A mean-field model of collective effects in beam dynamics

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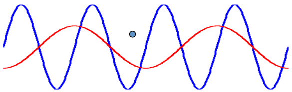
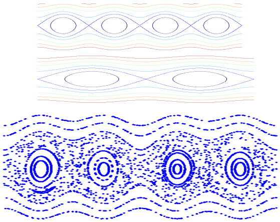
LOW DEGREES-OF-FREEDOM HAMILTONIAN SYSTEMS

- The simplest Hamiltonian systems with nontrivial (chaotic) dynamics are the well-understood 1-1/2 degrees-of-freedom systems
$$H(q,p,t) = \frac{p^2}{2m} + \phi(q,t)\,.$$
- A canonical example is a charged particle in 1-d in a time-dependent external electrostatic field

Chaotic motion in a two-waves field

$$\phi = \cos(k_1 x - \omega_1\, t) + \cos(k_2 x - \omega_2\, t)$$





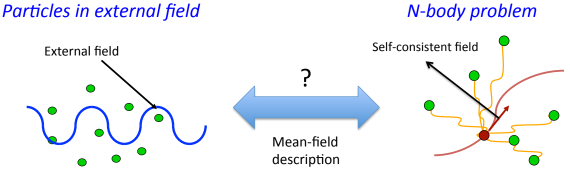
- When the spatial dimensionality increases, $d = 2\,,3$, this single particle problem complicates but relatively speaking (i.e., compared with what comes next) is a tractable problem.

VERY LARGE NUMBER OF DEGREES-OF-FREEDOM

- A canonical example is the (extremely difficult) *N*-body problem in which each particle interacts with each other, e.g.

$$H(q_i, p_i, t) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i<j} \phi(|q_i - q_j|)\,.$$

- The main motivation underlying mean-field models is to find a tractable description of intermediate complexity between the *N*-body problem and the dynamics in an external field.



- Among the key problems we would like to study is chaos and integrability in very large d.o.f. systems.

MEAN-FIELD MODELS

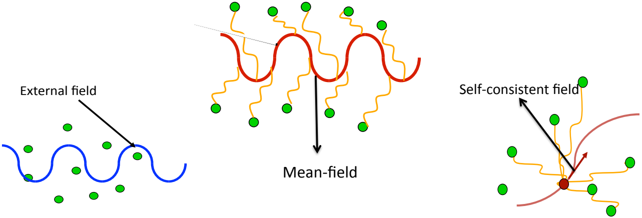
- Like in the external field problem, in the mean-field description all the particles "see" the same field

$$H(q_i, p_i, t) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_i \phi(q_i; \lambda) .$$

- But, like in the N-body problem there is a coupling between the particles that feeds-back onto the mean-field

$$\lambda = \mathcal{D}\left(q_1, q_2, \ldots q_N\right) .$$

Mean-field description



Particles in external field

N-body problem

THE SINGLE WAVE MODEL

- The mean-field model of interest here is the so-called Single-Wave-Model (SWM) which is a Hamiltonian system consisting of an ensemble of N-particles in one-dimension

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial u_j}, \qquad \frac{du_j}{dt} = -\frac{\partial H}{\partial x_j},$$

with a single-wave potential Hamiltonian

$$H(q_i, p_i, t) = \sum_{k=1}^{N} \left[\frac{u_k^2}{2} - a(t) e^{ix_k} - a^*(t) e^{-ix_k} \right].$$

- In this model the mean-field coupling determines the time evolution of the single-wave potential amplitude from

$$\frac{da}{dt} - iUa = \frac{i}{N} \sum_{k=1}^{N} \Gamma_k e^{-ix_k}.$$

where U and Γ_k, $k = 1, 2, \ldots N$ are constants.

THE SINGLE WAVE MODEL

- Writing

  $$a = \sqrt{J}e^{-i\theta}$$

  the SWM can be equivalently written as an ensemble of N globally coupled "pendulums"

  $$\frac{d^2 x_j}{dt^2} = -2\sqrt{J}\cos(x_j - \theta)\,, \qquad j = 1, \ldots N$$

- Where the mean-field coupling determines the time evolution of the amplitude J and the phase θ from

  $$\frac{dJ}{dt} = \frac{2\sqrt{J}}{N}\sum_{k=1}^{N}\Gamma_k \sin\left(x_k - \theta\right)$$

  $$\frac{d\theta}{dt} = -U - \frac{1}{N\sqrt{J}}\sum_{k=1}^{N}\Gamma_k \cos\left(x_k - \theta\right)$$

THE SINGLE WAVE MODEL: $N+1$ HAMILTONIAN FORMULATION

- Defining

$$a = \sqrt{J} e^{-i\theta} \,, \qquad p_k = \Gamma_k y_k \,,$$

the SWM can be equivalently written as an $N+1$, particles+field, Hamiltonian system

$$\frac{dx_k}{dt} = \frac{\partial \mathcal{H}}{\partial p_k} \,, \qquad \frac{dp_k}{dt} = -\frac{\partial \mathcal{H}}{\partial x_k} \,,$$

$$\frac{d\theta}{dt} = \frac{\partial \mathcal{H}}{\partial J} \,, \qquad \frac{dJ}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta} \,,$$

in which (x_k, p_k) are the canonical coordinates of the N particles, (θ, J) are the canonical coordinates of the mean-field, and

$$\mathcal{H} = \sum_{j=1}^{N} \left[\frac{1}{2\Gamma_j} \frac{p_j^2}{2} - 2\Gamma_j \sqrt{J} \cos(x_j - \theta) \right] - UJ \,.$$

THE SINGLE WAVE MODEL: $N \to \infty$ LIMIT

- In the $N \to \infty$ limit

$$(x_j, u_j) \to \text{single particle PDF } f(x, u)$$

- Where f satisfies the Liouville equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0\,,$$

with the single-wave potential

$$\phi = a(t)e^{ix} + a^*(t)e^{-ix}$$

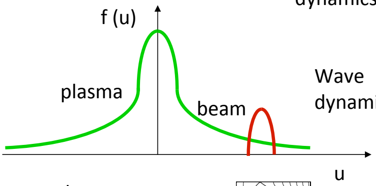
- The time evolution of the potential amplitude is given by the mean-field coupling

$$\frac{da}{dt} - iUa = i\frac{1}{2\pi}\int_0^{2\pi} dx \int_{-\infty}^{\infty} du\, f(x, u, t)\,.$$

THE SINGLE WAVE MODEL: THE ORIGINS

- As many cool ideas, the origins of the SWM go back to plasma physics!
- It was originally postulated and physically motivated (but not actually derived) in the study of the resonant wave-particle interaction in the beam-plasma instability

The beam-plasma instability

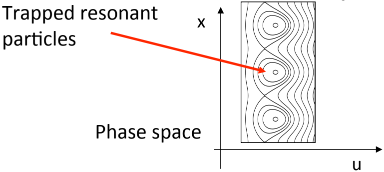


N-particle dynamics

$$\frac{d^2 x_j}{dt^2} = -2\rho(t)\sin\left[x_j - \theta(t)\right]$$

Wave dynamics

$$\frac{d}{dt}\left(\rho\, e^{-i\theta}\right) + iU\rho\, e^{-i\theta} = i\sum_k e^{-ix_k}$$



- Onischenko, Linetskii, et al. (1970)
- O' Neil, Winfrey &Malmberg,(1971)

THE SINGLE WAVE MODEL: DERIVATION AND GENERALIZATION

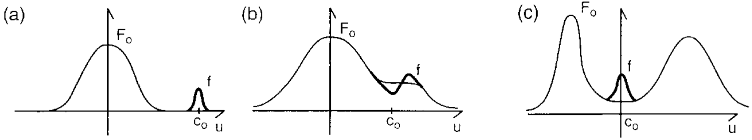
- In [dCN, Phys. Plasmas, 5, 3886 (1998)] the SWM was systematically derived as a a generic weakly nonlinear description of marginally stable Vlasov-Poisson type Hamiltonian systems of the from

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial \left(f_0+f\right)}{\partial v}=0\,, \qquad G(k)\tilde{\phi}(k,t)=-\int_{-\infty}^{\infty}\tilde{f}(k,u)du$$

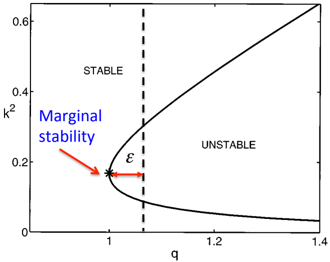
  where f_0 is a general marginally stable equilibrium, f is a small localized resonant perturbation, and $G(k)$ is the Fourier transform of a general self-consistent coupling.
- The SWM is universal in the sense that it is independent of f_0 (provided it is marginally stable), independent of the perturbation (provided is small and localized) and most importantly independent of $G(k)$.
- In the Vlasov-Poisson case $G(k)=-k^2=\mathcal{F}\left[\partial_x^2\right]$.

THE SINGLE WAVE MODEL: DERIVATION AND GENERALIZATION

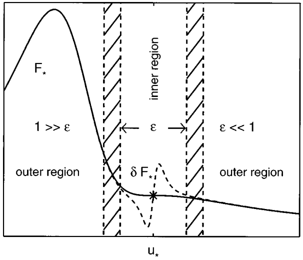
Some examples of marginally stable systems:
(a) Beam plasma; (b) Bump-on-tail; (c) Two streams



Stability diagram



Matched asymptotics



Trapping scaling $t = \varepsilon\, T$ $\quad k = k_*(1 - \varepsilon\, \Lambda)$ $\quad \Phi = \varepsilon^2 \phi$ $\quad F = \varepsilon^2 f$

THE SINGLE WAVE MODEL: DERIVATION AND GENERALIZATION

- The weakly nonlinear theory provides a systematic derivation of the previously stated SWM:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0\,, \qquad \phi = a(t)e^{ix} + a^*(t)e^{-ix}$$

$$\frac{da}{dt} - iUa = i\frac{1}{2\pi}\int_0^{2\pi} dx \int_{-\infty}^{\infty} du\, f(x,u,t)\,.$$

  and the corresponding discrete particle formulation

$$\frac{dx_k}{dt} = u_k\,, \qquad \frac{du_k}{dt} = -\frac{\partial \phi}{\partial x}\,, \qquad \frac{da}{dt} - iU = \frac{i}{N}\sum_{k=1}^{N} \Gamma_k e^{-ix_k}\,.$$

  as a universal model for marginal stable systems.
- Most importantly, going beyond the original formulation, the theory extends the SWM to $f > 0$ (clumps) and $f < 0$ (holes). In the discrete case this corresponds to $\Gamma_k > 0$ and $\Gamma_k < 0$. [dCN, Phys. Plasmas, 5 (1998); dCN, CHAOS, 10 (2000)]

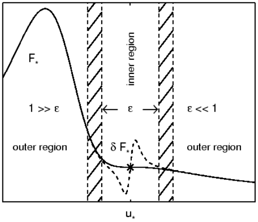
THE SINGLE WAVE MODEL: DERIVATION AND GENERALIZATION

Matched asymptotic expansion

$$\varepsilon\partial_t f + (u-c_*)\partial_x f + (F_0' + \varepsilon^2\partial_u f)\,\partial_x\phi = 0$$

$$G[nk_*(1-\varepsilon\Lambda)]\,\tilde\phi(n,t) = -\int_{-\infty}^{\infty} du'\,\tilde f(n,u',t)$$

$$\phi(x,t) = \phi_0 + \varepsilon\,\phi_1 + ...$$



•Inner region

$$v = \frac{u-c_*}{\varepsilon}$$

$$f(x,v,t) = f_0^i - \delta F_* + \varepsilon\, f_1^i + ...$$

$$\boxed{\partial_t f_0^i + v\partial_x f_0^i + \partial_v f_0^i\,\partial_x\phi_0 = 0}$$

•Outer region $\tilde f^o(n,u,t) = \tilde f_0^o + \varepsilon\,\tilde f_1^o + ...$

$$\tilde f_0^o(n,u,t) = -\frac{F_*'}{u-c_*}\tilde\phi_0(n,t) \qquad \tilde f_1^o(n,u,t) = -\frac{i}{n}\frac{F_*'}{(u-c_*)^2}\partial_t\tilde\phi_0 - \frac{F_*'}{u-c_*}\tilde\phi_1$$

THE SINGLE WAVE MODEL: DERIVATION AND GENERALIZATION

Self-consistent potential

Poisson equation: $$G[nk_*(1-\varepsilon\Lambda)]\,\tilde{\phi}(n,t) = -\int du\, \tilde{f}^{o}(n,u',t) - \varepsilon\int dv\, \tilde{f}^{i}(n,v,t)$$

•$O(1)$: $$G[nk_*]\,\tilde{\phi}_0(n,t) = -\int du\, \tilde{f}_0^{o}(n,u',t)$$

$$\tilde{f}_0^{o}(n,u,t) = -\frac{F_*'}{u-c_*}\tilde{\phi}_0(n,t)$$

$$\boxed{\phi_0(x,t) = a(t)\, e^{ix} + a^*(t)\, e^{-ix}}$$

•$O(\varepsilon)$: $$G[nk_*]\,\tilde{\phi}_1(n,t) - nk_*\Lambda G'[nk_*]\,\tilde{\phi}_0(n,t) = -\int du\, \tilde{f}_1^{o}(n,u,t) + \varepsilon\int dv\, \tilde{f}_0^{i}(n,v,t)$$

$$\tilde{f}_1^{o}(n,u,t) = -\frac{i}{n}\frac{F_*'}{(u-c_*)^2}\partial_t\tilde{\phi}_0 - \frac{F_*'}{u-c_*}\tilde{\phi}_1$$

$$\boxed{\frac{da}{dt} = iU\,a + i\int dx\, e^{-ix}\int dv\, f_0^{i}(x,v,t)}$$

THE SINGLE WAVE MODEL IN FLUIDS MECHANICS

- In the limit of strong shear, vorticity perturbation, f, can be described by the Vlasov-Poisson type system

$$\frac{\partial f}{\partial t}+y\frac{\partial f}{\partial x}+\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial y}=0\,, \qquad G(k)\tilde{\phi}(k,t)=-\int_{-\infty}^{\infty}\tilde{f}(k,y,t)dy$$

  where $G(k)=2k\coth k$ and $\psi=-y^2/2+\phi(x,t)$ is the fluid velocity stream-function. [Balmforth, dCN, Young, JFM, 333 (1997)].
- From here, application of the previously described weakly-nonlinear theory leads to a SWM description of vortex dynamics in strong, marginally stable shear flow flows. [Churilov and Shukhman, Geophys. Astrophys. Fluid. Dyn. 38 (1987); dCN, Phys. Plasmas, 5 (1998); dCN, CHAOS, 10 (2000)].
- For extensions of the single wave model for more general marginal stable perturbations in the context of shear flow dynamics see [Balmforth, Piccolo, JFM 2001]

THE SINGLE WAVE MODEL IN STATISTICAL MECHANICS

$$\frac{d^2 x_j}{dt^2} = ia(t)e^{ix_j} - ia^*(t)e^{-ix_j}\,, \qquad \frac{da}{dt} - iUa = \frac{i}{N}\sum_{k=1}^{N}\Gamma_k e^{-ix_k}\,.$$

- Neglecting the term da/dt, and assuming

  $$U = -2/\epsilon\,, \qquad \Gamma_k = 1 \qquad k = 1, \ldots N\,,$$

  the SWM model reduces to the Hamiltonian Mean Field Model [Antoni,Ruffo, 1995] used in the statistical mechanics of systems with long-range interactions

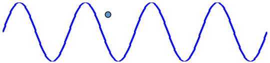
  $$\frac{d^2 x_j}{dt^2} = -\frac{\epsilon}{N}\sum_{k=1}^{N}\sin\left(x_j - x_k\right)$$

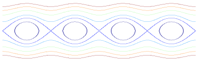
- This close connection allows the application of ideas and methods from plasma physics to statistical mechanics, including Landau damping, kinetic instabilities, BGK modes, etc. [dCN, Chapter in "Dynamics and thermodynamics of systems with Long-range interactions". Lecture Notes in Physics, Springer Vol. 602 (2002).]

THE SWM AND SELF-CONSISTENT CHAOS

Integrable motion in a one-wave field

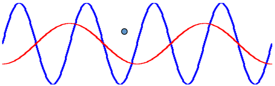
$$\phi = \cos(k_1 x - \omega_1 t)$$

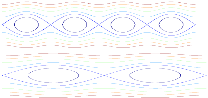


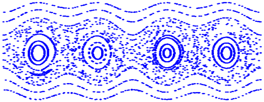


Chaotic motion in a two-waves field

$$\phi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$





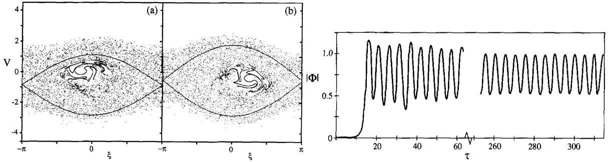


How does this well-understood picture change when we take into account self-consistency?

The SWM provides a tractable, powerful dynamical system of the right level of complexity to address this as well as many other Hamiltonian problems with many-degrees-of freedom.

THE SWM AND SELF-CONSISTENT CHAOS

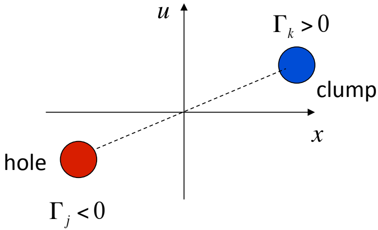
- Among the first papers to study self-consistent chaos from the point of view of dynamical systems in the SWM: Tennyson, Meiss and Morrison, Physica D 1994.



- "The system relaxes into a time asymptotic periodic state where only few collective degrees of freedom are active". "Self-consistency seems to effectively reduce the number of degrees of freedom". This motivates the concept of macro-particle.

COHERENT STRUCTURES AND SELF-CONSISTENT CHAOS

- The quasi-coherent macro particle oscillations in the beam-plasma instability are closely related to exact non-linear solutions known as BGK (Bernestein-Greene-Kruskal) modes and to the exact integrability of the $N = 1$ SWM.
- Motivated by this, in [DdCN, M.C. Firpo: CHAOS, 12, 496-507, (2002)], we studied the formation of more general coherent structures, different to the BGK modes.
- In particular we studied rotating dipole structures



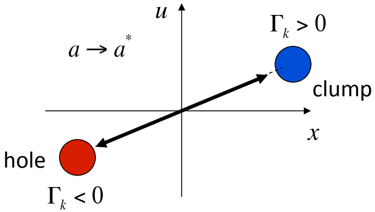
- The starting point was the study of the integrability of the $N = 2$ Single Wave Model.

INTEGRABILITY OF THE $N = 2$ SINGLE WAVE MODEL

- For N=2 the SWM has 3-degrees of freedom
- Accordingly, full integrability requires 3 constants of motion
- However, we only have 2: the Energy and the Momentum

$$H = \sum_{j=1}^{2} \left[\frac{1}{2\Gamma_j} p_j^2 - 2\,\Gamma_j \sqrt{J}\,\cos(x_j - \theta) \right] \qquad P = \sum_{j=1}^{N} p_j + J$$

- In general the N=2 problem is NOT integrable
- However, symmetric Hole-Clump states have an additional symmetry which allows integrability in this special case.



Hole-clump symmetry

$$\boxed{\left(x_k, u_k, a, \Gamma_k\right) \rightarrow \left(-x_k, -u_k, a^*, -\Gamma_k\right)}$$

INTEGRABILITY OF THE $N = 2$ SINGLE WAVE MODEL

$$x_1 = -x_2 = x \qquad u_1 = -u_2 = u$$

$$a = a^* \qquad \Gamma_1 = -\Gamma_2 = \Gamma$$

Hole-clump symmetry
reduce the system to 3 ODEs

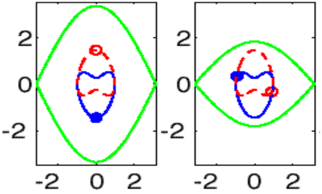
$$\boxed{\begin{aligned} \frac{dx}{dt} &= u \\ \frac{du}{dt} &= -2a\sin x \\ \frac{da}{dt} &= 2\,\Gamma\sin x \end{aligned}}$$

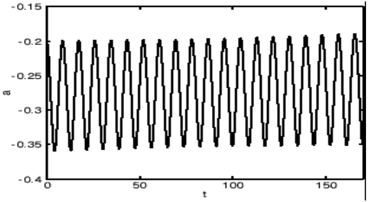
Momentum conservation

$$P = 2\Gamma u + a^2$$

Reduces the system to a
1-D Hamiltonian system

$$\boxed{\begin{gathered} \frac{dx}{d\tau} = \frac{\partial H}{\partial A} \qquad \frac{dA}{d\tau} = -\frac{\partial H}{\partial x} \\ H = \alpha A - \frac{A^3}{3} + \cos x \end{gathered}}$$



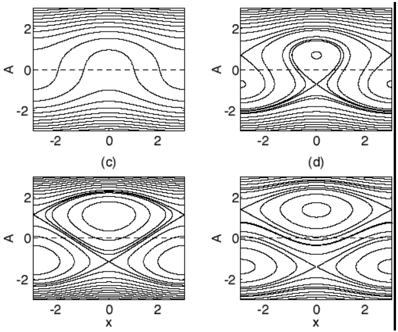


INTEGRABILITY OF THE $N = 2$ SINGLE WAVE MODEL

Remarkably, the reduced, integrable Hamiltonian system is a non-twist Hamiltonian system!

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial A} \qquad \frac{dA}{d\tau} = -\frac{\partial H}{\partial x}$$

$$H = \alpha A - \frac{A^3}{3} + \cos x$$



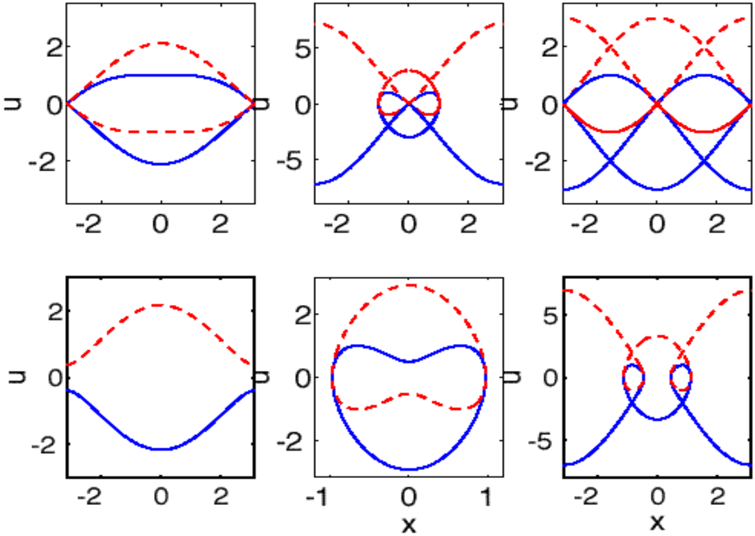
Separatrix reconnection

$$P < \left(\frac{3\Gamma^2}{2}\right)^{3/2}$$ Homoclinic

$$P > \left(\frac{3\Gamma^2}{2}\right)^{3/2}$$ Heteroclinic

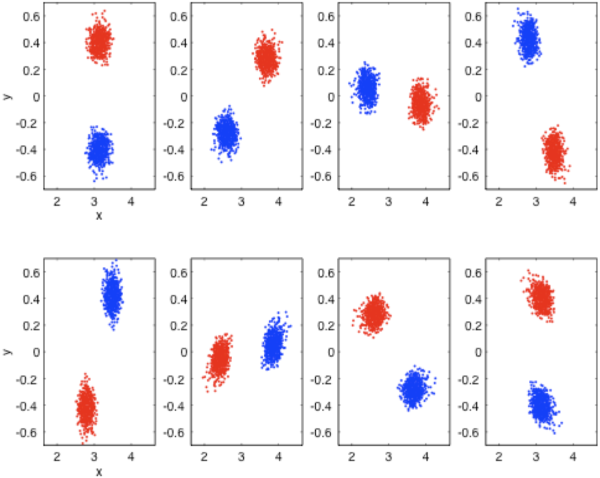
INTEGRABILITY OF THE $N = 2$ SINGLE WAVE MODEL

Some examples of symmetric hole-clump integrable orbits



DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

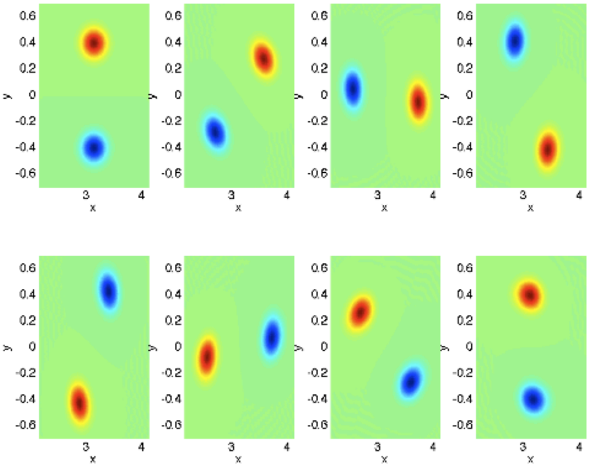
Numerical simulation of the finite-N single wave model



- D. del-Castillo-Negrete, M.C. Firpo: CHAOS, 12, 496-507, (2002).

DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

Numerical simulation of the continuum single wave model



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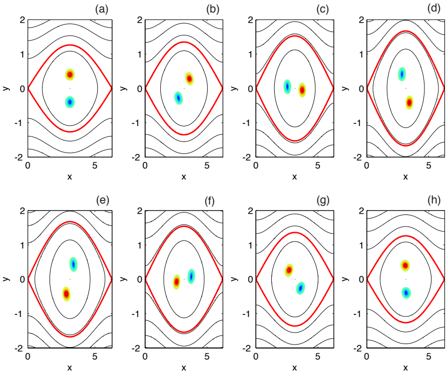
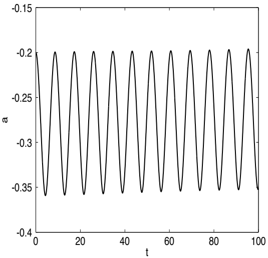
DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

Numerical simulation of the continuum single wave model



Self-consistent periodic evolution of wave mean field

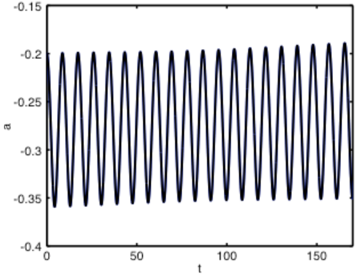
$a(t)$

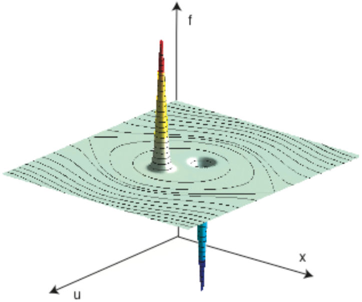


Separatrix “breathing” due to self-consistent wave-particle interaction

DIPOLE COHERENT STRUCTURES IN THE SINGLE WAVE MODEL

Time periodic mean field
Coherent rotating dipole



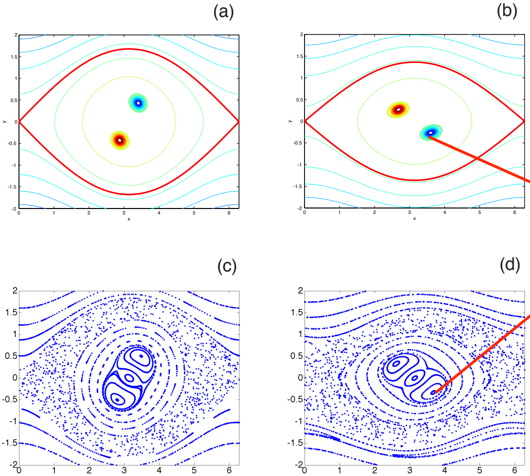


$$\boxed{\frac{da}{dt} = \frac{i}{2\pi}\int dx \int du\, e^{-ix}\, C(x,u,t)}$$

ROTATING DIPOLE COHERENT STRUCTURES AND SELF-CONSISTENT CHAOS

Rotating coherent dipole

Poincare section of time periodic self-consistent mean-field



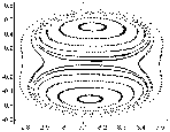
Coherence maintained by KAM surfaces

DIPOLE ROTATION AND SELF-CONSISTENT CHAOS

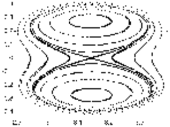
Parametric instability in Poincare section
for test particle in dipole field

$P > P_c$

$P = P_c$



$P < P_c$



$$\boxed{P_c = \left(\frac{5\Gamma^2}{6}\right)^{2/3}}$$

Particle mean field resonance

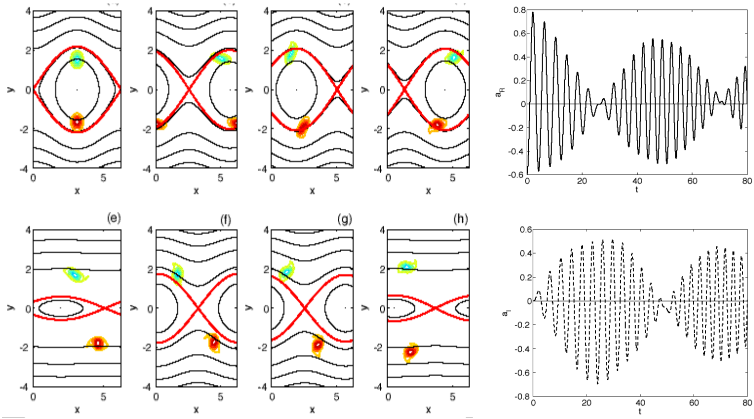
$$a(t) = \frac{\omega^2}{2}\left[1 - 2\,\varepsilon\cos\Omega t + \mathrm{K}\right]$$

$$x(t) = \omega^3\left[\,\varepsilon\sin\,\Omega t + \mathrm{K}\,\right]$$

Stability of the origin

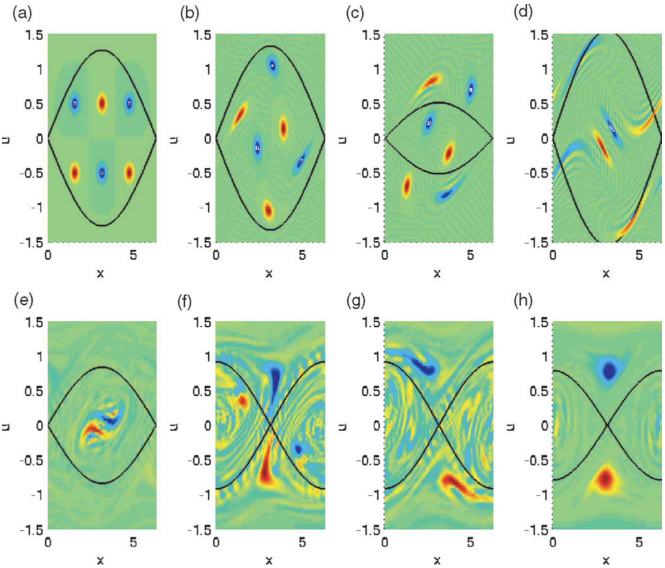
$$\boxed{q''(t) = 2\,a(t)\,q}$$

ASYMMETRIC DIPOLE STATE



- D. del-Castillo-Negrete, Plasma Physics and Controlled Fusion **47**, 1-11 (2005).

RELAXATION TOWARDS ROTATING DIPOLE STATE



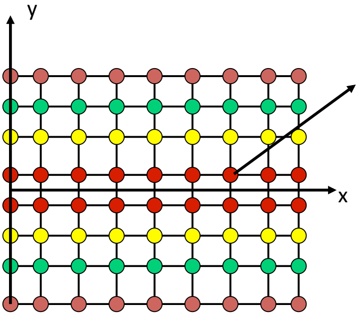
STANDARD MEAN FIELD MAP

$$\left.\begin{aligned} \frac{d\,x_j}{dt} &= y_j \qquad\qquad j = 1,2,\ldots\ldots N \\ \frac{dy_j}{dt^2} &= -2\rho(t)\sin\left[x_j - \theta(t)\right] \end{aligned}\right\} \text{particles} \quad \left\{\boxed{\begin{aligned} x_k^{n+1} &= x_k^n + y_k^{n+1} \\ y_k^{n+1} &= y_k^n - \kappa^{n+1}\sin\left(x_k^n - \theta^n\right) \end{aligned}}\right.$$

$$\left.\frac{d}{dt}\left(\rho\,\mathrm{e}^{-i\theta}\right) + i\,U\rho\,\mathrm{e}^{-i\theta} = i\sum_k \Gamma_k\, e^{-ix_k}\right\} \begin{matrix}\text{mean}\\ \text{field}\end{matrix} \quad \left\{\boxed{\begin{aligned} \kappa^{n+1} &= \sqrt{\left(\kappa^n\right)^2 + \left(\eta^n\right)^2} + \eta^n \\ \theta^{n+1} &= \theta^n + \frac{1}{\kappa^{n+1}}\frac{\partial\eta^n}{\partial\theta^n} \\ \eta^n &= \sum_{j=1}^{N}\gamma_j\sin\left(x_j^n - \theta^n\right) \end{aligned}}\right.$$

- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

STANDARD MEAN FIELD MAP



$\left(x_k^0\ ,y_k^0\right)$ Particles initial positions

$$k = 1,2,\ldots\ldots 2\times 10^5$$

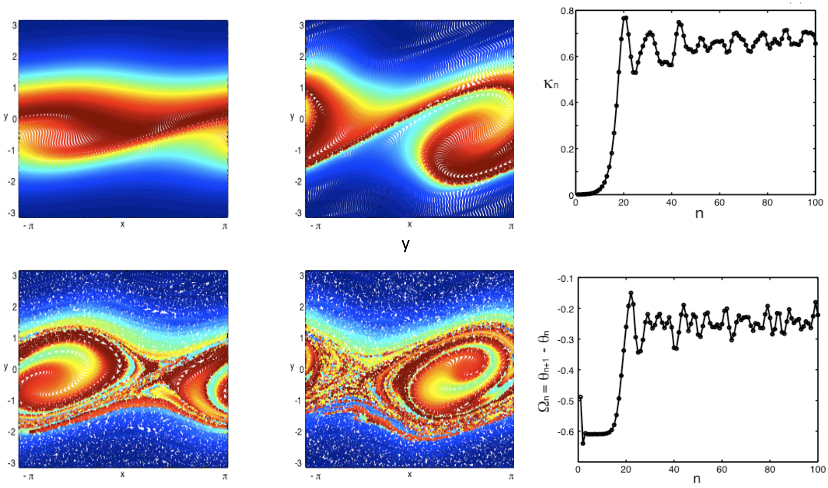
Particle's "charge"

$$\boxed{\gamma_k = \frac{\tau^3}{\pi}\exp\left(\frac{-y_k^2}{2}\right)}$$

Mean field initial condition

$$\boxed{\theta^0 = 0 \qquad \kappa^0 = 0.001}$$

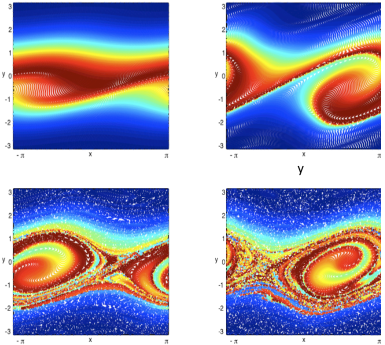
BEAM-PLASMA INSTABILITY AND COHERENT STRUCTURE FORMATION IN THE STANDARD MEAN-FIELD MAP



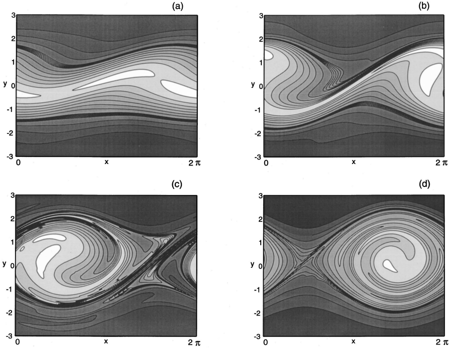
- D. del-Castillo-Negrete: CHAOS, 10, 75, (2000).

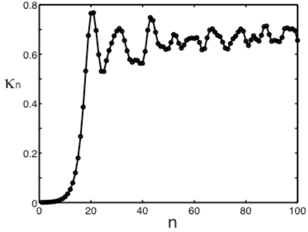
COMPARISON BETWEEN MAP AND CONTINUOUS SWM

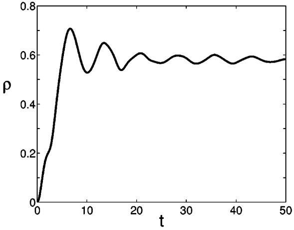
Meam Field Map



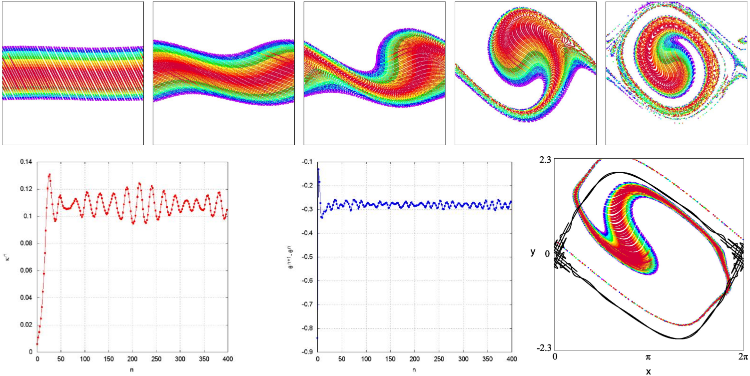
Continuous SWM







MORE RECENT SIMUALTIONS



D. Martinez-del-Rio, D. del-Castillo-Negrete, A. Olvera and R. Calleja Qualitative Theory of Dynamical Systems, **14** 313-335 (2016).

NONTWIST MEAN FIELD MAP

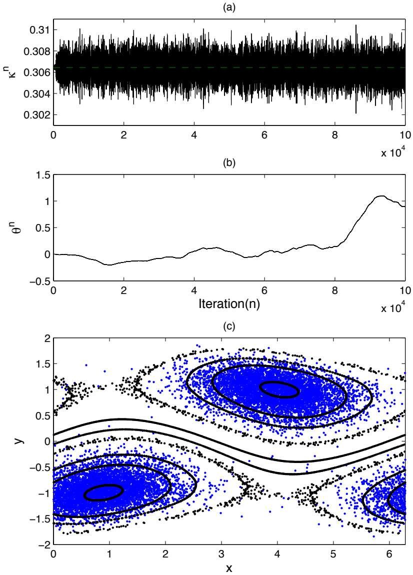
$k = 1,\, 2,\, \ldots N$

$$\begin{aligned}
x_k^{n+1} &= x_k^n + a\left[1 - \left(\frac{\tau}{\Gamma_k} p_k^{n+1}\right)^2\right], \\
p_k^{n+1} &= p_k^n - 2\tau\Gamma_k\sqrt{J^{n+1}}\sin\left(x_k^n - \theta^n\right), \\
\theta^{n+1} &= \theta^n - U\tau - \frac{\tau}{\sqrt{J^{n+1}}}\sum_{k=1}^{N}\Gamma_k\cos\left(x_k^n - \theta^n\right), \\
J^{n+1} &= J^n + 2\tau\sqrt{J^{n+1}}\sum_{k=1}^{N}\Gamma_k\sin\left(x_k^n - \theta^n\right),
\end{aligned} \tag{1}$$

• L. Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22 013137 (2012).

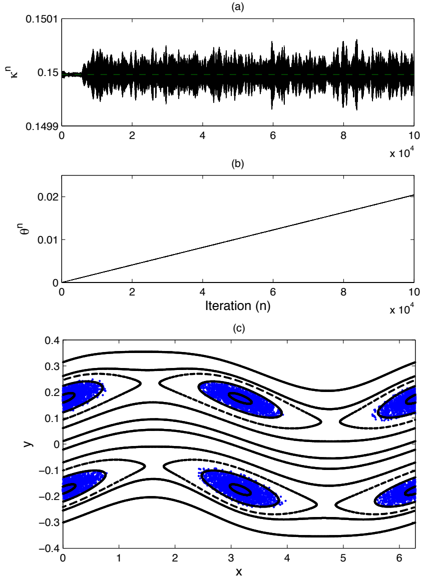
NONTWIST MEAN FIELD MAP

Period-one coherent structures



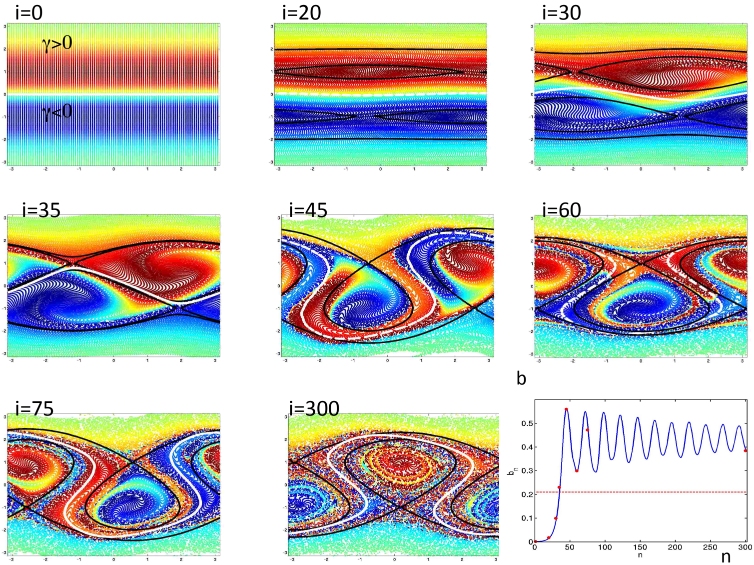
NONTWIST MEAN FIELD MAP

Period-two coherent structures



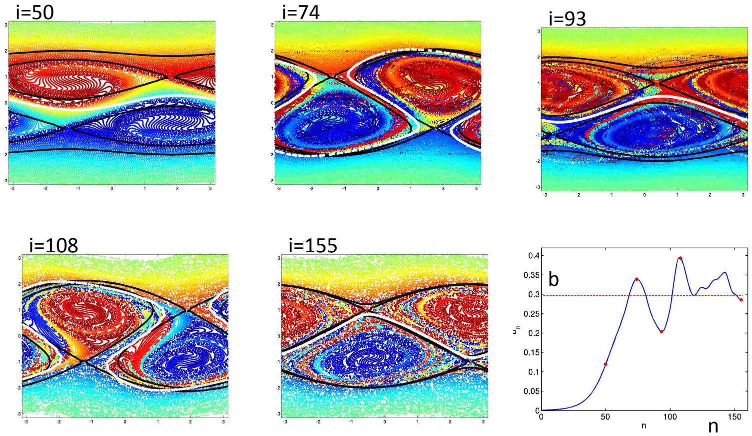
NONTWIST MEAN FIELD MAP

Separatrix reconnection and coherent structure formation



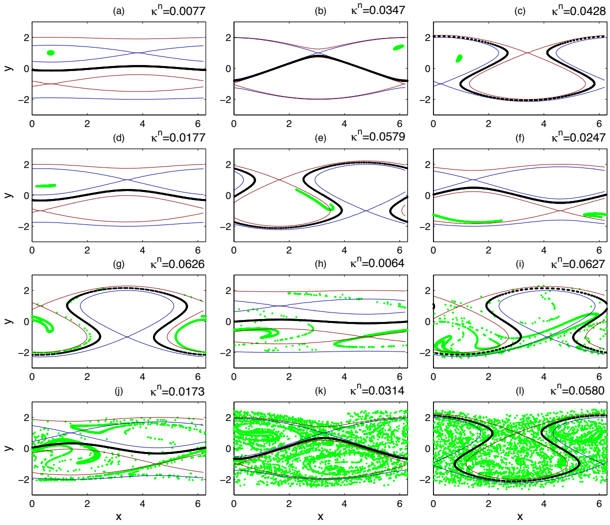
NONTWIST MEAN FIELD MAP

Separatrix reconnection and coherent structure formation



NONTWIST MEAN FIELD MAP

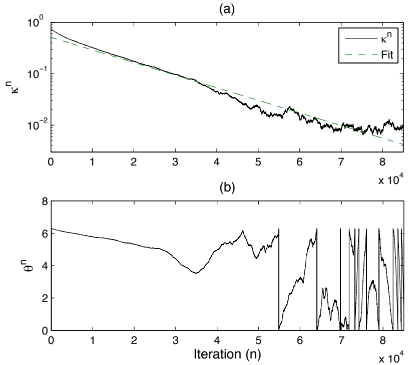
Self-consistent separatrix reconnection in the mean-field map

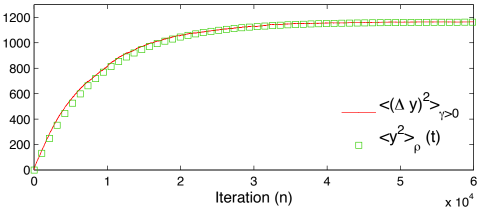


- L. Carbajal, D. del-Castillo-Negrete, and J. J. Martinell, Chaos, 22

NONTWIST MEAN FIELD MAP

Self-consistent suppression of diffusion

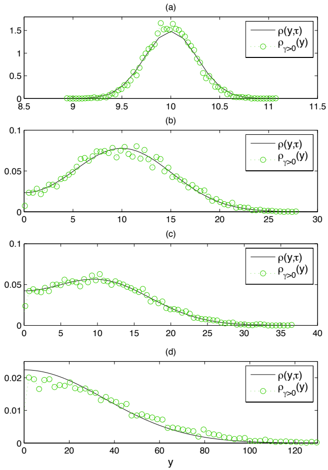




NONTWIST MEAN FIELD MAP

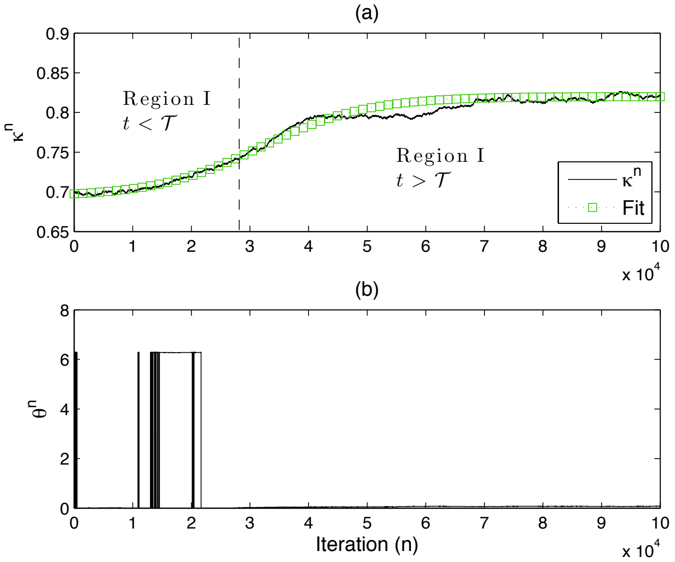
Self-consistent suppression of diffusion

$$\frac{\partial \rho}{\partial t} = D_{QL}(t)\,\frac{\partial^2 \rho}{\partial y^2}\,, \qquad D_{QL}(t) = \frac{\kappa_0^2}{4}\,e^{-2t\nu}\,.$$



NONTWIST MEAN FIELD MAP

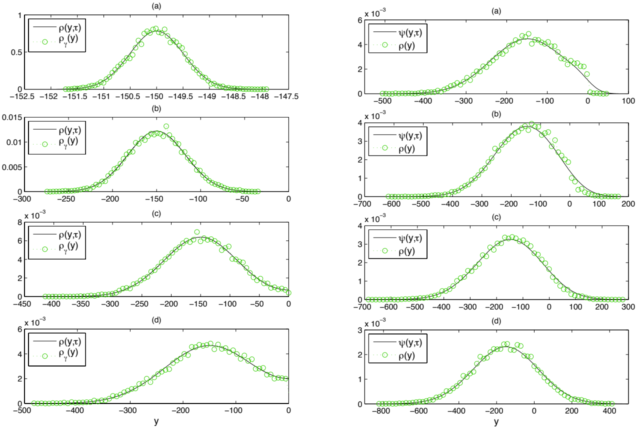
Self-consistent transition to global chaos



NONTWIST MEAN FIELD MAP

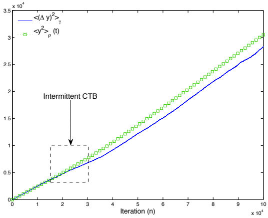
Self-consistent transport across shearless barrier

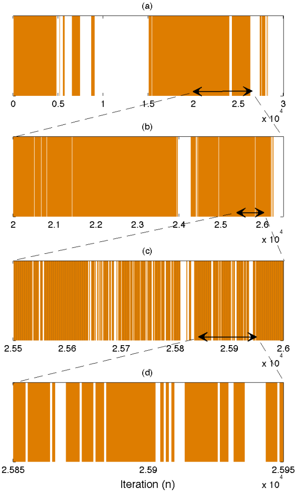
$$\frac{\partial \rho}{\partial t} = D_{QL}(t)\,\frac{\partial^2 \rho}{\partial y^2}\,, \qquad D_{QL}(t) = \frac{1}{4}\left[K + \alpha \tanh\left(\frac{t-\mu}{\beta}\right)\right]^2$$



NONTWIST MEAN FIELD MAP

Self-consistent intermittent transport near criticality





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