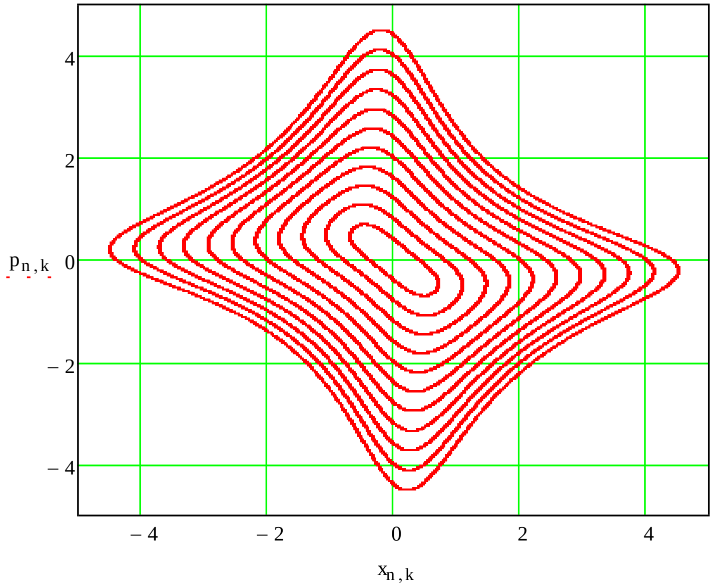
$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

Linear matrix: $\begin{pmatrix} 0 & 1 \\ -1 & -\frac{D}{C} \end{pmatrix}$ Bare tune: $\frac{1}{2\pi}\text{acos}\left(-\frac{D}{2C}\right)$

- At large *x:* $f(x) \to 0$

Linear matrix: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Tune: 0.25

A=1, *B* = 0, *C* = 1, D = 2



McMillan mapping in 2d

- We were unable to extend this mapping into 2d with magnets (Laplace equation).
- We have a solution on how to realize such a lens with a charge column (Poisson equation).

1. A ring with a transfer matrix

$$\begin{pmatrix} cI & sI \\ -sI & cI \end{pmatrix} \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\frac{1}{\beta} & 0 \end{pmatrix}$$

$$c = \cos(\phi)$$
$$s = \sin(\phi)$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. An axially-symmetric kick

$$f(r) = \frac{kr}{ar^2 + 1}$$

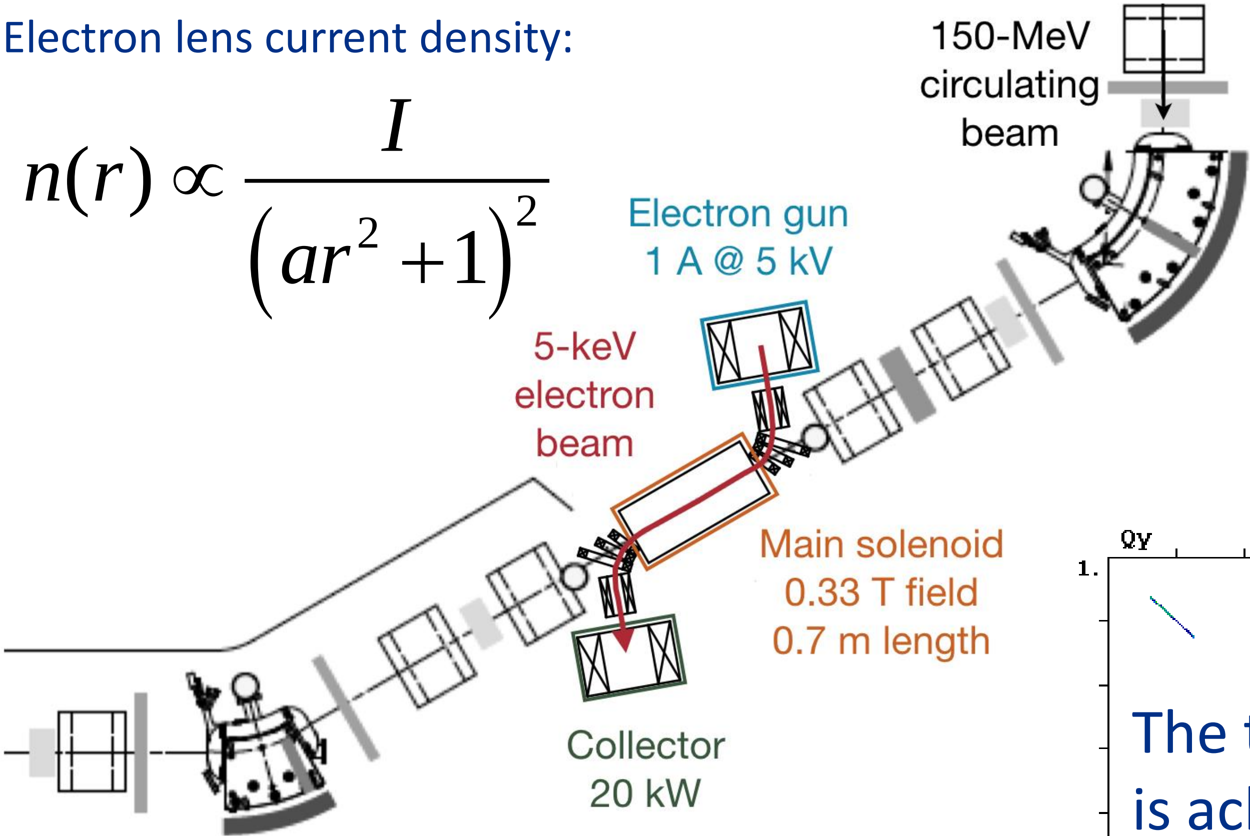
can be created with an electron lens

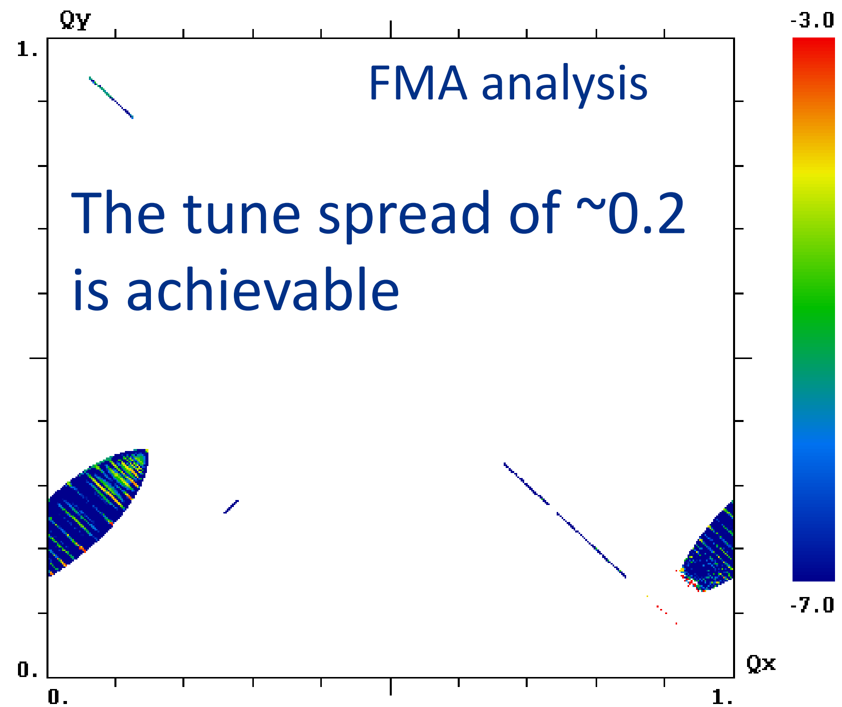
$$\boxed{\begin{aligned} x_i &= p_{i-1} \\ p_i &= -x_{i-1} + f(x_i) \end{aligned}}$$

McMillan electron lens

Electron lens current density:

$$n(r) \propto \frac{I}{\left(ar^2+1\right)^2}$$



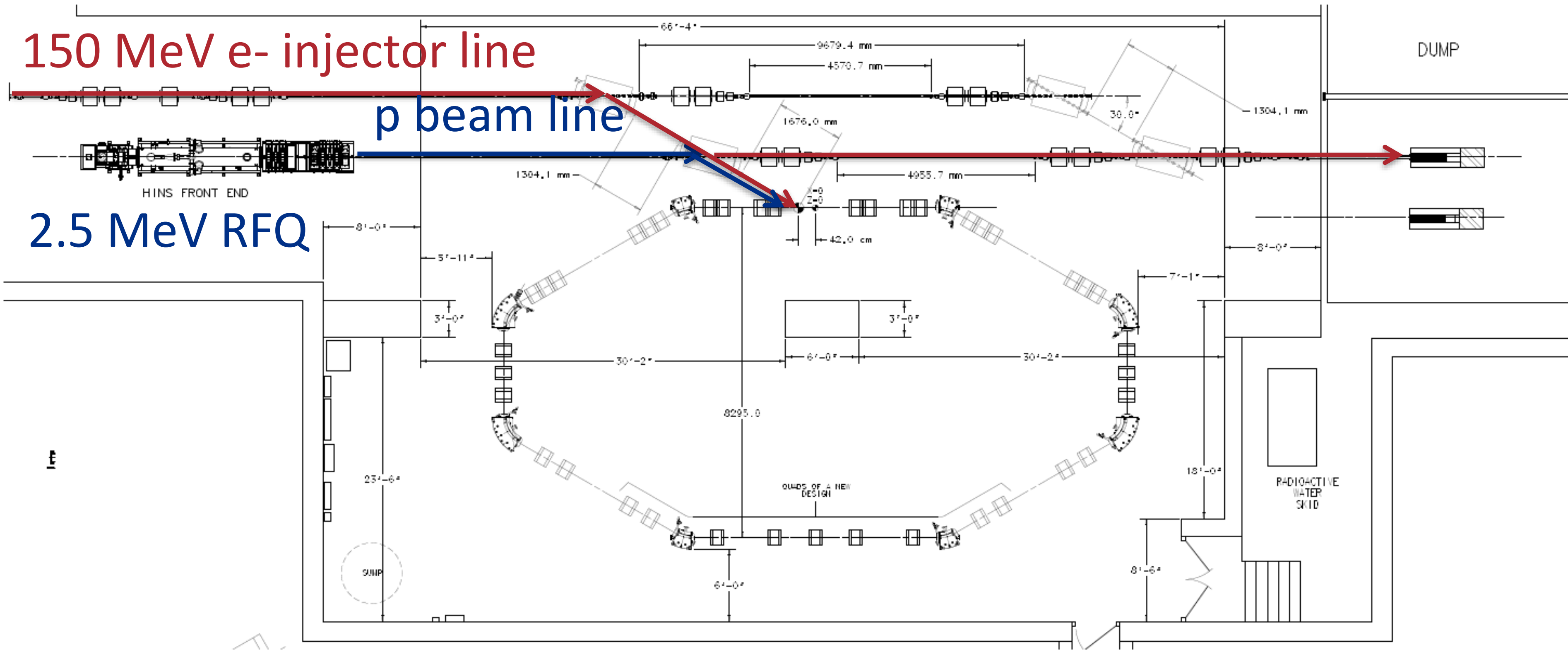


Enter the IOTA ring at Fermilab

- We have several **innovative ideas for Research**:
  - *Integrable Nonlinear Optics*
  - *Space Charge Compensation*

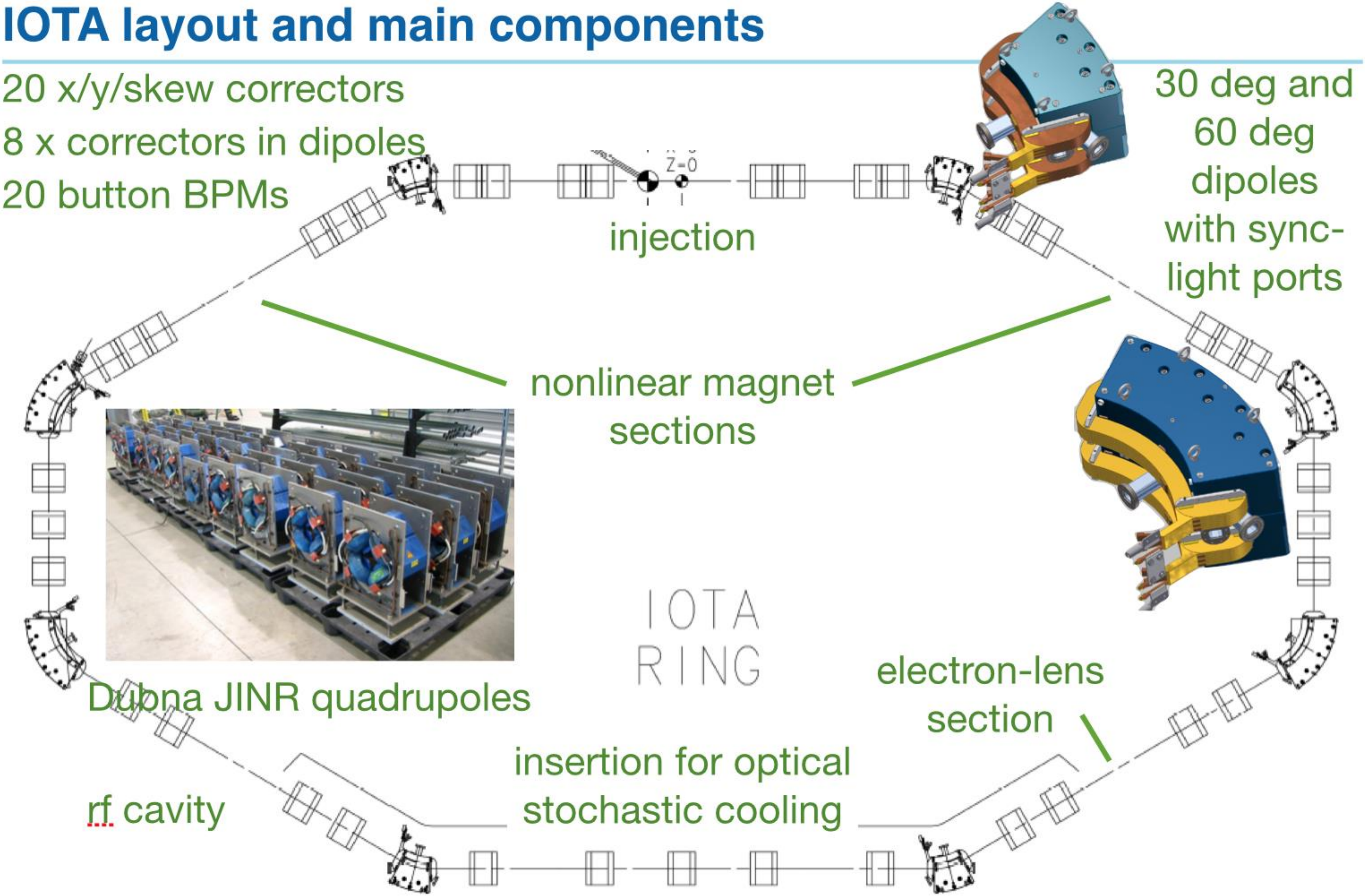
- To test them, we are building the **Integrable Optics Test Accelerator (IOTA)**

IOTA Ring

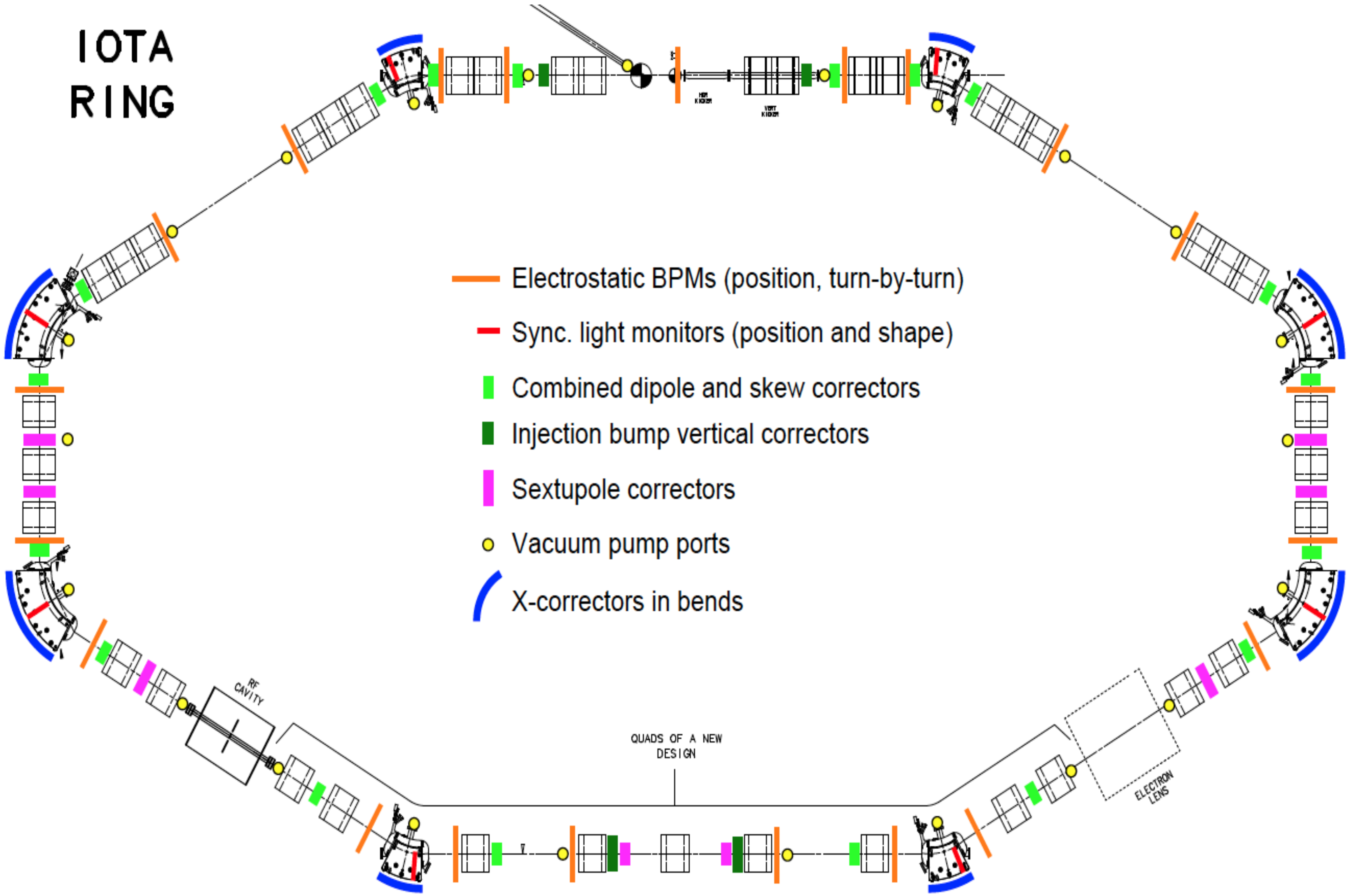


150 MeV e- injector line
p beam line
2.5 MeV RFQ
HINS FRONT END
DUMP
RADIOACTIVE WATER SKID
SUMP
QUADS OF A NEW DESIGN

IOTA layout and main components



IOTA Layout



S. Nagaitsev, Jan 23, 2017

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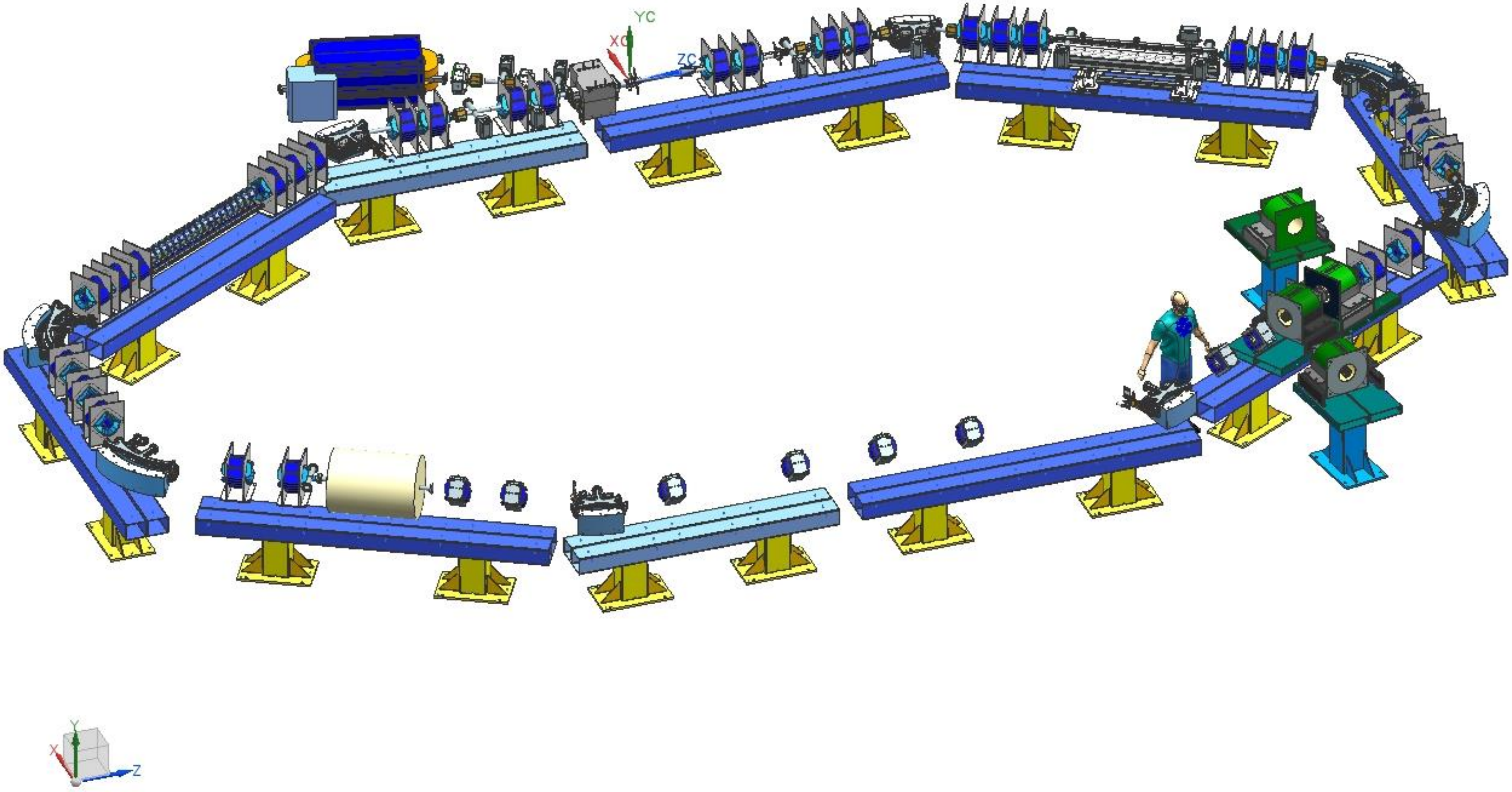
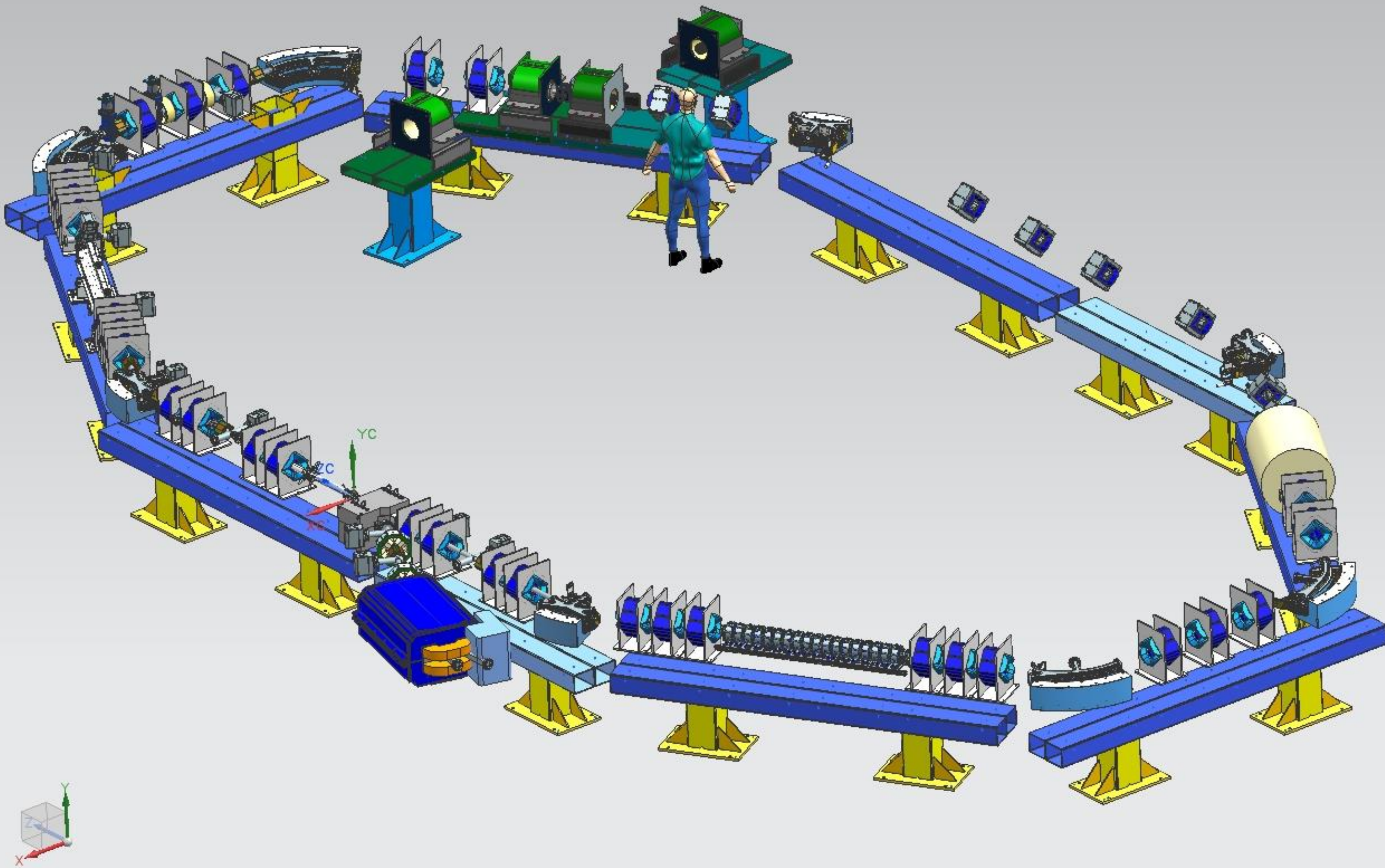
IOTA Layout



IOTA Layout



Summary

We (at Fermilab and UChicago) have a very exciting research program centered around nonlinear beam dynamics

1. Nonlinear Integrable Optics
2. Space Charge Compensation

- Inviting math collaborators to join us in advancing the accelerator focusing for the next generation machines.

Topics for this workshop

1. Some nonlinear integrable systems are better than others. Which ones are most suitable for accelerators?
   – Nekhoroshev's theory may be important here

2. We need more examples of accelerator-suitable 4D integrable mappings.

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